A note on the transformation of velocities under change of the reference system for deformable geodetic networks and its various linear approximations

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1. Introduction

The analysis of coordinate time series for a geodetic control network obtained by one of the space techniques (VLBI, SLR, GNSS, DORIS) usually involves a model where coordinates are assumed to be linear functions of time, which involves “initial” station coordinates at a particular reference epoch and their velocities. For the sake of convenience, we will refer here to this constant velocity model as “ITRF model”, since it is routinely used in the formulation of the ITRF by combining coordinate time series (stacking) from all four space technique. It is of great interest to know how the model parameters, namely the initial coordinates and the velocities transform under a general transformation of the reference system with respect to translation, rotation and scale (similarity transformation). Since only transformations “close to the identity”, i.e. ones with small transformation parameters (rotation angles, displacements, scale parameter) it is sufficient to use approximations where second and higher order terms in the small parameters are justifiably deleted. The coefficient matrix of the transformation parameters in particular, is of outmost importance because its transpose gives rise to the so called inner constraints, which is a set of minimal constraints necessary to resolve the datum problem, leading to a solution for the unknown parameters which has the minimum norm and covariance matrix with minimal trace. After giving the original non-linear exact transformation laws for coordinates and velocities two types of approximations will be presented in order to cover two approaches appearing in the literature. The first covers the case of “large” velocities, in the sense that velocity components are not small quantities with negligible square and higher order terms. The second covers the case of “small” velocities, in the sense that velocity components are small quantities, which along with the small transformation parameters have negligible square and higher order terms. In addition we will cover the most interesting case where the original coordinates to be transformed already belong to the ITRF model category. To arrive at the currently used approximate transformation laws for both coordinates and velocities, we show that one more assumption is necessary. This is the arbitrary assumption that the allowed transformation parameters are not only small and smooth (have small time derivatives) but they are in addition restricted to being linear functions of time. The arbitrariness of the last assumption is puzzling at first sight, but fortunately this issue has been resolved by Chatzinikos and Dermanis (2016) who have shown that different least squares solutions to the stacking problem of the ITRF formulation procedure differ only by differences in the reference system corresponding to linear with respect to time transformation parameters.

2. Transformation of coordinates and its linear approximation

The coordinates \( \mathbf{x} = [x_i, x_j, x_k]^T \) of a point in a deformable geodetic network refer to some particular reference system. Under the general change of the reference system by a rotation described by a proper orthogonal matrix \( \mathbf{R}(\mathbf{0}) \) where \( \mathbf{0} = [\theta_1, \theta_2, \theta_3]^T \) are rotation parameters (typically angles of rotation around the three system axes) a translation \( \mathbf{d} = [d_1, d_2, d_3]^T \) and a scale factor \( 1 + s \), is mathematically described by the similarity transformation

\[
\mathbf{x}(t) = [1 + s(t)]\mathbf{R}(\theta(t))\mathbf{x}(t) + \mathbf{d}(t) = \mathbf{x}(\mathbf{x}(t), \mathbf{p}(t)),
\]
where

\[ p(t) = \begin{bmatrix} \theta(t) \\ d(t) \\ s(t) \end{bmatrix} \]

are the seven transformation parameters. The symbol \([a \times]\) denotes here the \(3 \times 3\) antisymmetric matrix having \(a\) as its axial vector, i.e. the matrix \(A = [a \times]\) with elements \(A_{11} = A_{22} = A_{33} = 0\), \(A_{32} = -A_{23} = a_1\), \(A_{23} = -A_{32} = a_2\), \(A_{13} = -A_{31} = a_3\), \(A_{31} = -A_{13} = a_4\), \(A_{12} = -A_{21} = a_5\), \(A_{21} = -A_{12} = a_6\), and \(A_{13} = -A_{31} = a_7\).

### 2.1 Approximation for close to the identity coordinate transformations

For transformations “close to the identity”, i.e. ones with transformation parameters small enough for their second and higher order terms to be neglected, the following sufficient approximation may be used

\[ \bar{x}(t) = x(t) + [x(t) \times] \theta(t) + d(t) + s(t)x(t) = x(t) + \begin{bmatrix} \theta(t) \\ d(t) \\ s(t) \end{bmatrix} = x(t) + E(t)p(t), \]

where

\[ E(t) = \begin{bmatrix} [x(t) \times] \\ I_3 \end{bmatrix} x(t) \]

based on the approximation \(R(\theta) = I - [\theta \times]\) and subsequent neglect of squares and products of the small parameters \(\theta\), \(d\), \(s\).

When \(x(t)\) remains at all epochs close to an approximate value \(x^\text{ap}\), then neglecting second and higher order terms also in the small parameters \(\delta x(t) = x(t) - x^\text{ap}\), one may use the sufficient approximation

\[ \bar{x}(t) = x(t) + [x^\text{ap} \times] \theta(t) + d(t) + s(t)x^\text{ap} = x(t) + E^\text{ap}p(t), \]

where

\[ E^\text{ap} = \begin{bmatrix} [x^\text{ap} \times] \\ I_3 \end{bmatrix} x^\text{ap}. \]

### 2.2. The case of an ITRF type model linear with respect to time

When the original coordinates follow an ITRF-type linear-in-time (constant velocity) model \(x(t) = x_0 + (t - t_0)v\), the transformed coordinates

\[ \bar{x}(t) = [1 + s(t)]R(\theta(t))[x_0 + (t - t_0)v] + d(t), \]

are no more linear-in-time and the same holds for the corresponding linear approximation

\[ \bar{x}(t) = x_0 + [x_0 \times] \theta(t) + d(t) + s(t)x_0 + (t - t_0)\{v + [v \times] \theta(t) + s(t)v\} = \]
\[
\begin{align*}
\mathbf{x} &= \mathbf{x}_0 + \left( \left[ \mathbf{x}_0 \times \mathbf{I}_3 \right] \mathbf{x}_0 \right) \begin{bmatrix} \mathbf{0}(t) \\ \mathbf{d}(t) \end{bmatrix} + \left( t - t_0 \right) \mathbf{v} + \left( t - t_0 \right) \begin{bmatrix} [\mathbf{v} \times] & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0}(t) \\ \mathbf{d}(t) \end{bmatrix} \\
\equiv & \mathbf{x}_0 + \mathbf{E}_0 \mathbf{p}(t) + \left( t - t_0 \right) \left[ \mathbf{v} + \mathbf{H}_0 \mathbf{p}(t) \right] \\
&= \mathbf{x}_0 + \left( t - t_0 \right) \mathbf{v} + \left[ \mathbf{E}_0 + \left( t - t_0 \right) \mathbf{H}_0 \right] \mathbf{p}(t)
\end{align*}
\]

where

(9) \[ \mathbf{E}_0 = \left[ \left[ \mathbf{x}_0 \times \mathbf{I}_3 \right] \mathbf{x}_0 \right], \quad \mathbf{H}_0 = \left[ \left[ \mathbf{v} \times \mathbf{0} \right] \mathbf{v} \right]. \]

In the usual case of small velocities, with negligible second and higher terms in both \( \mathbf{v} \) and \( \mathbf{p}(t) \), the above approximation reduces to

(10) \[ \ddot{\mathbf{x}}(t) = \mathbf{x}_0 + \left[ \mathbf{x}_0 \times \right] \mathbf{0}(t) + \mathbf{d}(t) + s(t) \mathbf{x}_0 + \left( t - t_0 \right) \mathbf{v} = \]

\[ = \mathbf{x}_0 + \left( t - t_0 \right) \mathbf{v} + \left[ \mathbf{x}_0 \times \right] \mathbf{I}_3 \left[ \mathbf{x}_0 \right] \mathbf{d}(t) \equiv \mathbf{x}_0 + \left( t - t_0 \right) \mathbf{v} + \mathbf{E}_0 \mathbf{p}(t), \]

which is also not linear in time.

### 2.3. The case of an ITRF type model with restriction to linear-in-time transformation parameters

Only if the transformation parameters are arbitrarily restricted to be linear with respect to time functions \( \mathbf{p}(t) = \mathbf{p}_0 + \left( t - t_0 \right) \dot{\mathbf{p}} \), with small \( \dot{\mathbf{p}} \), the “large” velocities approximation (8) reduces to

(11) \[ \ddot{\mathbf{x}}(t) = \mathbf{x}_0 + \left[ \mathbf{x}_0 \times \right] \mathbf{0}(t) + \mathbf{d}_0 + s_0 \mathbf{x}_0 + \left( t - t_0 \right) \left[ \left[ \mathbf{x}_0 \times \right] \dot{\mathbf{0}} + \dot{\mathbf{d}} + \dot{s} \mathbf{x}_0 \right] + \mathbf{v} + \left[ \mathbf{v} \times \right] \mathbf{0}_0 + s_0 \mathbf{v} + \left( t - t_0 \right)^2 \left( \dot{\mathbf{v}} + \left[ \mathbf{v} \times \right] \dot{\mathbf{0}} \right), \]

which is of the “acceleration” form

(12) \[ \ddot{\mathbf{x}}(t) = \left[ \mathbf{x}_0 + \mathbf{E}_0 \mathbf{p}_0 \right] + \left( t - t_0 \right) \left[ \mathbf{v} + \mathbf{H}_0 \mathbf{p}_0 + \mathbf{E}_0 \dot{\mathbf{p}} \right] + \left( t - t_0 \right)^2 \left[ \mathbf{H}_0 \dot{\mathbf{p}} \right] = \ddot{\mathbf{x}}_0 + \left( t - t_0 \right) \ddot{\mathbf{v}} + \left( t - t_0 \right)^2 \ddot{\mathbf{a}}, \]

with

(13) \[ \ddot{\mathbf{x}}_0 = \mathbf{x}_0 + \left[ \mathbf{x}_0 \times \right] \mathbf{0}_0 + \mathbf{d}_0 + s_0 \mathbf{x}_0 = \mathbf{x}_0 + \mathbf{E}_0 \mathbf{p}_0. \]

(14) \[ \ddot{\mathbf{v}} = \mathbf{v} + \left[ \mathbf{v} \times \right] \mathbf{0}_0 + s_0 \mathbf{v} + \left[ \mathbf{x}_0 \times \right] \dot{\mathbf{0}} + \dot{\mathbf{d}} + \dot{s} \mathbf{x}_0 = \mathbf{v} + \mathbf{H}_0 \mathbf{p}_0 + \mathbf{E}_0 \dot{\mathbf{p}}. \]

(15) \[ \ddot{\mathbf{a}} = \dot{\mathbf{v}} + \left[ \mathbf{v} \times \right] \dot{\mathbf{0}} = \mathbf{H}_0 \dot{\mathbf{p}}. \]

The “small” velocities approximation (10) reduces to

(16) \[ \ddot{\mathbf{x}}(t) = \left\{ \mathbf{x}_0 + \left[ \mathbf{x}_0 \times \right] \mathbf{0}_0 + \mathbf{d}_0 + s_0 \mathbf{x}_0 \right\} + \left( t - t_0 \right) \left\{ \mathbf{v} + \left[ \mathbf{x}_0 \times \right] \dot{\mathbf{0}} + \dot{s} \mathbf{x}_0 + \dot{\mathbf{d}} \right\} = \]

\[ = \left( \mathbf{x}_0 + \mathbf{E}_0 \mathbf{p}_0 \right) + \left( t - t_0 \right) \left( \mathbf{v} + \mathbf{E}_0 \dot{\mathbf{p}} \right) \equiv \ddot{\mathbf{x}}_0 + \left( t - t_0 \right) \ddot{\mathbf{v}}, \]
with

\[(17) \quad \ddot{x}_0 = x_0 + [x_0 \times \theta_0] + d_0 + s_0 \theta_0 = x_0 + [x_0 \times I_3] \begin{bmatrix} \theta_0 \\ d_0 \\ s_0 \end{bmatrix} = x_0 + E_0 p_0,\]

\[(18) \quad \ddot{v} = v + [x_0 \times \dot{\theta}] + \dot{d} + s = v + [x_0 \times I_3] \begin{bmatrix} \theta \\ d \\ s \end{bmatrix} = v + E_0 \dot{p}.\]

and the ITRF linear-in-time model is preserved for the transformed coordinates.

Although (17) can be interpreted as the law of transformation of initial velocities, it would be inappropriate at this point to directly interpret (18) as a law of transformation of velocities, not only because of the arbitrary restriction to \( p(t) = p_0 + (t - t_0) \dot{p} \), but mainly because any approximation with side restrictions must be derived from the original non-linear law of the transformation of velocities, as it will be elaborated below in detail.

### 3. Transformation of velocities

To derive the law of transformation for (instantaneous) velocities \( v(t) = \dot{x}(t) \) (dots denoting derivatives with respect to time) it is sufficient to differentiate the similarity transformation relation (1) (and not its linearized form, as done sometimes), using the chain rule

\[(19) \quad \ddot{x}(t) = \dddot{x}(t) = \frac{\partial x}{\partial x_k(t)p(t)} \dot{x}(t) + \frac{\partial x}{\partial p} \dot{p}(t) = \frac{\partial x}{\partial x_k(t)p(t)} v(t) + \frac{\partial x}{\partial p} \dot{p}(t).\]

It must be made clear that by “velocities” \( v(t) \) we do not refer to the components of any physically meaningful velocity vector but rather to apparent velocities (rates of change of coordinates) as seen by an observer within the particular time varying reference system. Carrying out the necessary partial derivation in (19) we arrive at the (non-linear) law of transformation of velocities

\[(20) \quad \ddot{v}(t) = (1 + s(t))R(\theta(t))v(t) + (1 + s(t))R(\theta(t))(x(t) \times \Omega(\theta(t)))\dot{\theta}(t) + \ddot{d}(t) + \dot{s}(t)R(\theta(t))x(t)\]

where we have set

\[(21) \quad [\omega] = \frac{\partial R^T}{\partial \theta_i} R - \frac{\partial R}{\partial \theta_i} R^T, \quad i = 1, 2, 3\]

and

\[(22) \quad \Omega = [\omega_1, \omega_2, \omega_3].\]

For the usual choice of rotations around the axes,

\[(23) \quad R(\theta) = R_3(\theta_3)R_2(\theta_2)R_1(\theta_1),\]
it turns out that

\[(24) \quad \omega_1 = R_3(\theta_1) R_2(\theta_2) i_1, \quad \omega_2 = R_3(\theta_3), \quad \omega_3 = i_1, \]

where \(i_1, i_2)\), are the columns of the \(3 \times 3\) identity matrix \(I = [i_1, i_2, i_3]\), and hence

\[(25) \quad \Omega(\theta) = \begin{bmatrix} \cos \theta_3 \cos \theta_2 & \sin \theta_2 & -\sin \theta_3 \\ -\sin \theta_3 \cos \theta_2 & \cos \theta_2 & \cos \theta_3 \\ \sin \theta_2 & 0 & 1 \end{bmatrix}. \]

The linearization of the above transformation of velocities depends on whether the velocities \(v(t)\) are small quantities with negligible second and higher order quantities, or not. Thus we will distinguish between linearization for large angles and linearization for small angles. A further requirement is that in addition to the limitation to close to the identity transformations (\(p(t) \) small) the functions \(p(t)\) must be sufficiently smooth so that their derivatives \(\dot{p}(t)\) are also small quantities. In general small \(p(t)\) does not imply small \(\dot{p}(t)\), consider e.g. a monochromatic periodic \(p_i(t) = a_i \cos(\omega_i t + \phi_i)\) with small \(a_i\) where its derivative \(\dot{p}_i(t) = -\omega_i a_i \sin(\omega_i t + \phi_i)\) may become arbitrarily large by just increasing the angular frequency \(\omega_i\).

Note that is the coordinates at the original reference system follow the ITRF linear-in-time (constant velocity) model \(x(t) = x_0 + (t - t_0) v\), then in the new reference system the coordinates

\[(26) \quad \tilde{x}(t) = [1 + s(t)] R(\theta(t))[x_0 + (t - t_0) v] + d(t) \]

are no more linear functions of time, and the velocities

\[(27) \quad \tilde{v}(t) = (1 + s(t)) R(\theta(t)) v(t) + (1 + s(t)) R(\theta(t))[x_0 \times + (t - t_0)[v \times]] \Omega(\theta(t)) \hat{\theta}(t) + \dot{d}(t) + \\
+ \tilde{s}(t) R(\theta(t)) [x_0 + (t - t_0) v] \]

is no longer constant.

### 3.1. Linearized transformation law of “large” velocities

Assuming that all second and higher order terms in the small parameters \(p(t)\), \(\dot{p}(t)\) are negligible, we have \(R(\theta) = I - [\theta x]\) and \(\Omega(\theta) \approx \hat{\theta}\). The last easily verified for the use of rotations around the axes, but is also true for any parametrization of \(\Omega(\theta)\). Noting that \(\Omega(\theta) \approx I\) as \(\theta \approx 0\), so that \(\Omega(\theta) \approx I + \Delta(\theta)\) with \(\Delta(\theta)\) being of the first order in \(\theta\) and hence \(\Omega(\theta) \approx \hat{\theta} + \Delta(\theta) \approx \hat{\theta}\). Replacing these and further neglecting second order in transformation parameters and their derivatives, we arrive at the following linear approximation of the law of transformation of “large” velocities

\[(28) \quad \tilde{v}(t) = v(t) + [v(t) \times] \hat{\theta}(t) + s(t) v(t) + [x(t) \times] \hat{\theta}(t) + \dot{d}(t) + \tilde{s}(t) x(t) =

= v(t) + [[v(t)]]_0 v(t) \begin{bmatrix} \theta(t) \\ d(t) \end{bmatrix} + [x(t) \times]_s I s(t) \begin{bmatrix} \hat{\theta}(t) \\ \dot{d}(t) \end{bmatrix} = v(t) + H(t)p(t) + E(t) \dot{p}(t),\]
with

\[ H(t) = \begin{bmatrix} [v(t) \times] & 0 & v(t) \end{bmatrix}, \quad E(t) = \begin{bmatrix} [x(t) \times] & I_3 & x(t) \end{bmatrix}. \]

3.2. Linearized transformation law of “large” velocities for the ITRF model

As a special case we shall examine the above transformation in the case that \( x(t) = x_0 + (t - t_0)v \), when the coordinates in the original reference system obey an ITRF-type linear-in-time (or constant velocity) model. The linear approximation to (27) becomes

\[ \tilde{v}(t) = v + [v \times] \theta(t) + s(t)v + [(x_0 \times) + (t - t_0)(v \times)] \hat{\theta}(t) + \dot{d}(t) + \dot{s}(t) \{ x_0 + (t - t_0)v \} = \]

\[ = v + [v \times] 0 v \begin{bmatrix} \theta(t) \\ d(t) \\ s(t) \end{bmatrix} + [(x_0 \times) + (t - t_0)(v \times)] I_3 x_0 + (t - t_0)v \begin{bmatrix} \theta(t) \\ d(t) \\ s(t) \end{bmatrix} \]

or simply

\[ \tilde{v}(t) = v + H_0 p(t) + [E_0 + (t - t_0)H_0] \tilde{p}(t) \neq \text{const.} \]

with

\[ H_0 = \begin{bmatrix} [v \times] & 0 & v \end{bmatrix}, \quad E_0 = \begin{bmatrix} [x_0 \times] & I_3 & x_0 \end{bmatrix}. \]

From the corresponding transformation of coordinates (8) and the above approximation (31) we have

\[ \tilde{x}(t_0) = x_0 + E_0 p(t_0) \]

\[ \tilde{v}(t_0) = v + H_0 p(t_0) + E_0 \tilde{p}(t_0) . \]

Consequently, the linearized transformation of coordinates (8) is of the form

\[ \tilde{x}(t) = [x_0 + E_0 p(t)] + (t - t_0)\left[ v + H_0 p(t) \right] \neq \tilde{x}(t_0) + (t - t_0)\tilde{v}(t_0) \]

Therefore, within the linear approximation, the transformed coordinates are no more linear with respect to time (they do not conform with the ITRF model) and the transformed coordinates are not constant.

3.3. Linearized transformation law of “large” velocities for the ITRF model with restriction to linear-in-time transformation parameters

Only if the transformation parameters are \textbf{arbitrarily} restricted to be linear with respect to time functions \( p(t) = p_0 + (t - t_0)\tilde{p} \), with small \( \tilde{p} \), the “large” velocities approximation (31) reduces to

\[ \tilde{v}(t) = v + H_0 [p_0 + (t - t_0)\tilde{p}] + [E_0 + (t - t_0)H_0] \tilde{p} = \]
so that the transformed velocities are linear in time and not constant as in an ITRF type model.

### 3.4. Linearized transformation law of “small” velocities

In the usual case of small velocities, with negligible second and higher terms in both \( v \), \( \mathbf{p}(t) \) and \( \dot{\mathbf{p}}(t) \), the non-linear law of velocity transformation (27) can be approximated by

\[
(37) \quad \tilde{\mathbf{v}}(t) = \mathbf{v}(t) + [\mathbf{x}(t) \times \dot{\mathbf{0}}(t)] + \dot{\mathbf{d}}(t) + \dot{s}(t) \mathbf{x}(t) = \mathbf{v}(t) + [\mathbf{x}(t) \times \mathbf{I}_3 \mathbf{x}(t)] \begin{bmatrix} \dot{\mathbf{0}}(t) \\ \dot{\mathbf{d}}(t) \\ \dot{s}(t) \end{bmatrix} = \mathbf{v}(t) + \mathbf{E}(t) \dot{\mathbf{p}}(t).
\]

### 3.5. Linearized transformation law of “small” velocities for the ITRF model

When the original coordinates follow an ITRF-type linear-in-time model \( \mathbf{x}(t) = \mathbf{x}_0 + (t - t_0) \mathbf{v} \) with small velocities \( \mathbf{v} \), the approximation becomes

\[
(38) \quad \tilde{\mathbf{v}}(t) = \mathbf{v} + [\mathbf{x}_0 \times \dot{\mathbf{0}}(t)] + \dot{\mathbf{d}}(t) + \dot{s}(t) \mathbf{x}_0 = \mathbf{v} + [\mathbf{x}_0 \times \mathbf{I}_3 \mathbf{x}_0] \begin{bmatrix} \dot{\mathbf{0}}(t) \\ \dot{\mathbf{d}}(t) \\ \dot{s}(t) \end{bmatrix} = \mathbf{v} + \mathbf{E}_0 \dot{\mathbf{p}}(t).
\]

The dependence on \( \dot{\mathbf{p}}(t) \) results in transformed velocities which are not constant, in agreement with the fact that the transformed coordinates \( \tilde{\mathbf{x}}(t) \) are not linear functions of time (equation 5).

### 3.6. Linearized transformation law of “small” velocities for the ITRF model with restriction to linear-in-time transformation parameters

Only the arbitrary restriction to linear-in-time transformation parameters \( \mathbf{p}(t) = \mathbf{p}_0 + (t - t_0) \dot{\mathbf{p}} \), where the small \( \dot{\mathbf{p}} \) are constant, secures the preservation of the linear-in-time ITRF model (with \( \tilde{\mathbf{x}}(t) \) from equation (12) and constant velocities)

\[
(39) \quad \tilde{\mathbf{v}} = \mathbf{v} + [\mathbf{x}_0 \times \dot{\mathbf{0}}] + \dot{\mathbf{d}} + \dot{s} \mathbf{x}_0 = \mathbf{v} + [\mathbf{x}_0 \times \mathbf{I}_3 \mathbf{x}_0] \begin{bmatrix} \dot{\mathbf{0}} \\ \dot{\mathbf{d}} \\ \dot{s} \end{bmatrix} = \mathbf{v} + \mathbf{E}_0 \dot{\mathbf{p}}.
\]

When the differences between the unknown initial coordinates \( \mathbf{x}_0 \) and their known approximations \( \mathbf{x}_0^a \), is sufficiently small then the following simplified approximation can be used

\[
(40) \quad \tilde{\mathbf{v}} = \mathbf{v} + \mathbf{E}_0^a \dot{\mathbf{p}}, \quad \mathbf{E}_0^a = [\mathbf{x}_0^a \times \mathbf{I}_3 \mathbf{x}_0^a].
\]
This is coupled by the further simplification to the approximate transformation for the initial coordinates (17), which becomes

\[ \tilde{x}_0 = x_0 + E_0 p_0 \approx x_0 + E_0^{0p} p_0. \]

### 4. Discussion

Table 1 summarizes the obtained results, giving the transformation laws for coordinates and velocities, from their original rigorous non-linear form down to their different approximations. Of particular interest are the cases when the original coordinates follow an ITRF type model with coordinates, which are linear functions of time and constant velocities. It turns out that the linearity of the ITRF model is preserved only in the approximation which assumes small velocities and is also based on the restriction of the transformation parameters to be not only small and smooth (i.e. to have small derivatives) but to also linear functions of time. Fortunately all these assumptions are justified, because the station velocities are indeed small and the reference system is sought in a small neighborhood of the one implemented by approximate coordinate values. Smoothness is guaranteed by the linearity of transformation parameters, which may appear to be an arbitrary restriction, but as shown by Chatzinikos and Dermanis (2016) various least squares solutions to the stacking problem differ only in the definitions of their reference systems, which are connected by linear in time transformations.

Application of (40) and (41) to the coordinates to all stations \( P \) of a geodetic network give

\[ \tilde{x}_0 = x_0 + E_0^{0p} p_0, \quad \tilde{v}_i = v_i + E_0^{0p} \dot{p}_0, \] as derived e.g. by Altamimi and Dermanis (2012, 2013). For the vectors \( x_0 \) and \( v \) of all stations, the transformation law can be summarized into

\[ \begin{bmatrix} \tilde{x}_0 \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} x_0 \\ v \end{bmatrix} + \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} p_0 \\ \dot{p} \end{bmatrix}, \quad E = \begin{bmatrix} \vdots \\ E_0^{0p} \end{bmatrix}. \]

Davies and Blewitt have presented a different type of transformation (see their appendix A) of the form

\[ \begin{bmatrix} \tilde{x}_0 \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} x_0 \\ v \end{bmatrix} + \begin{bmatrix} E & 0 \\ H & E \end{bmatrix} \begin{bmatrix} p_0 \\ \dot{p} \end{bmatrix}, \quad H = \begin{bmatrix} \vdots \\ H_0^{0p} \end{bmatrix}, \quad H_0^{0p} = \begin{bmatrix} [v_i^{0p} \times] & 0 & v_i^{0p} \end{bmatrix}, \]

which is obviously, on the assumption of “large” velocities. Although the authors give no hint or reference to the literature concerning the derivation of (43) or the presumed relations \( \tilde{x}_0 = x_0 + E_0^{0p} p_0, \quad \tilde{v}_i = v_i + H_0^{0p} p_0 + E_0^{0p} \dot{p}_0 \), one may guess that the come from the differentiation of (12), which gives (36), and the subsequent evaluation of both (12) and (36) at \( t = t_0 \), which indeed gives

\[ \tilde{x}(t_0) = \tilde{x}_0 = x_0 + E_0 p_0 \approx x_0 + E_0^{0p} p_0 \]

\[ \tilde{v}(t_0) = \tilde{v}_i = v_i + H_0 p_0 + E_0^{0p} \dot{p}_0 \approx v_i + H_0^{0p} p_0 + E_0^{0p} \dot{p} \neq \tilde{v}(t) \quad \text{for} \quad t \neq t_0. \]

This type of approach ignores though the fact that for “large” velocities and small linear-in-time transformation parameters the ITRF model is not preserved. Indeed the original linear in time model for coordinates and constant velocities is transformed into a quadratic with respect to time model for coordinates (equation 12) and a non-constant linear in time model for velocities (equation 36). Thus the transformed
initial coordinates \( \mathbf{x}(t_0) \) and initial velocities \( \mathbf{v}(t_0) \) cannot be used to linearly reconstruct the transformed coordinates because \( \mathbf{x}(t_0) + (t-t_0)\mathbf{v}(t_0) \neq \mathbf{\bar{x}}(t) \). Apart from its lack of proper theoretical foundation the transformation (43) suffers from lack of relevancy to the actual geodetic calculations. To see that assume initial epoch transformation parameters that correspond to a change on the surface of the earth of the order of \( e_0 \), with derivatives of the order of \( \dot{e} \) and realistic velocities to be of the order of \( v_0 \). With coordinates being of the order of the earth radius \( (x_{0}^{i} \sim R) \), and transformation parameters of the orders \( \theta_0 \sim e_0 / R \), \( d_0 \sim e_0 \), \( s_0 \sim e_0 / R \) and \( \dot{\theta} \sim \dot{e} / R \), \( \dot{d} \sim e \), \( \dot{s} \sim \dot{e} / R \), the term \( \mathbf{E}_0 \mathbf{p} \approx \mathbf{E}_0^{\theta} \mathbf{p} \) has elements \( [\mathbf{x}_0^{i} \times] \dot{\theta} + \dot{d} + \dot{s} \mathbf{x}_0^{i} \) of the order of \( \dot{e} \). At the same time the term \( \mathbf{H}_0 \mathbf{p} \approx \mathbf{H}_0^{\theta} \mathbf{p} \) has elements \( [\mathbf{v}_0^{i} \times] \theta_0 + s_0 \mathbf{v}_0^{i} \) of the order of \( e_0 / R \) and therefore differs from \( \mathbf{E}_0 \mathbf{p} \) by a factor of \( (e_0 / R)(e_0 / \dot{e}) \). The known order of observed velocities is 0.1 m/yr and in any they do not exceed the order of \( v_0 \approx 1 \) m/yr and \( R = 6371 \) km, while it is plausible to assume that \( \dot{e} = e_0 \), i.e. that transformation parameter derivatives cause within a year the same displacements as their initial values. To be safe, let us exaggerate and set \( e_0 / \dot{e} \approx 100 \), thus leading to the conclusion that \( \mathbf{H}_0 \mathbf{p} \) is smaller of \( \mathbf{E}_0 \mathbf{p} \) by a factor of 63710 in a worst case scenario Therefore the term \( \mathbf{H}_0 \mathbf{p} \) is negligible in the transformation of velocities given by the second part of (43), \( \mathbf{\bar{v}} = \mathbf{v} + \mathbf{H}_0 \mathbf{p} + \mathbf{E}_0 \mathbf{p} \) and thus one can securely use instead the second part of (41) \( \mathbf{\bar{v}} = \mathbf{v} + \mathbf{E}_0 \mathbf{p} \). These practical computational considerations resolve in a simple way the theoretical worries about the validity and relevancy of equation (43) discussed above.

References


Chatzinikos M, Dermanis A, 2016. A coordinate-free ITRF model for understanding the nature of rank deficiencies, the non-estimability of velocities and the effect of the choice of minimal constraints on obtained velocity estimates (submitted to the Journal of Geodesy).


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<table>
<thead>
<tr>
<th>Transformation parameters:</th>
<th>( \mathbf{p}(t) = \begin{bmatrix} \theta(t)^T &amp; \mathbf{d}(t)^T &amp; s(t) \end{bmatrix} ), ( \dot{\mathbf{p}}(t) = \begin{bmatrix} \dot{\theta}(t)^T &amp; \dot{\mathbf{d}}(t)^T &amp; \dot{s}(t) \end{bmatrix} )</th>
</tr>
</thead>
</table>

Original non-linear transformations:

\[
\begin{aligned}
\ddot{x}(t) &= [(1+s(t))\mathbf{R}(\theta(t))\mathbf{x}(t) + \mathbf{d}(t), \\
\ddot{v}(t) &= (1+s(t))\mathbf{R}(\theta(t))\mathbf{v}(t) + (1+s(t))\mathbf{R}(\theta(t))[\mathbf{x}(t)\times]\Omega(\theta(t))\dot{\theta}(t) + \mathbf{d}(t) + \dot{s}(t)\mathbf{R}(\theta(t))\mathbf{x}(t) \\
[\omega_i \times] &= \frac{\partial \mathbf{R}^T}{\partial \theta_i} - \frac{\partial \mathbf{R}}{\partial \theta_i} \mathbf{R}^T, \\
\Omega &= [\omega_1 \omega_2 \omega_3].
\end{aligned}
\]

Close to identity approximation for “large” velocities:

\[
\begin{aligned}
\ddot{x}(t) &= \mathbf{x}(t) + \mathbf{E}(t)\mathbf{p}(t), \\
\ddot{v}(t) &= \mathbf{v}(t) + \mathbf{H}(t)\mathbf{p}(t) + \mathbf{E}(t)\dot{\mathbf{p}}(t), \\
\mathbf{E}(t) &= \begin{bmatrix} [\mathbf{x}(t)\times] & \mathbf{I}_3 & \mathbf{x}(t) \end{bmatrix}, \\
\mathbf{H}(t) &= \begin{bmatrix} [\mathbf{v}(t)\times] & 0 & \mathbf{v}(t) \end{bmatrix}.
\end{aligned}
\]

Close to identity approximation for “large” velocities and ITRF model:

\[
\begin{aligned}
\ddot{x}(t) &= \mathbf{x}_0 + \mathbf{E}_0\mathbf{p}(t) + (t-t_0)[\mathbf{v} + \mathbf{H}_0\mathbf{p}(t)] = \mathbf{x}_0 + (t-t_0)\mathbf{v} + \mathbf{E}_0 + (t-t_0)\mathbf{H}_0\mathbf{p}(t), \\
\ddot{v}(t) &= \mathbf{v} + \mathbf{H}_0\mathbf{p}(t) + [\mathbf{E}_0 + (t-t_0)\mathbf{H}_0]p(t) + \mathbf{p}(t) \dot{\mathbf{p}}(t), \\
\mathbf{E}_0 &= \begin{bmatrix} [\mathbf{x}_0\times] & \mathbf{I}_3 & \mathbf{x}_0 \end{bmatrix}, \\
\mathbf{H}_0 &= \begin{bmatrix} [\mathbf{v}\times] & 0 & \mathbf{v} \end{bmatrix}.
\end{aligned}
\]

Close to identity approximation for “large” velocities, ITRF model and restriction to linear transformation parameters:

\[
\begin{aligned}
\ddot{x}(t) &= \mathbf{x}_0 + \mathbf{E}_0\mathbf{p}(t) + (t-t_0)(\mathbf{v} + \mathbf{H}_0\mathbf{p}_0 + \mathbf{E}_0\dot{\mathbf{p}}) + (t-t_0)^2(\mathbf{H}_0\dot{\mathbf{p}}), \\
\ddot{v}(t) &= \mathbf{v} + \mathbf{H}_0\mathbf{p}_0 + \mathbf{E}_0\dot{\mathbf{p}} + (t-t_0)[2\mathbf{H}_0\dot{\mathbf{p}}] \\
\end{aligned}
\]

Close to identity approximation for “small” velocities:

\[
\begin{aligned}
\ddot{x}(t) &= \mathbf{x}(t) + \mathbf{E}(t)\mathbf{p}(t), \\
\ddot{v}(t) &= \mathbf{v}(t) + \mathbf{E}(t)\dot{\mathbf{p}}(t).
\end{aligned}
\]

Close to identity approximation for “small” velocities and ITRF model:

\[
\begin{aligned}
\ddot{x}(t) &= \mathbf{x}_0 + (t-t_0)\mathbf{v} + \mathbf{E}_0\mathbf{p}(t), \\
\ddot{v}(t) &= \mathbf{v} + \mathbf{E}_0\mathbf{p}(t).
\end{aligned}
\]

Close to identity approximation for “small” velocities, ITRF model and restriction to linear transformation parameters:

\[
\begin{aligned}
\ddot{x}(t) &= \mathbf{x}_0 + \mathbf{E}_0\mathbf{p}_0 + (t-t_0)(\mathbf{v} + \mathbf{E}_0\dot{\mathbf{p}}), \\
\ddot{v} &= \mathbf{v} + \mathbf{E}_0\dot{\mathbf{p}}.
\end{aligned}
\]