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STRAIN ANALYSIS OF MAP PROJECTIONS

(A. Δερμάνης και E. Λιβιέρατος: Ἄναλυση παραμορφώσεων στίς χαρτογραφικές ἀπεικονίσεις)

Abstract

A strain analysis of map projections is carried out, focusing our interest on some scalar strain parameters, i.e., dilatation, maximum shear strain, rotation and strain energy. These parameters, being of obvious geometrical and physical meaning, allow a deeper understanding of map distortions and provide additional tools for comparing maps of even the same traditional properties (conformality, equivalence, etc.).
Mapping differentially the local tangent plane (azimuthal plane) to the plane of representation, the local strain tensors are obtained according to known relations from the Theory of Elasticity. From such strain tensors, local dilatation and maximum shear strain can be computed for any desired point.

If the deformation is studied on the objective surface (sphere), the Lagrangian approach is followed and the dilatation $\Delta_L$ and maximum shear strain $\gamma_L$ are given by

$$\Delta_L = \frac{1}{2} \left( \frac{x_\lambda^2 + y_\lambda^2}{\cos^2 \varphi} + x_\varphi^2 + y_\varphi^2 \right) - 1$$

$$\gamma_L = \frac{1}{2 \cos \varphi} \sqrt{\left[ x_\lambda^2 + y_\lambda^2 - \cos^2 \varphi \left( x_\varphi^2 + y_\varphi^2 \right) \right]^2 + \left[ 2 \cos \varphi \left( x_\lambda x_\varphi + y_\lambda y_\varphi \right) \right]^2}$$

subscripts denoting partial differentiation with respect to the corresponding variable.

For the study of deformation on the plane of representation, the Eulerian approach is followed obtaining

$$\Delta_E = \frac{\cos^2 \varphi \left( \lambda_x^2 + \lambda_y^2 \right) + \varphi_x^2 + \varphi_y^2}{2} - 1$$

$$\gamma_E = \frac{1}{2} \sqrt{\left[ \cos^2 \varphi \left( \lambda_x^2 - \lambda_y^2 \right) + \varphi_x^2 - \varphi_y^2 \right]^2 + 4 \left[ \cos^2 \varphi \lambda_x \lambda_y + \varphi_x \varphi_y \right]^2}$$

It is of cartographic interest to analyze the strain involved when comparing two different maps $(x,y)$ and $(x',y')$. In this case the relative dilatation $\Delta_R$ and relative maximum shear strain $\gamma_R$ are given by
$$\Delta_R = x^i_\lambda \lambda_x + x^i_\psi \psi_x + y^i_\lambda \lambda_y + y^i_\psi \psi_y - 2$$

$$\nu_R = \sqrt{(x^i_\lambda \lambda_x + x^i_\psi \psi_x - y^i_\lambda \lambda_y - y^i_\psi \psi_y)^2 + (x^i_\lambda \lambda_x + x^i_\psi \psi_x + y^i_\lambda \lambda_y + y^i_\psi \psi_y)^2}$$

Rotations can be computed by solving a system of non-linear equations. In the vicinity of tangent points or lines of the mapping rotations are infinitesimal and are approximately given by the rotation angle

$$\omega = x_\psi - \frac{y_\lambda}{\cos \psi} \cdot$$

Another interesting measure of deformation is the strain energy (per unit area), W, dissipated in the fictitious process of deforming a material objective surface into the map plane. The local strain energy is computed at any point by

$$W = \frac{\lambda + \mu}{2} \Delta^2 + \frac{\mu}{2} \nu^2$$

where \( \lambda \) and \( \mu \) are the Lamé constants characterizing a fictitious material. For the particular convenient choice of \( \lambda = \mu = 1 \) the strain energy becomes

$$W = \Delta^2 + \frac{1}{2} \nu^2 \cdot$$

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