Derivation of Engineering-relevant Deformation Parameters from Repeated Surveys of Surface-like Constructions

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Surface-like Construction:
One dimension insignificant compared to the other two.
Behaves like a thin-shell
Usual surveying approach to “deformation”: Determination of 3-dimensional displacements between two epochs \( t \) and \( t' \) (usually at particular discrete points)
The construction engineer approach to “deformation” (Strength of material point of view):

Local deformation (strain) related to forces (stresses) through the constitutive equations of the material (stress-strain relations)
Dilatation (change of area) at a point $P$:

$$\Delta = \lim_{E \to 0} \frac{E' - E}{E}$$

(relative change of area)

Dilatation may be positive (expansion) or negative (shrinkage)
Three types of relevant local deformation parameters

Dilatation (change of area) at a point P:

Relates to central stresses (forces)

positive dilatation (expansion)
Three types of relevant local deformation parameters

Dilatation (change of area) at a point P:

Relates to central stresses (forces)

negative dilatation (shrinking)
Three types of relevant local deformation parameters

Shear strain at a point $P$ in a particular direction:

$$\gamma = \tan \omega$$

We seek the direction where maximum shear occurs.
Three types of relevant local deformation parameters

Maximum shear strain at a point P:

Relates to shearing stresses (tearing forces)
Bending at a point P in a particular direction: Realized by the change of the radius of curvature $R$ of the corresponding normal section (radius of curvature = radius of best fitting circle)

We seek the direction where maximum bending $R' - R$ occurs
Three types of relevant local deformation parameters

Bending at a point $P$ in a particular direction:

Relates to bending torques
Planar deformation is completely determined by the gradient matrix $F$. 

$$F = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{bmatrix}$$
Planar deformation is completely determined by the gradient matrix $F$.

![Diagram showing planar deformation before and after transformation](image)

The transformation matrix $F$ is given by the gradient matrix:

$$
F = \begin{bmatrix}
\frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\
\frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y}
\end{bmatrix}
$$

Diagonalization of $F$:

$$
F = \begin{bmatrix}
\cos \theta' & -\sin \theta' \\
\sin \theta' & \cos \theta'
\end{bmatrix}
\begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix}
\begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}
$$
Planar deformation is completely determined by the gradient matrix $F$.

**Diagonalization of $F$**

Before

$$\begin{bmatrix}
\frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y'} \\
\frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y}
\end{bmatrix}$$

$$F = 
\begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix}
$$

After

$$F = \begin{bmatrix}
\cos \theta' & -\sin \theta' \\
\sin \theta' & \cos \theta'
\end{bmatrix}
\begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix}
\begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}$$
Planar deformation is completely determined by the gradient matrix $F$.

$F$ is the gradient matrix of the deformation:

$$F = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{bmatrix}$$

The diagonalization of $F$ is:

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Deformation only from the eigenvalues of $F$!
Planar deformation is completely determined by the gradient matrix $F$.

\[
F = \begin{bmatrix}
\frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\
\frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y}
\end{bmatrix}
\]

Diagonalization of $F$:

\[
\begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix}
\]

\[
\Delta = \lambda_1 \lambda_2 - 1
\]

\[
\gamma = \frac{\lambda_1 - \lambda_2}{\sqrt{\lambda_1 \lambda_2}}
\]
For dilatation and maximum shear the surface is approximated by the local tangent plane at $P$ (best fitting plane).
Deformation of a curved surface

For dilatation and maximum shear the surface is approximated by the local tangent plane at P (best fitting plane)

For maximum bending the surface is approximated by the local oscillating ellipsoid at P (best fitting ellipsoid)
Deformation of a curved surface – Dilatation and maximum shear strain

On a curved surface only curvilinear coordinates \((u,v)\) can be used

Coice of curvilinear coordinates: horizontal cartesian coordinates
\[ u = X, \quad v = Y \]

at the original epoch \(t\)

At second epoch \(t'\):
Use as coordinates those of original epoch \(t\)
\[ u' = X, \quad v' = Y \]

(\textit{convected coordinates})

Curvilinear deformation gradient
\[
\mathbf{F}_q = \begin{bmatrix}
\frac{\partial u'}{\partial u} & \frac{\partial u'}{\partial v} \\
\frac{\partial v'}{\partial u} & \frac{\partial v'}{\partial v}
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = \mathbf{I}
\]

\(\mathbf{F}_q = \mathbf{I}\) : simple but inappropriate because it refers to oblique axes and non unit basis vectors
Deformation of a curved surface – Dilatation and maximum shear strain

original epoch $t$

$F_q = I$

second epoch $t'$

$\nu' = \nu$

$u' = u$
Deformation of a curved surface – Dilatation and maximum shear strain

original epoch $t$

$F_q = I$

second epoch $t'$

transformation to orthonormal systems

$S^{-1}$

$q = FI$

$S'^{-1}$
Deformation of a curved surface – Dilatation and maximum shear strain

original epoch $t$

$F_q = I$

second epoch $t'$

$v' = v$

$u' = u$

$S'$

$S'^{-1}$
Deformation of a curved surface – Dilatation and maximum shear strain

original epoch $t$

second epoch $t'$

Deformation of grid elements:

$F_q = I$

$F = S'^{-1} F_q S = S'^{-1} S$
Deformation of a curved surface – Dilatation and maximum shear strain

Diagonalization \( \mathbf{F} = \mathbf{R}(-\theta') \mathbf{L} \mathbf{R}(\theta) = \begin{bmatrix} \cos \theta' & -\sin \theta' \\ \sin \theta' & \cos \theta' \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \)

Invariant (independent of coordinate systems) deformation parameters

\( \lambda_1 > \lambda_2 \) principal elongations

\( \theta, \theta + 90^\circ \) directions of principal elongations \( \lambda_1, \lambda_2 \), respectively

\( \Delta = \lambda_1 \lambda_2 - 1 \) dilatation

\( \gamma = \frac{\lambda_1 - \lambda_2}{\sqrt{\lambda_1 \lambda_2}} \) maximum shear strain at direction angle \( \phi = \theta - \frac{1}{2} \arctan \left( \frac{2}{-\gamma} \right) \)
Deformation of a curved surface – Bending

Normal section at P:
intersection of surface with any plane containing the surface normal at P

R = radius of circle best fitting to normal section
k = 1/R curvature of normal section at P

Among all normal sections there are two perpendicular principal directions
where the curvature obtains its maximum value $k_1$ and its minimum value $k_2$
(principal curvatures)
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The curvature of any normal section at angle $\theta$ from the first principal direction is given by

$$k(\theta) = k_1 \cos^2 \theta + k_2 \sin^2 \theta$$
Deformation of a curved surface – Bending

Computation of principal curvatures

Tangent vectors

$$x_u = \frac{\partial x}{\partial u} = \begin{bmatrix} \frac{\partial u}{\partial u} X \\ \frac{\partial u}{\partial u} Y \\ \frac{\partial u}{\partial u} Z \end{bmatrix} \quad x_v = \frac{\partial x}{\partial v} = \begin{bmatrix} \frac{\partial v}{\partial v} X \\ \frac{\partial v}{\partial v} Y \\ \frac{\partial v}{\partial v} Z \end{bmatrix}$$

Normal vector

$$n = \frac{x_u \times x_v}{|x_u \times x_v|}$$

First Fundamental Form

$$G = \begin{bmatrix} x_u^T x_u & x_u^T x_v \\ x_u^T x_v & x_v^T x_v \end{bmatrix}$$

Mean curvature

$$H = \frac{2G_{22}L_{11} - 2G_{12}L_{12} + G_{11}L_{22}}{2 \det G}$$

Second Fundamental Form

$$L = \begin{bmatrix} n^T \frac{\partial^2 x}{\partial u^2} & n^T \frac{\partial^2 x}{\partial u \partial v} \\ n^T \frac{\partial^2 x}{\partial u \partial v} & n^T \frac{\partial^2 x}{\partial v^2} \end{bmatrix}$$

Gaussian curvature

$$K = \frac{\det L}{\det G}$$

$$k_1 = H + \sqrt{H^2 - K}$$

$$k_2 = H - \sqrt{H^2 - K}$$
Deformation of a curved surface – Bending

original epoch $t$

$\phi$

$k(\theta) = k_1 \cos^2 \theta + k_2 \sin^2 \theta$

$k'(\theta) = k'_1 \cos^2 \theta'(\theta) + k'_2 \sin^2 \theta'(\theta)$

$\Delta k(\theta) = k'(\theta) - k(\theta) = \text{max}$

Value $\hat{\theta}$ for maximum from numerical solution of a non-linear equation

$\hat{k} = k(\hat{\theta}) = \frac{1}{\hat{R}}$

$\hat{k}' = k'(\hat{\theta}) = \frac{1}{\hat{R}'}$

Most differing radii of curvature over all normal sections through P

second epoch $t'$

$\theta' = \theta'(\theta)$
Deformation of a curved surface - Interpolation

To compute dilatation, maximum shear strain and maximum bending we need the following functions

\[ X(u, v) \quad X'(u', v') \quad u'(u, v) \]
\[ Y(u, v) \quad Y'(u', v') \quad v'(u, v) \]
\[ Z(u, v) \quad Z'(u', v') \]

Using convective coordinates the required functions reduce to

\[ u = X, v = Y, \quad u' = u = X, v' = v = Y \]

They can be obtained from the interpolation of the available discrete data

\[ Z_i = Z(X_i, Y_i) \]
\[ \Delta X(X_i, Y_i) = X_i' - X_i \]
\[ \Delta Y(X_i, Y_i) = Y_i' - Y_i \]
\[ \Delta Z(X_i, Y_i) = Z_i' - Z_i \]
Interpolation of the available discrete data

\[ Z(X_i, Y_i), \Delta X(X_i, Y_i), \Delta Y(X_i, Y_i), \Delta Z(X_i, Y_i) \]

In general:
Interpolate a function \( f(X, Y) \) from discrete data \( f_i = f(X_i, Y_i) \)

One possibility: \textbf{Collocation}
(Minimum norm interpolation = Minimum Mean Square Error Prediction)

Use two-point covariance function: \( C(X, Y; X', Y') \)

\[
f(X, Y) = c^T C^{-1} f
\]

\[
f_i = f(X_i, Y_i) \quad C_{ik} = C(X_i, Y_i; X_k, Y_k) \quad c_i = C(X, Y; X_i, Y_i)
\]

\[
\frac{\partial f}{\partial X}(X, Y) = \frac{\partial c^T}{\partial X} C^{-1} f, \quad \frac{\partial^2 f}{\partial X \partial Y}(X, Y) = \frac{\partial^2 c^T}{\partial X \partial Y} C^{-1} f, \quad \text{etc.}
\]