The definition of a geophysically meaningful International Terrestrial Reference System Problems and prospects

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Objective (theory): Express deformation of the earth by the temporal variation of earth point coordinates with respect to an optimally selected International Terrestrial Reference System (ITRS)

Objective (practice): Provide a realization of the ITRS by an International Terrestrial Reference Frame (ITRF), consisting of the coordinate functions $x_i(t), y_i(t), z_i(t)$ of points $P_i$ in a global geodetic network.
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Available data: Coordinates $x_i(t_k)$, $y_i(t_k)$, $z_i(t_k)$ of points in subnetworks at discrete epochs $t_k$ (e.g. daily or weekly solutions) with respect to a different reference systems for each subnetwork
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<td>Problem (theory):</td>
<td>Introduce a principle leading to an optimal choice of the reference frame</td>
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<td>Problem (practice):</td>
<td>Combine the known shapes of the overlapping subnetworks into the shape of a global network expressed by coordinates with respect to the already defined optimal reference system</td>
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The optimal choice of reference system
General characteristic: Apparent motion (coordinate variation) of mass points should be minimal

For the whole earth

\[ \text{Tisserand principle:} \]

Vanishing relative angular momentum

\[ \mathbf{h}_R = \int_{\text{Earth}} [\mathbf{x} \times] \frac{d\mathbf{x}}{dt} \, dm = 0 \]

Minimal relative kinetic energy

\[ T_R = \int_{\text{Earth}} \left( \frac{d\mathbf{x}}{dt} \right)^T \frac{d\mathbf{x}}{dt} \, dm = \min \]
The optimal choice of reference system
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\[ \mathbf{h}_R = \int_{\text{Earth}} [\mathbf{x} \times] \frac{d\mathbf{x}}{dt} \, dm = 0 \]

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Discrete Tisserand principle:

Vanishing relative angular momentum

\[ \mathbf{h}_R = \sum_i [\mathbf{x}_i \times] \frac{d\mathbf{x}_i}{dt} \, m_i = 0 \]

Minimal relative kinetic energy

\[ T_R = \sum_i \left( \frac{d\mathbf{x}_i}{dt} \right)^T \frac{d\mathbf{x}_i}{dt} \, m_i = \text{min} \]
The optimal choice of reference system
General characteristic: Apparent motion (coordinate variation) of mass points should be minimal

For the whole earth

\[ M = \int_{\text{Earth}} dm \]

\[ \mathbf{m} = \frac{1}{M} \int_{\text{Earth}} \mathbf{x} \, dm = \text{const.} \]

Tisserand principle:

For a network of discrete points

\[ M = \sum_{i} m_{i} \]

\[ \mathbf{m} = \frac{1}{M} \sum_{i} \mathbf{x}_{i} m_{i} = \text{const.} \]

Discrete Tisserand principle:

Define orientation of the reference system.

For the position:

Preservation of the center of mass:

Trivial choice:

\[ m_{i} = 1 \]
The “linear” model

Is there a linear model for deformation? **No!**

It is not possible to have all shape related parameters which are coordinate invariants as linear functions of time!
Is there a linear model for deformation? **No!**

A planar counterexample:

The 5 shape defining sides are linear functions of time:

\[ s_i = a_i \, t + b_i, \quad i = 1,2,3,4,5 \]

It is not possible to have all shape related parameters which are coordinate invariants as linear functions of time!
Is there a linear model for deformation? **No!**

A planar counterexample:

The 5 shape defining sides are linear functions of time:

\[ s_i = a_i t + b_i, \quad i = 1,2,3,4,5 \]

\[ \Rightarrow \quad d = \text{nonlinear in } t \]

The 2nd diagonal of the quadrilateral is not a linear function of time!
The “linear” model

What is the meaning of linear model for coordinate functions?

Linear coordinates in one reference system

\[ x(t) = x_0 t + v_x , \]
\[ y(t) = y_0 t + v_y , \]
\[ z(t) = z_0 t + v_z , \]

Non-linear coordinates in another

\[ x'(t) = R(t) x_0 t + R(t) v + c(t) \neq x_0'(t) t + v' \]
The “linear” model

What is the meaning of linear model for coordinate functions?

\[ x(t) = x_0 \ t + v_x , \]
\[ y(t) = y_0 \ t + v_y , \]
\[ z(t) = z_0 \ t + v_z , \]

\[ \mathbf{x}(t) \]
\[ \mathbf{x}'(t) = \mathbf{R}(t) \ \mathbf{x}(t) + \mathbf{c}(t) \]
are both Tisserand

\[ \mathbf{R}(t) = \mathbf{R} = \text{const.} \]
\[ \mathbf{c}(t) = \mathbf{c} = \text{const.} \]
The “linear” model

What is the meaning of linear model for coordinate functions?

\[
x(t) = x_0 + t \, v_x, \\
y(t) = y_0 + t \, v_y, \\
z(t) = z_0 + t \, v_z,
\]

Ad hoc definition of “linear deformation” model:

- Coordinates in one (and hence all) Tisserand system are linear functions of time
- \( x(t) = x(t) \)
- \( x'(t) = R(t) \, x(t) + c(t) \)
- \( R(t) = R = \text{const.} \)
- \( c(t) = c = \text{const.} \)

Note: **Tisserand reference system** here means:
- Orientation by Tisserand principle (zero relative angular momentum)
- Position by center of mass preservation principle
The Tisserand reference system conditions for the linear model

\[ x_i(t) = x_{0i} + tv_i \quad i = 1, 2, \ldots, N \]

Vanishing relative angular momentum:

\[ h_R(t) = \sum_i m_i [x_{0i} \times v_i] = 0 \]

Constant (zero) center of mass coordinates:

\[ m(t) = \frac{1}{M} \sum_i m_i x_{0i} + t \frac{1}{M} \sum_i m_i v_i = \text{const.} = 0 \]

Tisserand conditions:

\[ \sum_i m_i [x_{0i} \times v_i] = 0 \]
\[ \sum_i m_i x_{0i} = 0 \]
\[ \sum_i m_i v_i = 0 \]

Usual choice:

\[ m_i = 1 \]

Tisserand conditions:

\[ \sum_i [x_{0i} \times v_i] = 0 \]
\[ \sum_i x_{0i} = 0 \]
\[ \sum_i v_i = 0 \]
The Tisserand reference system conditions for the linear model

Used at the stage of data analysis as additional constraints

Usual inner constraints for the definition of reference frame at epoch $t_0$ =

Selection of a single Tisserand frame out of a set of dynamically equivalent ones

Tisserand conditions:

$$\sum_i m_i [x_{0i} \times] v_i = 0$$

$$\sum_i m_i x_{0i} = 0$$

$$\sum_i m_i v_i = 0$$

Usual choice:

$$m_i = 1$$

$$_i = 1,2,\ldots,N$$

$$[x_{0i} \times] v_i = 0$$

$$\sum_i m_i v_i = \text{const.} = 0$$

$$x_{0i} + t \frac{1}{M} \sum_i m_i v_i = 0$$

Usual inner constraints for the definition of reference frame at epoch $t_0$ +

Selection of a single Tisserand frame out of a set of dynamically equivalent ones
Accessing a geophysically meaningful reference system

\[
\tilde{x}(t) = R(t) \; x(t)
\]

\[
\tilde{h}_R = \int_{\text{Earth}} [\tilde{x} \times] \frac{d\tilde{x}}{dt} \; dm = 0
\]

To access \( R(t) \) transforming \( h_R(\text{Earth}) \) to \( \tilde{h}_R(\text{Earth}) = 0 \) we need an approximation of \( h_R(\text{Earth}) \).
Accessing a geophysically meaningful reference system

\[ \mathbf{x}(t) \]  
(discrete) Tisserand

\[ \mathbf{\tilde{x}}(t) = \mathbf{R}(t) \mathbf{x}(t) \]  
(earth) Tisserand

\[ \mathbf{h}_R = \sum_i [\mathbf{x}_i \times] \frac{d\mathbf{x}_i}{dt} m_i = 0 \]

\[ \mathbf{\tilde{h}}_R = \int_{\text{Earth}} [\mathbf{\tilde{x}} \times] \frac{d\mathbf{\tilde{x}}}{dt} \, dm = 0 \]

To access \( \mathbf{R}(t) \) transforming \( \mathbf{h}_R\)(Earth) to \( \mathbf{\tilde{h}}_R\)(Earth)=0 we need an approximation of \( \mathbf{h}_R\)(Earth)

\[ \text{Earth} \equiv E = P_1 + \ldots + P_K + \ldots + P_L + P_0 \]

\[ \text{Network} \equiv D = D_1 + \ldots + D_K + \ldots + D_L \]

\[ \mathbf{\tilde{h}}_R = \sum_K \mathbf{\tilde{h}}_{P_K} + \mathbf{\tilde{h}}_{P_0} = 0 \]

Use \( D_K \) to access \( \mathbf{h}_{P_K}(t) \)!
Estimation of the rigid motion of a plate

\[ x' = R_K x \]

Transformation such that

\[ h'_R(D_K) = 0 \]

original velocities
Estimation of the rigid motion of a plate

\[ x' = R_K x \]  
\[ h'_R(D_K) = 0 \]

Transformation such that the original velocities \( h'_R(D_K) \) are transformed to the reference system best fitted to \( D_K \).
Estimation of the rigid motion of a plate

$x' = R_K x$  \quad \text{Transformation such that} \quad h'_R (D_K) = 0$

- Original velocities
- Transformed velocities
- Determine rotation $R_K$ to reference system best fitted to $D_K$
- Transformed velocities
- Assume rigid motion of plate $P_K$ due only to rotation $R_K$
- Velocities due to rigid plate rotation only
Estimation of the rigid motion of a plate

\[ \mathbf{x}' = R_K \mathbf{x} \]

Transformation such that

\[ \mathbf{h}'_R(D_K) = 0 \]

original velocities

determine rotation \( R_K \) to reference system best fitted to \( D_K \)

transformed velocities

assume rigid motion of plate \( P_K \) due only to rotation \( R_K \)

velocities due to rigid plate rotation only

compute relative angular momentum of plate \( P_K \) due only to rotation \( R_K \)

\[ \mathbf{h}_R(P_K) = \int_{P_K} [\mathbf{x} \times \mathbf{v}] \, dm \]
Final transformation to an approximate Earth-Tisserand reference system

\[ \mathbf{h}_R = \sum_{K} \mathbf{h}_R(P_K) \]

\[ \mathbf{\tilde{x}} = \mathbf{R} \mathbf{x} \]

\[ \tilde{\mathbf{h}}_R = 0 \]

Previously estimated
Final transformation to an approximate Earth-Tisserand reference system

\[ \mathbf{h}_R = \sum_{K} \mathbf{h}_R(P_K) \]

\[ \tilde{\mathbf{x}} = \mathbf{R} \mathbf{x} \]

\[ \tilde{\mathbf{h}}_R = 0 \]

Previously estimated

Approximate (numerically sufficient) solution for the linear model:

\[ x_i(t) = x_{0i} + t \mathbf{v}_i \]
Final transformation to an approximate Earth-Tisserand reference system

\[ h_R = \sum_K h_R(P_K) \]

\[ \tilde{x} = R x \]

\[ \tilde{h}_R = 0 \]

Previously estimated

Approximate (numerically sufficient) solution for the linear model:

\[ x_i(t) = x_{0i} + t v_i \]

1. Subnetwork \( D_K \)
   - Relative angular momentum
     \[ h_K = \sum_{i \in D_K} [x_{0i} \times] v_i \]
   - Inertia matrix
     \[ C_{0K} = -\sum_{i \in D_K} [x_{0i} \times][x_{0i} \times] \]
Final transformation to an approximate Earth-Tisserand reference system

**1. Subnetwork $D_K$**

- Relative angular momentum
  \[ h_K = \sum_{i \in D_K} [x_{0i} \times] v_i \]

- Inertia matrix
  \[ C_{0K} = - \sum_{i \in D_K} [x_{0i} \times][x_{0i} \times] \]

**2. Plate $P_K$**

- Rotation vector
  \[ \omega_{K0} = C_{0K}^{-1} h_K \]

- Inertia matrix
  \[ C_{P_K} \equiv - \int_{P_K} [x_0 \times][x_0 \times] dm \]

Previously estimated

Approximate (numerically sufficient) solution for the linear model:

\[ x_i(t) = x_{0i} + t v_i \]
Final transformation to an approximate Earth-Tisserand reference system

\[ h_R = \sum_K h_R(P_K) \]

\[ \tilde{\mathbf{x}} = R \mathbf{x} \]

\[ \tilde{h}_R = 0 \]

Previously estimated

Approximate (numerically sufficient) solution for the linear model:

\[ \mathbf{x}_i(t) = \mathbf{x}_{0i} + t \mathbf{v}_i \]

1. Subnetwork \( D_K \)
   - Relative angular momentum
     \[ \mathbf{h}_K = \sum_{i \in D_K} [\mathbf{x}_{0i} \times] \mathbf{v}_i \]
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2. Plate \( P_K \)
   - Rotation vector
     \[ \mathbf{\omega}_{K0} = \mathbf{C}_{0K}^{-1} \mathbf{h}_K \]
   - Inertia matrix
     \[ \mathbf{C}_{P_K} \equiv -\int_{P_K} [\mathbf{x}_0 \times][\mathbf{x}_0 \times] dm \]

3. Total rotation vector
   \[ \mathbf{\omega}_0 = \left( \sum_K \mathbf{C}_{P_K} \right)^{-1} \sum_K \mathbf{C}_{P_K} \mathbf{\omega}_{K0} \]
Final transformation to an approximate Earth-Tisserand reference system

\[ \mathbf{h}_R = \sum_K \mathbf{h}_R(P_K) \]

\[ \tilde{\mathbf{x}} = \mathbf{R} \mathbf{x} \]

\[ \tilde{\mathbf{h}}_R = 0 \]

Previously estimated

Approximate (numerically sufficient) solution for the linear model:

\[ \mathbf{x}_i(t) = \mathbf{x}_{0i} + t \mathbf{v}_i \]

1. **Subnetwork** \( D_K \)
   - Relative angular momentum:
     \[ \mathbf{h}_K = \sum_{i \in D_K} [\mathbf{x}_{0i} \times] \mathbf{v}_i \]
   - Inertia matrix:
     \[ \mathbf{C}_{0K} = -\sum_{i \in D_K} [\mathbf{x}_{0i} \times][\mathbf{x}_{0i} \times] \]

2. **Plate** \( P_K \)
   - Rotation vector:
     \[ \mathbf{\omega}_{0K} = \mathbf{C}^{-1}_{0K} \mathbf{h}_K \]
   - Inertia matrix:
     \[ \mathbf{C}_{P_K} \equiv -\int_{P_K} [\mathbf{x}_0 \times][\mathbf{x}_0 \times] dm \]

3. Total rotation vector:
   \[ \mathbf{\omega}_0 = \left( \sum_K \mathbf{C}_{P_K} \right)^{-1} \sum_K \mathbf{C}_{P_K} \mathbf{\omega}_{K0} \]

4. New linear model:
   \[ \tilde{\mathbf{x}}_i(t) \approx \tilde{\mathbf{x}}_{0i} + t \tilde{\mathbf{v}}_i \]

   \[ \tilde{\mathbf{x}}_{0i} \approx \mathbf{x}_{0i} \]

   \[ \tilde{\mathbf{v}}_i \approx \mathbf{v}_i - [\mathbf{\omega}_0 \times] \mathbf{x}_{0i} \]

   only velocities updated!
Thanks for your attention!

A copy of this presentation can be downloaded from:

http://der.topo.auth.gr
The Intercommission Committee on Theory (ICCT) of the International Association of Geodesy (IAG) invites you to the

VI Hotine-Marussi Symposium of Theoretical and Computational Geodesy: Challenge and Role of Modern Geodesy

May 29 - June 2, 2006, Wuhan, China

hope to see you there!