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# Modeling Alternatives in Four-dimensional Geodesy

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## Summary

Four-dimensional integrated geodesy deals with the analysis of observations for the study of network geometry and its variation with time, when these observations depend on the gravity field of the earth and its temporal variation. Parameters appearing in the functional part of the mathematical model can be alternatively treated in three different ways: (a) as independent deterministic parameters, (b) as interrelated deterministic parameters expressed as functions of fewer independent parameters with the use of empirical models and (c) as correlated stochastic parameters when they depend on underlying unknown functions which are in turn modeled as stochastic processes. These possibilities can be applied to parameters related to the gravity field, its time variation and the time variation of the geometry of geodetic networks, in order to provide a manifold of alternative models for four dimensional geodesy. Previous work is easily classified in relation to such a framework. In addition to the modeling possibilities, different solution concepts can be applied on the basis of alternative criteria of best (minimum mean square error) estimation and prediction.

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## 1. Introduction

Integrated geodesy has its roots in the long tradition of geodetic work, which is related to the efforts of avoiding the classical separation of data analysis in horizontal and vertical parts with the use of reduced observations, as well as to the desire for mathematical rigor in model formulation. Krarup himself gives the following definition (Krarup, 1971):

> " ... I will define integrated Geodesy as a system for geodetic calculation which handles all forms of geodetic observations in unreduced form ... " .

However, integrated geodesy also departs from the previous concepts of three-dimensional and mathematical geodesy in various aspects. The most important of course is the inclusion of the gravity potential function as an "unknown" in the mathematical model of the observations. The second aspect is the need for a more advanced solution concept (norm minimization), in order to obtain a unique solution to an adjustment problem with too many unknowns, infinite in fact.

Four-dimensional integrated geodesy, i.e. integrated geodesy where the time variation of geometry and gravity are taken into account, poses some modeling problems whose rich structure gives the opportunity for a deeper understanding of the many new ideas which followed the original concept of integrated geodesy.

A proper way to classify methods for the analysis of geodetic observations, is with respect to the discrete or continuous character of both the observations and the model unknowns. An attempt to classify geodetic data analysis techniques in this respect is given in table 1. Integrated geodesy differs from the classical network adjustment concept, where both observations and unknowns are discrete, in that it deals with discrete observations and both discrete and continuous unknowns.

However, the continuous unknown (the gravity potential) appears in the mathematical model only through the discrete values of certain functionals, which we call signals. When the dependence of the signals on the underlying unknown function is ignored, one still has a usual adjustment problem with discrete observations and unknowns. In the case of integrated geodesy, the interrelation of the signals which is a consequence of their common dependence on the same unknown function, is taken into account in either a stochastic (covariance function) or deterministic (reproducing kernel) formulation, and this leads to an adjustment problem with signals (Dermanis, 1979). The solution of this problem leads not only to estimates of the unknown signal, but also to an implicit estimate of the unknown function, thus essentially also solving an interpolation problem (Dermanis, 1987). When a stochastic formulation is used, this a problem of prediction rather than one of interpolation.

The introduction of the time dimension leads to models where gravity related signals are interrelated both space- and time-wise, since the underlying gravity potential is now a function of both space and time. Furthermore, the instantaneous coordinates of network points appearing in the model are now signals depending on the coordinates as functions of time. To make things more realistic, and unfortunately more complicated too, the spatial interrelation between coordinate variations of different network points must also be taken into account. Indeed, displacements from initial positions at two different places and epochs are likely be more "similar" the closest in space and time they are, provided they are not separated by seismic faults and epochs of great seismic activity. This similarity is a consequence of the expected smoothness of the process of crustal deformation at time periods between main seismic events. It resembles the similarity between the deflections of the vertical at nearby points and the same instance, which of course is also a consequence of the spatial smoothness of the gravity field.

As in the case of three-dimensional integrated geodesy, the dependence of the observations in a geodetic network on signals, which in turn depend on underlying unknown functions, gives rise to the possibility of choosing between several modeling alternatives. All these alternatives aim at the determination of unique solutions, not only for the signals appearing in the observational model, but also for the underlying functions through an implicit interpolation.

When signal interrelation is expressed by covariance functions within a stochastic formulation, the problem is one of prediction (determination of estimates for the values of random variables). In addition to the predictions for the signals appearing in the model, it is also possible to predict other signals depending on the same unknown functions, now modeled as stochastic processes, and especially any desired values of these functions, thus solving again an interpolation problem.

| **Data Analysis** | **Observations** | **Unknowns** |
|---|---|---|
| Classical Least Squares Adjustment | discrete | discrete |
| Integrated Geodesy | discrete | discrete + continuous |
| Mathematics (e.g. GBVP) | continuous | continuous |
| ? | discrete + continuous | discrete + continuous |

Table 1: Data analysis methods in relation to the discrete or continuous character of observations and unknowns.

Recent advances in the theory of linear prediction (Schaffrin, 1983, 1985a,b,c, 1986a,b), provide the possibility of choosing between several solution alternatives for the same stochastic model, in addition to the, now classical, least squares collocation principle, usually used in integrated geodesy.

We shall use here, only the stochastic interpretation of least squares collocation, for the sake of simplicity. The relation with the deterministic interpretation where the minimum norm principle is used, is given in several papers, e.g., Sansó (1986), Dermanis (1987). We shall reserve the term deterministic approach for the case where the signals are treated as deterministic parameters ignoring their dependence on the unknown functions. Another approach, which is essentially deterministic, is to model these functions through empirical models with deterministic unknown coefficients. We shall call this approach here the analytical approach (compare with the "model function approach" of Dermanis and Grafarend (1981), Dermanis (1984)).

| **INTEGRATED GEODESY** | |
|---|---|
| ~ 1969 | ~ 1988 |
| Use of unreduced observations | More rigorous functional models |
| Combined treatment of geometric and gravity related observations | Integration of observations from different instrumentation systems |
| Include gravity field as unknown function | Include any unknown function related to the model |
| New solution concept: (least squares) collocation | Linear "best" (MMSE) estimation and prediction |

Table 2: The evolution of the concept of integrated geodesy

Different modeling alternatives arise from combinations where different types of signals are treated in different ways as deterministic, stochastic quantities or with the analytical approach mentioned above. In fact, many such partial approaches have been used in the past by several authors, mostly for three-dimensional problems and less for the determination of crustal motions.

We shall show here, how these approaches find their place in a general scheme of classification for the various possible modeling alternatives.

Three-dimensional integrated geodesy in its various aspects has been treated by several authors, e.g. Ashkenazi and Grist (1983), Benciolini et al. (1986), Dermanis (1984), Eeg and Krarup (1975), Grafarend (1978), Hein (1982a,b, 1983, 1985, 1986), Hein and Landau (1983), Hein et al. (1984), Krarup (1971), Landau et al. (1985), Reilly (1982b, 1984). Four-dimensional integrated geodesy has been treated in Collier et al. (1988), Hein (1984, 1986), Reilly (1981, 1982a), Rossikopoulos (1986), Zhou Zhongmo and Chao Dingbo (1987).

Table 2 is an attempt to show how our concept of integrated geodesy has evolved in the last twenty years.

## 2. The problem of modeling and the solution concept

A model is a description of physical reality, in the language of mathematics. Mathematical equations are the result of abstraction which leads to relative simplicity, while physical reality is extremely complex. The art of modeling is therefore a game of balance between simplicity and fidelity.

The real world is a unified entity whose parts are in continuous interaction. Models try, by necessity, to deal each time with a particular part of the real world. As a result of the unavoidable shortcomings in the description of such a part, as well as of the ignored interaction with the rest of the real world, discrepancies arise which we call errors. These include what is usually called modeling errors, but also the observational errors, since the measuring instrument is a part of the real world, which interacts with the part described by the model.

In a sufficient model, the observations are explained as the effect of a number of unknown parameters which describe the world part under study with sufficient fidelity, and the remaining discrepancies between observations and observables, the observational errors, are modeled as random variables with zero means. In more advanced models, some of the parameters are also modeled as random variables. The mathematical relations relating observations to unknown quantities, which include the deterministic or random model parameters and the observational errors, are usually called the mathematical model. The known or assumed statistical characteristics (means, variances, covariances, and perhaps probability distributions) of the observations and the stochastic parameters comprise the stochastic model.

Alternative models can be obtained from the same functional model by treating each unknown quantity in one of three different ways: (a) as completely known by assigning fixed values to them, (b) as completely unknown parameters, (c) as stochastic parameters with only their statistical behavior known or partially known. In reality, the original functional model may also vary depending on whether the effect of certain parameters on the observations has been recognized or not. However, such particular models can be considered as resulting from a more general model, which takes all effects into account, by fixing the values of the ignored parameters to zero.

An essential feature of the concept of integrated geodesy is the simultaneous treatment of all available observations, relevant to the part of the real world we are interested in. This calls for integrated models and combined adjustment solutions (Dermanis, 1986), which, though very attractive from the theoretical point of view, may face serious operational difficulties. Each new type of observations usually depends not only on the parameters of interest, but also on additional parameters of no primary interest (nuisance parameters). The increasing number of parameters makes the combined adjustment less tractable from the computational point of view, although the continuous improvements in computer power make these difficulties less important. In this respect we quote from Krarup (1974):

> *" ... I think it is in continuation of good geodetic tradition to work with such an adjustment model which involves all forms of geodetic observations even if there may be practical reasons for realizing the computations in sections each of which involves parts of the measurements, only in this way can arrive at a theory for the correct joining together of partial results. .... "*

Single solutions where each set of observations is treated separately, have the disadvantage that they lead to different estimates for the same common parameters, although the comparison of these estimates leads to important conclusions about the validity of the assumptions in the models used.

Sequential solutions may be either rigorous, which are algorithmic alternatives to the combined adjustment, or non-rigorous. In non-rigorous sequential solutions, estimates obtained from the adjustment of one set of observations are typically used as fixed values of the parameters in the adjustment of another set.

The need for an integrated model and combined adjustment solution may also arise in a case where no common parameters appear in the functional models for two separate sets of observations. This is the case where different parameters in the two separate models are considered to be distinct but correlated random variables. This correlation expresses, e.g., the common dependence of different signals on the same underlying function.

## 3. Functional model and unknown functions in four-dimensional geodesy

In the most general case of four-dimensional integrated geodesy, a single observation is a function of the instantaneous positions of one, two or more material points and the instantaneous gravity field at some of the same points. Considering only one point $P$ for simplicity, the observed quantity b must relate to position and gravity through the general functional form

$$b = F\big(\mathbf{r}(P,t), W(P,t)\big) \tag{1}$$

where $\mathbf{r}(P,t)$ is the coordinate vector of point $P$ at the epoch $t$, and $W(P,t)$ the gravity potential at the same point and epoch. For the identification of point $P$ one may use its coordinates $\mathbf{r}_0 = \mathbf{r}(P,t_0)$ at some fixed reference epoch $t_0$.

In order to linearize equation (1), the unknowns $\mathbf{r}(P,t)$ and $W(P,t)$ must be analyzed into approximate parts and small corrections

$$\mathbf{r}(P,t) = \mathbf{r}(P,t_0) + \delta_t\mathbf{r}(P,t) = \mathbf{r}^0(P,t_0) + \delta\mathbf{r}(P,t_0) + \delta_t\mathbf{r}(P,t) \tag{2}$$

Here $\mathbf{r}_0$ is a known approximate value, $\delta\mathbf{r}$ is the unknown correction for the position at epoch $t_0$ and $\delta_t\mathbf{r}$ is an unknown displacement signal. The potential is first analyzed into a normal-known and a disturbing-unknown part

$$W\big(\mathbf{r}(P,t),t\big) = U\big(\mathbf{r}(P,t),t\big) + T\big(\mathbf{r}(P,t),t\big)\,, \tag{3}$$

where we have assumed that it is sufficient to use a time-invariant normal field. Linearization with respect to the approximate position $\mathbf{r}^0$ gives

$$U\big(\mathbf{r}(P,t)\big) \approx U\big(\mathbf{r}^0(P,t_0)\big) + \boldsymbol{\gamma}\big(\mathbf{r}^0(P,t_0)\big)^T[\delta\mathbf{r}(P,t_0) + \delta_t\mathbf{r}(P,t)] \tag{4}$$

$$T\big(\mathbf{r}(P,t),t\big) \approx T\big(\mathbf{r}(P,t),t_0)\big) + \delta_t T\big(\mathbf{r}(P,t),t\big) \tag{5}$$

$$T\big(\mathbf{r}(P,t),t_0\big) \approx T\big(\mathbf{r}^0(P,t_0),t_0)\big) \tag{6}$$

$$\delta_t T\big(\mathbf{r}(P,t),t\big) \approx \delta_t T\big(\mathbf{r}^0(P,t_0),t_0\big) \tag{7}$$

With the help of the last equations the potential becomes

$$W(\mathbf{r}(P,t),t) = U(\mathbf{r}^0(P,t_0)) + \boldsymbol{\gamma}(\mathbf{r}^0(P,t_0))^T \delta\mathbf{r}(P,t_0) + \boldsymbol{\gamma}(\mathbf{r}^0(P,t_0))^T \delta_t\mathbf{r}(P,t) +$$

$$+ T(\mathbf{r}^0(P,t_0),t_0) + \delta_t T(\mathbf{r}^0(P,t_0),t_0) \tag{8}$$

where $\delta\mathbf{r}(P,t_0)$ is a deterministic unknown, $\delta_t\mathbf{r}(P,t)$ is a displacement signal, $T(\mathbf{r}^0(P,t_0),t_0)$ is a signal related to the gravity field at $t_0$ and $\delta_t T(\mathbf{r}^0(P,t_0),t_0)$ is a signal related to the time variation of the gravity field. It must be noted that, as a result of the linearization, all known terms, unknowns and signals refer to the approximate position $\mathbf{r}^0(P,t_0)$. Similar equations hold for any other gravity related quantity, such as $\Lambda(\mathbf{r}(P,t),t)$, $\Phi(\mathbf{r}(P,t),t)$ and $g(\mathbf{r}(P,t),t)$.

The detailed form of the linearized observation equations for the most common types of observations is given in Appendix A.

A simpler way to understand the structure of the linearized equations, is to consider separate linearization of the observations at each epoch $t_i$, where it is possible to even have only one observation for each epoch. The linearization is performed using the same approximate coordinates and the same normal potential for all epochs. The linearized observation equations at each epoch $t_i$ have the form

$$\mathbf{w}_i = \mathbf{A}_i\mathbf{x}_i + \mathbf{G}_i\mathbf{s}_i + \mathbf{v}_i, \qquad i = 0,1,2,\ldots,n \tag{9}$$

where $\mathbf{w}_i$ are the reduced observations (observed minus computed), $\mathbf{x}_i$ are now the coordinate corrections at epoch $t_i$, $\mathbf{s}_i$ are the gravity signals at the same epoch, $\mathbf{v}_i$ are the observational errors and $\mathbf{A}_i$, $\mathbf{G}_i$ are coefficient matrices.

Taking into account that coordinate differences are equal to the differences of coordinate corrections when the same approximate values are used, we may set

$$\mathbf{x}_i = \mathbf{x}_0 + \mathbf{p}_i, \qquad i = 1,2,\ldots,n \tag{10}$$

where the vector $\mathbf{p}_i = \mathbf{x}_i - \mathbf{x}_0$ contains displacements $\delta\mathbf{r}(\mathbf{r}_0,t_i)$ from the positions at epoch $t_0$ to those at epoch $t_i$. We can also set

$$\mathbf{s}_i = \mathbf{s}_0 + \mathbf{q}_i \tag{11}$$

where $\mathbf{q}_i$ are the gravity signal variations at the same point, from epoch $t_0$ to epoch $t_i$.

With the introduction of the vectors $\mathbf{p}_i$ and $\mathbf{q}_i$ the collective observation equations (9) for all epochs become

$$\mathbf{w}_0 = \mathbf{A}_0\mathbf{x}_0 + \mathbf{G}_0\mathbf{s}_0 + \mathbf{v}_0$$

$$\mathbf{w}_1 = \mathbf{A}_1\mathbf{x}_0 + \mathbf{A}_1\mathbf{p}_1 + \mathbf{G}_1\mathbf{s}_0 + \mathbf{G}_1\mathbf{q}_1 + \mathbf{v}_1$$

....................................................

$$\mathbf{w}_n = \mathbf{A}_n\mathbf{x}_0 + \mathbf{A}_n\mathbf{p}_n + \mathbf{G}_n\mathbf{s}_0 + \mathbf{G}_n\mathbf{q}_n + \mathbf{v}_n, \tag{12}$$

which are combined into

$$\begin{bmatrix}\mathbf{w}_0\\ \mathbf{w}_1\\ \mathbf{w}_2\\ \vdots\\ \mathbf{w}_n\end{bmatrix}=\begin{bmatrix}\mathbf{A}_0\\ \mathbf{A}_1\\ \mathbf{A}_2\\ \vdots\\ \mathbf{A}_n\end{bmatrix}\mathbf{x}_0+\begin{bmatrix}\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{A}_1 & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{A}_2 & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_n\end{bmatrix}\begin{bmatrix}\mathbf{p}_1\\ \mathbf{p}_2\\ \vdots\\ \mathbf{p}_n\end{bmatrix}+\begin{bmatrix}\mathbf{G}_0\\ \mathbf{G}_1\\ \vdots\\ \mathbf{G}_n\end{bmatrix}\mathbf{s}_0+\begin{bmatrix}\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{G}_1 & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{G}_2 & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{G}_n\end{bmatrix}\begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}_2\\ \vdots\\ \mathbf{q}_n\end{bmatrix}+\begin{bmatrix}\mathbf{v}_0\\ \mathbf{v}_1\\ \mathbf{v}_2\\ \vdots\\ \mathbf{v}_n\end{bmatrix} \qquad (13)$$

or simply

$$\mathbf{w}=\mathbf{A}^0\mathbf{x}_0+\mathbf{Hp}+\mathbf{G}^0\mathbf{s}_0+\mathbf{Dq}+\mathbf{v}\,. \qquad (14)$$

The above fundamental linearized functional model (14) is the basis for any further particular model which depends on assumptions about the displacements $\mathbf{p}$, the gravity related signals $\mathbf{s}_0$ at the reference epoch $t_0$ and their variations $\mathbf{q}$. The coordinate corrections $\mathbf{x}_0$ for reference epoch $t_0$ are always treated as deterministic completely unknown parameters. The matrix $\mathbf{A}^0$ has a rank deficiency due to the lack of any natural reference frame definition at epoch $t_0$. Minimal constraints are used to remove this deficiency. All elements of the vectors $\mathbf{p}$, $\mathbf{s}_0$ and $\mathbf{q}$ depend on the unknown underlying functions $\delta_t\mathbf{r}(\mathbf{r}^0,t)$, $T(\mathbf{r}^0,t_0)$ and $\delta_t T(\mathbf{r}^0,t)$, respectively, and they are therefore characterized as "signals".

## 4. Alternative signal treatments

### 4.1. Deterministic approach

Signals may be treated as independent deterministic unknown parameters, in which case their relation to an underlying function is ignored. This approach has the advantage that it is free from any dubious assumptions about the structure of the underlying function, essentially about its smoothness characteristics. The disadvantage lies in the fact that one ends up with too many parameters, which limit the degrees of freedom and the power of the adjustment to be performed. In fact, if no simultaneous observations exist, each observation brings in at least the displacements of the points involved at the current instant, not to mention the gravity signals, which results in more unknowns than observations.

The only way to make this approach possible is to extend the notion of epoch from one single instant to a relatively short time period, say one week of campaign, short enough for temporal variations in position and gravity to be negligible. In this case the epochs $t_0$, $t_1$, ..., $t_n$ refer to repeated campaigns rather than time instants. Of course periodic variations of non-negligible amplitude must be either separately computed and subtracted from the observations, or properly modeled. Temporal variations considered here will be only the secular ones.

Examples of the deterministic treatment can be found, e.g., in Engler et al. (1981), Fubara (1974), Grafarend (1981), Grafarend et al. (1987), Hradilec (1984), Körner (1968), Ramsayer (1972), Reilly (1980), Stolz (1970), Wolf (1963).

### 4.2. Analytical approach

The dependence of signals on underlying functions can be taken into account in two different ways. The first is to introduce a more or less empirical model for the underlying function, which involves unknown parameters to be estimated from the adjustment of the observations. Typical choices are linear combinations of known base functions with unknown coefficients. For example displacements can be modeled as

$$\delta_t\mathbf{r}(\mathbf{r}^0,t)=\mathbf{a}_0(\mathbf{r}^0)+\mathbf{a}_1(\mathbf{r}^0)(t-t_0)+\mathbf{a}_2(\mathbf{r}^0)(t-t_0)^2+\ldots \qquad (15)$$

with a finite number of terms. Since $\delta_t \mathbf{r}(\mathbf{r}^0, t_0) = \mathbf{a}_0(\mathbf{r}^0) = \mathbf{0}$ by the definition of displacements, the first term must be dropped. Usually only one or two terms are used, $\mathbf{a}_1(\mathbf{r}^0) = d(\delta_t \mathbf{r}) / dt = d\mathbf{r} / dt = \mathbf{v}$ being the displacement velocity at $\mathbf{r}^0$ and $\mathbf{a}_2(\mathbf{r}^0) = d^2(\delta \mathbf{r}) / dt^2 = \mathbf{a}$ being the displacement acceleration. The coefficients $\mathbf{a}_i(\mathbf{r}^0)$ are in general, but not necessarily, different for each point $\mathbf{r}^0$.

Analytical models for the displacement can also be used with respect to space,

$$\delta_t \mathbf{r}(\mathbf{r}^0, t) = \mathbf{a}_1(t) f_1(\mathbf{r}^0) + \mathbf{a}_2(t) f_2(\mathbf{r}^0) + \ldots \quad (16)$$

where $f_1(\mathbf{r}^0)$, $f_2(\mathbf{r}^0)$, … are known base functions of space, or with respect to both space and time

$$\delta_t \mathbf{r}(\mathbf{r}^0, t) = \mathbf{a}_1 f_1(\mathbf{r}^0, t) + \mathbf{a}_2(t) f_2(\mathbf{r}^0, t) + \ldots \quad (17)$$

Similar empirical analytical models with respect to time, space or space-time can be used for the gravity signal variations, which are elements of the vector $\mathbf{q}$. A general form of such empirical models is

$$q_i(\mathbf{r}^0, t) = b_1 \zeta_1(\mathbf{r}^0, t) + b_2 \zeta_2(\mathbf{r}^0, t) + \ldots. \quad (18)$$

The role of such analytical models for the unknown functions is to ensure a certain degree of smoothness in the final function estimate, in agreement with the smooth structure that the real function is expected to have.

As a result of the introduction of such analytical models, the signals vectors $\mathbf{p}$ and $\mathbf{q}$ are replaced in the model (14) by vectors of coefficients $\mathbf{a}$ and $\mathbf{b}$, respectively, using the relations

$$\mathbf{p} = \mathbf{F}\mathbf{a}, \qquad \mathbf{q} = \mathbf{\Phi}\mathbf{b}. \quad (19)$$

resulting from the application of equations like (17) and (18). The model becomes

$$\mathbf{w} = \mathbf{A}^0 \mathbf{x}_0 + \mathbf{H}\mathbf{F}\mathbf{a} + \mathbf{G}^0 \mathbf{s}_0 + \mathbf{D}\mathbf{\Phi}\mathbf{b} + \mathbf{v}, \quad (20)$$

where $\mathbf{r}^0$, $\mathbf{a}$ and $\mathbf{b}$ are deterministic unknown parameters to be estimated from the adjustment of the observations.

Examples of the analytical treatment can be found, e.g., in Bibby (1975), Brunner (1979), Chrzanowski et al. (1983), Collier et al. (1988), Hein and Kisterman (1981), Holdahl (1978, 1980), Holdahl and Hardy (1979), Mälzer et al. (1979), Margrave and Nyland (1980), Palmason (1980), Papo and Perelmuter (1983), Reilly (1986), Schmidt (1981), Snay et al. (1978), Snay et al. (1983), Vanicek (1975), Vanicek et al. (1979), Zhou Zhongmo and Chao Dingbo (1987), Whitten (1967).

### 4.3. Stochastic approach

The second way for taking into account the dependence of the signals on the unknown function is the use of stochastic models. The smooth structure of this function can be related to the similarity of function values, which are close in space and time, and the stochastic counterpart of similarity is correlation. The function is modeled as a stochastic process with known (usually zero) mean function and known covariance function. As a consequence the signals themselves become random variables. From the known dependence of the signals on the underlying function, the variances and covariances of the signals can be computed with application of the law of covariance propagation.

Since the displacement $\delta_t \mathbf{r}(\mathbf{r}_0, t) = \delta_t \mathbf{r}(P, t)$ of point $P$ is a vector function, each component $\delta_t r^i(P,t)$, $i = 1,2,3$, must be considered separately. It can be shown that invariance with respect to the arbitrary chosen reference frame poses the conditions

$$\sigma(\delta_t r^1(P,t), \delta_t r^1(Q,t')) = \sigma(\delta_t r^2(P,t), \delta_t r^2(Q,t')) = \sigma(\delta_t r^3(P,t), \delta_t r^3(Q,t')) \qquad (21)$$

$$\sigma(\delta_t r^i(P,t), \delta_t r^j(Q,t')) = 0 \qquad \text{for} \qquad i \neq j, \qquad (22)$$

on the relevant covariance functions. The further desired properties of homogeneity and isotropy with respect to space, as well as, stationarity with respect to time, are satisfied only when

$$\sigma(\delta_t r^i(P,t), \delta_t r^i(Q,t')) = C_{\delta r}(P,Q,t,t') = C_{\delta r}(s,\tau), \qquad i = 1,2,3 \qquad (23)$$

where $s$ is the distance between the positions $\mathbf{r}_{0,P}$ and $\mathbf{r}_{0,Q}$ occupied by points $P$ and Q, respectively at the reference epoch $t_0$, and $\tau = |t - t'|$.

When the stochastic approach is used, it is not necessary to separate the gravity signals $\mathbf{s}_0$ from their variations $\mathbf{q}$. Using instead the original gravity signals $\mathbf{s}_i = \mathbf{s}_0 + \mathbf{q}_i$, at each observation epoch $t_i$, the term $\mathbf{G}^0\mathbf{s}_0 + \mathbf{Dq}$ in equation (14) must be replaced by

$$\mathbf{Gs}_0 + \mathbf{Dq} = \mathbf{Gs} = \begin{bmatrix} \mathbf{G}_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{G}_n \end{bmatrix} \begin{bmatrix} \mathbf{s}_0 \\ \mathbf{s}_1 \\ \mathbf{s}_2 \\ \vdots \\ \mathbf{s}_n \end{bmatrix}, \qquad (24)$$

and the model now becomes

$$\mathbf{w} = \mathbf{A}^0\mathbf{x}_0 + \mathbf{Hp} + \mathbf{Gs} + \mathbf{v}. \qquad (25)$$

The covariances $\sigma(s_i(P,t), s_j(Q,t'))$ between any signals $s_i, s_j \in \{\delta\Lambda, \delta\Phi, \delta g, T\}$, can be computed by applying the law of covariance propagation on the "space-time" covariance function of the disturbing potential

$$\sigma(T(P,t), T(Q,t')) = C_T(r_P, r_Q, \gamma, \tau) \qquad (26)$$

assuming homogeneity, isotropy and stationarity as usual. The gravity signals in $\mathbf{s}$ can be disturbances in longitude $\delta\Lambda$, latitude $\delta\Phi$, gravity $\delta g$ or values of the disturbing potential $T$. Here $r_P$, $r_Q$ are the radial distances from the frame origin and $\gamma = \gamma_{PQ}$ is the angular distance of the points at epoch $t_0$. Space-time covariance functions have been considered in Kanngieser (1983) for vertical displacements only and in Rossikopoulos (1986) for the more general case of equation (26).

One possibility which appears only in the stochastic approach for the gravity signals is the incorporation in a combined adjustment of observations which are not directly tied to the network, such as observations of $\Lambda$, $\Phi$, $g$ or potential differences (dynamic heights) from leveling, even at points other than the network stations.

We have considered above stochastic models where randomness is considered for both the space and time parts of the domain involved. However it is possible to have a combination of stochastic and deterministic or analytical treatment with respect to different parts of the function domain. For example, we may model displacements to be stochastic only with respect to space and use an analytical model for their temporal behavior

$$\delta_t r^i(P,t) = a_1^i(P)(t-t_0) + a_2^i(P)(t-t_0)^2 , \qquad i = 1, 2, 3 . \tag{27}$$

The coefficients $a_1^i(P)$, $a_2^i(P)$ (velocity and acceleration) are here spatial stochastic processes with known covariance and cross-covariance functions. As a result the displacements $\mathbf{p}$ are replaced in the original functional model (14) by $\mathbf{p} = \mathbf{Fa}$, where, however the elements of $\mathbf{a}$ are now random variables with known variances and covariances. A similar combination of space-stochastic and time-analytical approach can be also used for the signals $\mathbf{q}$, resulting in random signals $\mathbf{b}$ in the model (20).

The displacement function $\delta_t \mathbf{r}(P,t)$ and the gravity potential $W(P,t)$ have so far been treated as independent functions in both their analytical or stochastic treatment. This is not true however, since temporal variations, of both the external gravity field and isolated points on the earth surface, are the result of motion and redistribution of masses in the earth's interior. This interrelation must be taken into account in both the analytical and stochastic treatment. The transformation from gravity to densities is an improperly posed problem and this poses great difficulties as far as the analytical approach is concerned.

In the stochastic treatment, the most difficult problem is to find cross-covariances between displacements and gravity signal variations. A departing point for the solution of this problem is the study of the stochastic behavior of densities, in relation to that of gravity, for the much simpler case where time variations are not considered. (Forsberg, 1984, Hein, 1984, 1986, Jordan, 1978, Krarup, 1970, Sansó and Tscherning, 1982, Sansó et al. 1986, Tscherning, 1976, 1977).

Even when the temporal variation of the gravity field is ignored, displacements are related to the values of gravity signals as a result of the change of position only. Usually only the change of potential due to vertical displacements is considered. Examples can be found in Biro (1983), Heck (1982), Heck and Mälzer (1983, 1986), Jachens (1978), Johnsen et al. (1980), Kanngieser (1983), Walsh (1982), Walsh and Rice (1979), Whitecomb (1976).

When a covariance function is assumed for displacements, the covariance of signal variations, which are elements of the vector $\mathbf{q}$, must also take into consideration the potential variation which is due to the displacements $\delta\mathbf{r}$ only.

A combined analytical-stochastic treatment is also possible for the displacements $\mathbf{p}$ or the gravity signal variations $\mathbf{q}$. One part of the underlying function, the trend, is modeled with the help of a linear combination of known base functions and the remaining part is modeled as a stochastic process. As a result the signals $\mathbf{p}$ or $\mathbf{q}$ are replaced by

$$\mathbf{p} = \mathbf{Fa} + \delta\mathbf{p} , \qquad \mathbf{q} = \mathbf{\Phi b} + \delta\mathbf{q} , \tag{28}$$

where $\mathbf{a}$ and $\mathbf{b}$ are unknown deterministic parameters and $\delta\mathbf{p}$ and $\delta\mathbf{q}$ are random variables.

## 5. Possible models for four-dimensional geodesy

In order to combine the possibilities presented above, to obtain alternatives for the final linearized model of the observation equations, two cases must be distinguished.

When the epochs of the individual observations are scattered in time, we have different displacement and gravity signal values in each observation. Therefore $\mathbf{p}$ and $\mathbf{s}$ cannot be treated as deterministic unknown parameters, because they are simply too many. A least squares adjustment is not possible since the unknowns exceed in number the available observations.

When groups of observations are performed "simultaneously", then the situation for each "epoch" is equivalent to that of three-dimensional integrated geodesy. In practice simultaneity is replaced by a set of neighboring epochs, and the common epoch by the middle of the period of a short-timed campaign. In this case both $\mathbf{p}$ and $\mathbf{s}$ can be treated either as unknown parameters, or one may treat $\mathbf{p}$ as unknown and $\mathbf{s}$ as stochastic parameters. In the first case we have a set of three-dimensional problems, one for each campaign:

$$\mathbf{w}_i = [\mathbf{A}_i \quad \mathbf{G}_i] \begin{bmatrix} \mathbf{x}_i \\ \mathbf{s}_i \end{bmatrix} + \mathbf{v}_i, \qquad \mathbf{v}_i^T \mathbf{P}_i \mathbf{v}_i = \min. \tag{29}$$

In the second case the approach is four-dimensional treatment only when the correlation between gravity signals $\mathbf{s}_i$, $\mathbf{s}_j$, referring to different time epochs $t_i$, $t_j$, is not ignored, and their non-zero cross-covariance matrices $\mathbf{C}_{\mathbf{s}_i \mathbf{s}_j}$ are taken into account. Another possibility is to ignore the temporal variation of gravity, setting $\mathbf{q}_i = \mathbf{0}$ and $\mathbf{s}_i = \mathbf{s}_0 + \mathbf{q}_i = \mathbf{s}_0$ for all epochs $t_i$. The observation sets of different campaigns depend on the same signals $\mathbf{s}_0$ and they are combined in a single adjustment. However, even if the gravity field is considered time invariant, the gravity signals do change with time as a result of the displacements of the relevant points.

When no simultaneous, or close to simultaneous, observations are available, space-time analytical models can be used for the displacements $\mathbf{p}$, while space-time stochastic, or space-stochastic and time-analytical models can be used for either the displacements $\mathbf{p}$ or the gravity signals $\mathbf{s}$. Space-analytical and time-stochastic models should not be considered, since time variation is typically smoother than space variation and can be sufficiently modeled by few polynomial terms. When the gravity signals $\mathbf{s}$ are separated into gravity signals $\mathbf{s}_0$ at the reference epoch $t_0$ and gravity signal variations $\mathbf{q}$, the same types of models used for $\mathbf{s}$ can also be used for $\mathbf{q}$. The signals $\mathbf{s}_0$ should be treated only stochastically.

Using the abbreviations A for (space-time) analytical, S for (space-time) stochastic and SA for space-stochastic and time-analytical treatment, we have the following possibilities resulting from the initial functional model (15) $\mathbf{w} = \mathbf{A}^0 \mathbf{x}_0 + \mathbf{H}\mathbf{p} + \mathbf{G}^0 \mathbf{s}_0 + \mathbf{D}\mathbf{q} + \mathbf{v}$. For convenience, deterministic parameters are separated from the stochastic ones:

**p**-A, **q**-S :

$$\mathbf{w} = [\mathbf{A}^0 \quad \mathbf{HF}] \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{a} \end{bmatrix} + [\mathbf{G}^0 \quad \mathbf{D}] \begin{bmatrix} \mathbf{s}_0 \\ \mathbf{q} \end{bmatrix} + \mathbf{v} \tag{30}$$

**p**-A, **q**-SA :

$$\mathbf{w} = [\mathbf{A}^0 \quad \mathbf{HF}] \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{a} \end{bmatrix} + [\mathbf{G}^0 \quad \mathbf{D\Phi}] \begin{bmatrix} \mathbf{s}_0 \\ \mathbf{b} \end{bmatrix} + \mathbf{v} \tag{31}$$

**p**-S, **q**-S :

$$\mathbf{w} = \mathbf{A}^0 \mathbf{x}_0 + [\mathbf{H} \quad \mathbf{G}^0 \quad \mathbf{D}] \begin{bmatrix} \mathbf{p} \\ \mathbf{s}_0 \\ \mathbf{q} \end{bmatrix} + \mathbf{v} \tag{32}$$

**p**-S, **q**-SA :

$$\mathbf{w} = \mathbf{A}^0 \mathbf{x}_0 + [\mathbf{H} \quad \mathbf{G}^0 \quad \mathbf{D}\boldsymbol{\Phi}] \begin{bmatrix} \mathbf{p} \\ \mathbf{s}_0 \\ \mathbf{b} \end{bmatrix} + \mathbf{v} \tag{33}$$

**p**-SA, **q**-S :

$$\mathbf{w} = \mathbf{A}^0 \mathbf{x}_0 + [\mathbf{H}\mathbf{F} \quad \mathbf{G}^0 \quad \mathbf{D}] \begin{bmatrix} \mathbf{a} \\ \mathbf{s}_0 \\ \mathbf{q} \end{bmatrix} + \mathbf{v} \tag{34}$$

**p**-SA, **q**-SA :

$$\mathbf{w} = \mathbf{A}^0 \mathbf{x}_0 + [\mathbf{H}\Phi \quad \mathbf{G}^0 \quad \mathbf{D}\boldsymbol{\Phi}] \begin{bmatrix} \mathbf{a} \\ \mathbf{s}_0 \\ \mathbf{b} \end{bmatrix} + \mathbf{v}\,. \tag{35}$$

The above cases do not exhaust all possibilities. We have not included at all cases where signals are treated in a combined analytical-stochastic treatment according to equations (28). Table 3 presents all possibilities for the treatment of displacements and gravity signals, which can be further combined in all possible ways.

| **Further treatment of the initial functional model :** $\mathbf{w} = \mathbf{A}^o \mathbf{x}_o + \mathbf{H}\,\mathbf{p} + \mathbf{G}^o \mathbf{s}_o + \mathbf{D}\,\mathbf{q} + \mathbf{v}$ or $\mathbf{w} = \mathbf{A}^o \mathbf{x}_o + \mathbf{H}\,\mathbf{p} + \mathbf{G}\,\mathbf{s} + \mathbf{v}$ | |
|---|---|
| treatment of displacements<br>**p** | treatment of **q** or **s**<br>$\mathbf{s}_0$ always stochastic $\mathbf{s}_0 \sim (\mathbf{0}, \mathbf{C}_{\mathbf{s}_0})$ |
| analytical : $\mathbf{p} = \mathbf{F}\,\mathbf{a}$ | analytical : $\mathbf{q} = \boldsymbol{\Phi}\,\mathbf{b}$ |
| stochastic : $\mathbf{p} \sim (\mathbf{0}, \mathbf{C}_\mathbf{p})$ | stochastic : $\mathbf{s} \sim (\mathbf{0}, \mathbf{C}_\mathbf{s})$ |
| space-stochastic & time-analytical :<br>$\mathbf{p} = \mathbf{F}\,\mathbf{a}$ , $\mathbf{a} \sim (\mathbf{0}, \mathbf{C}_\mathbf{a})$ | space-stochastic & time-analytical :<br>$\mathbf{q} = \boldsymbol{\Phi}\,\mathbf{b}$ , $\mathbf{b} \sim (\mathbf{0}, \mathbf{C}_\mathbf{b})$ |
| analytical + stochastic :<br>$\mathbf{p} = \mathbf{F}\,\mathbf{a} + \delta\mathbf{p}$ , $\delta\mathbf{p} \sim (\mathbf{0}, \mathbf{C}_{\delta\mathbf{p}})$ | analytical + stochastic :<br>$\mathbf{q} = \boldsymbol{\Phi}\,\mathbf{b} + \delta\mathbf{q}$ , $\delta\mathbf{q} \sim (\mathbf{0}, \mathbf{C}_{\delta\mathbf{q}})$ |

Table 3: A summary of modeling alternatives.

All the above models, resulting from the combination of different treatments of the signals, are finally of the form

$$\mathbf{w} = \mathbf{A}\mathbf{x} + \mathbf{G}\mathbf{s} + \mathbf{v} \tag{36}$$

where $\mathbf{x}$ contains all the deterministic parameters, $\mathbf{s}$ contains all the stochastic parameters and $\mathbf{v}$ are the observational errors. The adjustment problem is one of estimation with respect to $\mathbf{x}$ and prediction with respect to $\mathbf{s}$ and $\mathbf{v}$. For the stochastic parameters it is assumed that the means $E\{\mathbf{s}\} = \boldsymbol{\mu}$, $E\{\mathbf{v}\} = \mathbf{0}$ and the covariance matrices $E\{(\mathbf{s}-\boldsymbol{\mu})(\mathbf{s}-\boldsymbol{\mu})^T\} = \mathbf{C_s}$, $E\{\mathbf{v}\mathbf{v}^T\} = \mathbf{C_v}$ and $E\{(\mathbf{s}-\boldsymbol{\mu})\mathbf{v}^T\} = \mathbf{C_{sv}} = \mathbf{0}$ are known.

The estimates **x** and predictions **s** and **v** depend on the estimation and prediction principles used. In the most usual case Best Linear Unbiased Estimation (BLUE) is used. The solution for various linear prediction principles used has the general form (Schaffrin, 1983, 1985a,b,c, 1986a,b)

$$\mathbf{M} = \mathbf{G}\mathbf{C_s}\mathbf{G}^T \tag{37}$$

$$\mathbf{L} = (\mathbf{A}^T\mathbf{M}^{-1}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{M}^{-1} \tag{38}$$

$$\mathbf{x} = \mathbf{L}(\mathbf{w} - \mathbf{G}\boldsymbol{\mu}) \tag{39}$$

$$\mathbf{s} = \alpha\boldsymbol{\mu} + \mathbf{C_s}(\mathbf{I} + \mathbf{M}\mathbf{C_s})^{-1}\mathbf{G}^T\mathbf{C_s}^{-1}(\mathbf{I} - \mathbf{A}\mathbf{L})(\mathbf{w} - \alpha\mathbf{G}\boldsymbol{\mu})\,. \tag{40}$$

The value of the parameter α varies according to the prediction principle:

**inhomBLIP** (Best inhomogeneously Linear Prediction), which is identical to
**inhomBLUP** ( Best inhomogeneously Linear weakly Unbiased Prediction)

$$\alpha = 1\,. \tag{41}$$

This is the usual "collocation solution", widely used in practice. Less used are the robust alternatives

**homBLIP** (Best Linear homogeneously Linear Prediction)

$$\alpha = \frac{\lambda}{\delta} \tag{42}$$

**homBLUP** (Best Linear homogeneously weakly Unbiased Linear Prediction)

$$\alpha = \frac{\lambda}{1+\delta} \tag{43}$$

where

$$\lambda = \boldsymbol{\mu}^T(\mathbf{I} + \mathbf{M}\mathbf{C_s})^{-1}\mathbf{G}^T\mathbf{C_v}^{-1}(\mathbf{I} - \mathbf{A}\mathbf{L})\mathbf{w} \tag{44}$$

$$\delta = \boldsymbol{\mu}^T(\mathbf{I} + \mathbf{M}\mathbf{C_s})^{-1}\mathbf{G}^T\mathbf{C_v}^{-1}(\mathbf{I} - \mathbf{A}\mathbf{L})\mathbf{G}\boldsymbol{\mu}\,. \tag{45}$$

A detailed analysis of the least squares collocation solutions for the various models summarized here by equations (30) to (35) can be found in Rossikopoulos (1986).

## Appendix A: Observations equations of four-dimensional networks.

The equations presented here are directly adopted from those of the three-dimensional case given in Dermanis (1984). The time dimension is introduced by considering the time-invariant equations to hold for the observation epoch $t$. Coordinate corrections are then analyzed into coordinate corrections at $t_0$ and displacements, and gravity signals into gravity signals at $t_0$ and their corresponding variations.

The following notation is used:

| | |
|---|---|
| $i$ | station point |
| $j$ | target point |
| $\mathbf{r}_i$ | position vector of point $i$ in global frame |
| $\mathbf{r}_{ij}^*$ | position vector of $j$ with respect to local frame at $i$ |
| $\mathbf{R}(\Lambda_i, \Phi_i)$ | rotation matrix relating local frame at $i$ with global frame |
| $\Lambda_i, \Phi_i$ | astronomic longitude and latitude |
| $s_{ij}$ | distance between points $i$ and $j$ |
| $\delta_{ij}$ | horizontal direction of $j$ observed from $i$ |
| $A_{ij}$ | azimuth of $j$ from $i$ |
| $Z_{ij}$ | zenith angle of $j$ from $i$ |
| $\theta_i$ | orientation unknown at $i$ |
| $g$ | gravity |
| $W$ | gravity potential |
| $U$, $\lambda$, $\phi$, $\gamma$, $a$ | normal counterparts of $W$, $\Lambda$, $\Phi$, $g$ and $A$ respectively. |

The global cartesian frame is any frame common for all network points and not necessarily the conventional geocentric frame. The local cartesian frame has its third axis in the direction of the local vertical and its second axis coplanar with the vertical and the third axis of the global frame. The relation between the two frames is

$$\mathbf{r}_{ij}^* = \mathbf{R}(\Lambda_i, \Phi_i)(\mathbf{r}_j - \mathbf{r}_i) \tag{A1}$$

where

$$\mathbf{R} = \mathbf{R}(\Lambda, \Phi) = \mathbf{R}_1(90^\circ - \Phi)\mathbf{R}_3(90^\circ + \Lambda) \tag{A2}$$

The coordinates at epoch t can be analyzed into

$$\mathbf{r}_i(t) = \mathbf{r}_i(t_0) + \delta_t \mathbf{r}_i(t) = \mathbf{r}_i^0 + \mathbf{x}_i(t_0) + \delta_t \mathbf{r}_i(t) \tag{A3}$$

where $\mathbf{r}_i^0$ are approximate coordinates and $\mathbf{x}_i(t_0)$ the corrections for the epoch $t_0$.

As a result of the linearization all signals at epoch t are referring to the positions corresponding to the approximate coordinates $\mathbf{r}_i^0$. They can be further analyzed into signals at the reference epoch $t_0$ and their variations, as follows:

$$\delta\Lambda_i(t) = \Lambda(\mathbf{r}_i^0, t) - \lambda(\mathbf{r}_i^0) = \delta\Lambda_i(t_0) + \delta_t \delta\Lambda_i(t) \tag{A4}$$

$$\delta\Phi_i(t) = \Phi(\mathbf{r}_i^0, t) - \phi(\mathbf{r}_i^0) = \delta\Phi_i(t_0) + \delta_t \delta\Phi_i(t) \tag{A5}$$

$$\delta g_i(t) = g(\mathbf{r}_i^0, t) - \gamma(\mathbf{r}_i^0) = \delta g_i(t_0) + \delta_t \delta g_i(t) \tag{A6}$$

$$T_i(t) = W(\mathbf{r}_i^0, t) - U(\mathbf{r}_i^0) = T_i(t_0) + \delta_t T_i(t) \tag{A7}$$

## 1. Observation equations of geometric type

1. Horizontal direction

$$\delta_{ij} = \arctan \frac{x_{ij}^*}{y_{ij}^*} - \theta_i \tag{A8}$$

$$\delta_{ij} - \delta_{ij}^0 = [-\mathbf{a}^T \mathbf{a}^T]\mathbf{R}\begin{bmatrix}\mathbf{x}_i(t_0)\\ \mathbf{x}_j(t_0)\end{bmatrix} + [-\mathbf{a}^T \mathbf{a}^T]\mathbf{R}\begin{bmatrix}\delta_t \mathbf{r}_i(t)\\ \delta_t \mathbf{r}_j(t)\end{bmatrix} +$$

$$+\left[-\mathbf{a}^T \mathbf{R}_\Lambda(\mathbf{r}_j - \mathbf{r}_i) \quad \mathbf{a}^T \mathbf{R}_\Phi(\mathbf{r}_j - \mathbf{r}_i)\right]_0 \begin{bmatrix}\delta\Lambda_i(t_0)\\ \delta\Phi_i(t_0)\end{bmatrix} +$$

$$+\left[-\mathbf{a}^T \mathbf{R}_\Lambda(\mathbf{r}_j - \mathbf{r}_i) \quad \mathbf{a}^T \mathbf{R}_\Phi(\mathbf{r}_j - \mathbf{r}_i)\right]_0 \begin{bmatrix}\delta_t\delta\Lambda_i(t_0)\\ \delta_t\delta\Phi_i(t_0)\end{bmatrix} - \delta\theta_i \tag{A9}$$

where

$$\mathbf{a}^T = \begin{bmatrix}\frac{y_{ij}^*}{(d_{ij}^*)^2} & \frac{-x_{ij}^*}{(d_{ij}^*)^2} & 0\end{bmatrix}_0, \qquad d_{ij}^* = \sqrt{(x_{ij}^*)^2 + (y_{ij}^*)^2} \tag{A10}$$

$$\mathbf{R}_\Lambda = \frac{\partial}{\partial\Lambda}\mathbf{R}, \qquad \mathbf{R}_\Phi = \frac{\partial}{\partial\Phi}\mathbf{R}, \qquad \delta\theta_i = \theta_i - \theta_i^0 \tag{A11}$$

2. Zenith angle

$$Z_{ij} = \arctan \frac{d_{ij}^*}{z_{ij}^*} \tag{A12}$$

$$Z_{ij} - Z_{ij}^0 = [-\mathbf{b}^T \mathbf{b}^T]\mathbf{R}\begin{bmatrix}\mathbf{x}_i(t_0)\\ \mathbf{x}_j(t_0)\end{bmatrix} + [-\mathbf{b}^T \mathbf{b}^T]\mathbf{R}\begin{bmatrix}\delta_t \mathbf{r}_i(t)\\ \delta_t \mathbf{r}_j(t)\end{bmatrix} +$$

$$+\left[-\mathbf{b}^T \mathbf{R}_\Lambda(\mathbf{r}_j - \mathbf{r}_i) \quad \mathbf{b}^T \mathbf{R}_\Phi(\mathbf{r}_j - \mathbf{r}_i)\right]_0 \begin{bmatrix}\delta\Lambda_i(t_0)\\ \delta\Phi_i(t_0)\end{bmatrix} +$$

$$+\left[-\mathbf{b}^T \mathbf{R}_\Lambda(\mathbf{r}_j - \mathbf{r}_i) \quad \mathbf{b}^T \mathbf{R}_\Phi(\mathbf{r}_j - \mathbf{r}_i)\right]_0 \begin{bmatrix}\delta_t\delta\Lambda_i(t)\\ \delta_t\delta\Phi_i(t)\end{bmatrix}, \tag{A13}$$

where

$$\mathbf{b}^T = \begin{bmatrix}\frac{x_{ij}^* z_{ij}^*}{d_{ij}^2 s_{ij}^*} & \frac{y_{ij}^* z_{ij}^*}{d_{ij}^* s_{ij}^2} & \frac{-d_{ij}^*}{s_{ij}^2}\end{bmatrix}_0 \tag{A14}$$

3. distance

$$s_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2} \tag{A15}$$

$$s_{ij} - s_{ij}^0 = [-\mathbf{c}^T \mathbf{c}^T] \begin{bmatrix} \mathbf{x}_i(t_0) \\ \mathbf{x}_j(t_0) \end{bmatrix} + [-\mathbf{c}^T \mathbf{c}^T] \begin{bmatrix} \delta_t \mathbf{r}_i(t) \\ \delta_t \mathbf{r}_j(t) \end{bmatrix}, \tag{A16}$$

where

$$\mathbf{c}^T = \begin{bmatrix} \frac{x_j^0 - x_i^0}{s_{ij}^0} & \frac{y_j^0 - y_i^0}{s_{ij}^0} & \frac{z_j^0 - z_i^0}{s_{ij}^0} \end{bmatrix} \tag{A17}$$

## 2. Additional Observations

1. Astronomic longitude, latitude and gravity.

$$\begin{bmatrix} \Lambda(\mathbf{r},t) - \lambda(\mathbf{r}^0) \\ \Phi(\mathbf{r},t) - \phi(\mathbf{r}^0) \\ g(\mathbf{r},t) - \gamma(\mathbf{r}^0) \end{bmatrix} = \frac{\partial[\lambda\phi\gamma]^T}{\partial\boldsymbol{\gamma}}\bigg|_0 \frac{\partial\boldsymbol{\gamma}}{\partial\mathbf{r}}\bigg|_0 \left(\mathbf{x}(t_0) + \delta\mathbf{r}_t(t)\right) + \begin{bmatrix} \delta\Lambda(t_0) \\ \delta\Phi(t_0) \\ \delta g(t_0) \end{bmatrix} + \begin{bmatrix} \delta_t\delta\Lambda(t) \\ \delta_t\delta\Phi(t) \\ \delta_t\delta g(t) \end{bmatrix} \tag{A18}$$

where

$$\frac{\partial[\lambda\varphi\gamma]^{\mathrm{T}}}{\partial\boldsymbol{\gamma}} = \begin{bmatrix} -\gamma\cos\varphi & 0 & 0 \\ 0 & -\gamma & 0 \\ 0 & 0 & -1 \end{bmatrix}^{-1} \mathbf{R}_1(90^\circ - \varphi)\mathbf{R}_3(90^\circ + \lambda) \tag{A19}$$

2. Potential differences $\Delta W_{ij} = W_j - W_i$ (from leveling)

$$\Delta W_{ij} - \Delta U_{ij} = [-\boldsymbol{\gamma}_i^T \ \boldsymbol{\gamma}_j^T] \begin{bmatrix} \mathbf{x}_i(t_0) \\ \mathbf{x}_j(t_0) \end{bmatrix} + [-\boldsymbol{\gamma}_i^T \ \boldsymbol{\gamma}_j^T] \begin{bmatrix} \delta_t \mathbf{r}_i(t_0) \\ \delta_t \mathbf{r}_j(t_0) \end{bmatrix} + T_i(t_0) + \delta_t T_i(t) \tag{A20}$$

where $\Delta U_{ij} = U(\mathbf{r}_j^0) - U(\mathbf{r}_i^0)$.

3. Astronomic azimuth

$$\begin{aligned} A_{ij} - a_{ij}^0 = {} & [-\mathbf{a}^T \mathbf{a}^T] \mathbf{R} \begin{bmatrix} \mathbf{x}_i(t_0) \\ \mathbf{x}_j(t_0) \end{bmatrix} + [-\mathbf{a}^T \mathbf{a}^T] \mathbf{R} \begin{bmatrix} \delta_t \mathbf{r}_i(t) \\ \delta_t \mathbf{r}_j(t) \end{bmatrix} + \\ & + \left[-\mathbf{a}^T \mathbf{R}_\Lambda (\mathbf{r}_j - \mathbf{r}_i) \quad \mathbf{a}^T \mathbf{R}_\Phi (\mathbf{r}_j - \mathbf{r}_i)\right]_0 \begin{bmatrix} \delta\Lambda_i(t_0) \\ \delta\Phi_i(t_0) \end{bmatrix} + \\ & + \left[-\mathbf{a}^T \mathbf{R}_\Lambda (\mathbf{r}_j - \mathbf{r}_i) \quad \mathbf{a}^T \mathbf{R}_\Phi (\mathbf{r}_j - \mathbf{r}_i)\right]_0 \begin{bmatrix} \delta_t\delta\Lambda_i(t_0) \\ \delta_t\delta\Phi_i(t_0) \end{bmatrix} \end{aligned} \tag{A21}$$

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