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THE ROLE OF FRAME DEFINITIONS IN THE GEODETIC DETERMINATION OF CRUSTAL DEFORMATION PARAMETERS

Abstract

A study of the role of coordinate frame definitions in the determination of crustal deformation parameters is first carried out for the theoretical case where displacement information between two discrete time epochs is continuously available for all area points. The obtained results are next applied to the realistic case where the required continuous information is derived by means of an interpolation of the known coordinate variations at the points of a horizontal geodetic network.

The problem is different from the usual one of frame–invariant interpolation, since not only the domain of definition, but also the interpolated quantities, depend on independent choices of coordinate frames.

Specific necessary and sufficient conditions for the invariance of derived crustal deformation parameters are given for linear type of interpolations of either the coordinates at the second epoch or of the displacements.

With the help of the above conditions the invariance characteristics of two commonly used types of linear interpolations are finally derived, in order to illustrate the practical significance of the results.

1. Introduction

In recent years there has been a continuously rising interest in studying the deformation of the earth's crust in view of its significance for the understanding and eventual prediction of earthquakes (US NRC, 1981, Rikitake, 1976).

Since deformation, roughly speaking, means change of geometric configuration, classical as well as space geodetic techniques play an important role in the determination of crustal deformation parameters.

Our interest here focuses on some problems related to the analysis of classical two–dimensional triangulation results for the extraction of information about crustal deformation. From repeated triangulations different plane coordinates of stations are available describing the different network configurations at the two epochs under comparison. The discrete nature of such information is in contrast with the nature of geophysically meaningful plane deformation parameters, such as dilatation Δ , maximum shear strain γ , etc. These are local differential quantities defined pointwise in the area

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of interest and their values depend on the behaviour of points in an infinitesimal neighbourhood of the point of interest.

The answer to the question whether such local differential parameters can be determined, or estimated, from discrete geodetic information, must be flatly negative, at least if one has to stick to the theoretical aspects of the problem.

However such an attitude relegates all physical quantities having the character of derivatives with respect to space or time, to the realm of the nondeterminable, nonestimable quantities, something in contrast to everyday practice in the various fields of applied science.

The way out of such a problem is found through the following compromise. It is assumed that discrete information can be reasonably extended to be continuous by means of appropriate interpolation techniques.

Geodetic information is usually described by means of coordinates which depend not only on the geometric configuration of the network, but also on the coordinate frame used.

It is well known to geodesists that estimable quantities in geodetic networks are equivalent to quantities invariant under coordinate transformations that also leave the relevant available geodetic observations invariant. This equivalence has been established through the pioneering work of W. Baarda (1962, 1967, 1971, 1973) and was later elaborated by Bossler (1973) and Grafarend and Schaffrin (1974, 1976).

By studying the invariance properties of plane deformation parameters in the limiting case when continuous position information is at hand, we shall derive here necessary and sufficient conditions for the same invariance properties to hold when continuous information is obtained by means of interpolation techniques.

Within the standard least–squares adjustment techniques used in geodesy, nonestimable parameters, such as coordinates, are determined with the introduction of arbitrary reference frames through appropriate (minimal) constraints. A quantity that can be expressed as a function of the nonestimable parameters is estimable, if its estimated value is independent of the particular choice of coordinate frame. This is a somewhat vague and restricted–to–geodetic–problems definition of estimability. Within the usual probabilistic setup of the observation equations (Gauss–Markov model), estimability is defined as the property of a quantity to have an unbiased estimate ; see, e.g., Koch (1980), p. 169. An equivalent algebraic definition of estimability is also possible ; see, e.g., Rauhala (1980), p. 2.

Here we look upon the estimability of a quantity that can be expressed as the value of a functional on a certain unknown underlying function which is obtained from coordinate estimates from a more or less arbitrarily chosen interpolation scheme. Therefore the quantity in question becomes the value of an equally arbitrary functional on the vector of coordinates. It is the estimability of the value of this well–defined, after the choice of interpolation scheme, functional that we are after here. The estimated value is *not* to be compared directly with the actual physical quantity it aspires to be an estimate of. The actual quantity cannot be estimated as already stated from the limited discrete information at hand. This problem will be discussed in a more rigorous mathematical setup elsewhere.

In a previous paper (Dermanis, 1981) an analysis of the estimability of crustal deformation parameters was carried out, based on the above–mentioned concepts of invariance. This analysis has been restricted in two ways. First, only the case of crustal deformation parameters obtained through the most commonly used finite element method has been considered. According to this method displacements at the three vertices of each network triangle are interpolated following a piecewise linear interpolation scheme. The second restriction was that the computed crustal deformation parameters were infinitesimal in the sense that second–order terms of the displacement gradients have been neglected.

Here the notion of estimability concerns quantities derived from coordinates through prescribed interpolation schemes. Network coordinates are typically nonestimable quantities. Crustal deformation parameters, being local differential quantities defined in terms of displacement derivatives, cannot naturally be determined from discrete coordinates and are therefore nonestimable quantities. However, the approximate counterparts of these parameters obtained through interpolation techniques are the values of known functions of the coordinates and they may be estimable if they share certain well–known invariance characteristics.

Instead of studying here the invariance properties of crustal deformation parameters defined through one or more specific interpolation techniques, we rather obtain necessary and sufficient conditions such that the general linear interpolation scheme leads to estimates of deformation parameters having the desired invariance properties. In each case the appropriate invariance characteristics are those of invariance under coordinate transformations shared by the original geodetic observations in the network.

2. Invariance characteristics of crustal deformation parameters when continuous information is available.

Suppose that the plane horizontal coordinates of every material point P within some region are known at two different epochs. We denote by

$$\mathbf{r} = [x \ y]^T \tag{1}$$

and

$$\mathbf{r}' = [x' \ y']^T \tag{2}$$

the position vectors of a point P at the two epochs under consideration. The two coordinate sets (x, y) and (x', y') generally refer to independently defined frames. Since they are known for each point of an open subset of a two–dimensional Euclidean manifold, the functions

$$x' = x'(x, y) \qquad y' = y'(x, y)$$

as well as the inverse functions

$$x = x(x', y') \qquad y = y(x', y')$$

are also known. It is also assumed that the Jacobian determinant

$$\left| \frac{\partial (x', y')}{\partial (x, y)} \right| = \left| \frac{\partial \mathbf{r}'}{\partial \mathbf{r}} \right| \neq 0 \tag{3}$$

never vanishes at all points of interest. If ds , ds′ are the metric elements at the two epochs the Lagrangian strain tensor (matrix) **E** is defined by means of (Dermanis & Livieratos, 1983)

$$ds'^2 - ds^2 = 2\, d\mathbf{r}^T \mathbf{E}\, d\mathbf{r} \tag{4}$$

Recalling that in Euclidean spaces $ds^2 = d\mathbf{r}^T d\mathbf{r}$, $ds'^2 = (d\mathbf{r}')^T d\mathbf{r}'$.

it follows that

$$\mathbf{E}(\mathbf{r}) = \begin{bmatrix} E_{xx} & E_{xy} \\ & \\ E_{xy} & E_{yy} \end{bmatrix} = \frac{1}{2}\left[\left(\frac{\partial \mathbf{r}'}{\partial \mathbf{r}}\right)^T \left(\frac{\partial \mathbf{r}'}{\partial \mathbf{r}}\right) - \mathbf{I} \right] \tag{5}$$

Relevant deformation parameters are the dilatation

$$\Delta = \mathit{trace}\,\mathbf{E} = E_{xx} + E_{yy} \tag{6}$$

the shear components

$$\gamma_1 = E_{xx} - E_{yy} \tag{7}$$

$$\gamma_2 = 2\,E_{xy} \tag{8}$$

the maximum shear strain

$$\gamma = \sqrt{\gamma_1^2 + \gamma_2^2} \tag{9}$$

the principal strains

$$e_{max} = \frac{\Delta + \gamma}{2} \tag{10}$$

$$e_{min} = \frac{\Delta - \gamma}{2} \tag{11}$$

and the azimuth φ of e_{max}

$$\varphi = \frac{1}{2}\,\mathit{arctan}\left(\frac{-\gamma_2}{\gamma_1}\right) \tag{12}$$

Let us consider the case when both frames to which the position vectors **r** and **r′** refer undergo independent similarity transformations

$$\tilde{\mathbf{r}} = \lambda\, \mathbf{R}\, \mathbf{r} + \mathbf{b} \tag{13}$$

$$\tilde{\mathbf{r}}' = \lambda'\, \mathbf{R}'\, \mathbf{r}' + \mathbf{b}' \tag{14}$$

where λ , λ' are scale factors,

$$\mathbf{b} = [b_x \; b_y]^T \tag{15}$$

$$\mathbf{b}' = [b'_x \; b'_y]^T \tag{16}$$

are parallel translations and

$$\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \tag{17}$$

$$\mathbf{R}' = \begin{bmatrix} \cos\theta' & \sin\theta' \\ -\sin\theta' & \cos\theta' \end{bmatrix} = \begin{bmatrix} c' & s' \\ -s' & c' \end{bmatrix} \tag{18}$$

are rotations with obvious notations for c , s and c', s' .

The strain matrix with respect to the new frames after the similarity transformations is given by

$$\tilde{\mathbf{E}} = \frac{1}{2}\left[\left(\frac{\partial \tilde{r}'}{\partial \tilde{r}}\right)^T \frac{\partial \tilde{r}'}{\partial \tilde{r}} - \mathbf{I}\right] = \begin{bmatrix} \tilde{E}_{xx} & \tilde{E}_{xy} \\ \tilde{E}_{xy} & \tilde{E}_{yy} \end{bmatrix} . \tag{19}$$

Since

$$\frac{\partial \tilde{r}'}{\partial \tilde{r}} = \frac{\partial \tilde{r}'}{\partial r'} \frac{\partial r'}{\partial r} \frac{\partial r}{\partial \tilde{r}} = \lambda' \, \mathbf{R}' \frac{\partial r'}{\partial r} (\lambda \, \mathbf{R})^{-1} = \mu \, \mathbf{R}' \frac{\partial r'}{\partial r} \mathbf{R}^T \tag{20}$$

where $\mu = \lambda'/\lambda$, the strain matrix becomes

$$\tilde{\mathbf{E}} = \frac{1}{2}\left\{(\mu \, \mathbf{R}' \frac{\partial r'}{\partial r} \mathbf{R}^T)^T \; (\mu \, \mathbf{R}' \frac{\partial r'}{\partial r} \mathbf{R}^T) - \mathbf{I}\right\} . \tag{21}$$

After some matrix operations and recalling equation (5) we obtain

$$\tilde{\mathbf{E}} = \left\{\mu^2 \, \mathbf{R} \, \mathbf{E} \, \mathbf{R}^T + \frac{(\mu^2 - 1)}{2} \, \mathbf{I}\right\} \tag{22}$$

It is immediately seen that the components of the strain matrix are invariant under the translations $\mathbf{b}$, $\mathbf{b}'$ and the rotation by an angle θ' of the second frame. In the particular case that $\lambda = \lambda'$, $\mu = 1$, i.e., only rigid transformations or similarity with the same scale factor are allowed, the well–known rule for tensor transformation follows

$$\tilde{\mathbf{E}} = \mathbf{R} \, \mathbf{E} \, \mathbf{R}^T \tag{23}$$

For the more general case when $\mu \neq 1$, we have

$$\tilde{E}_{xx} = \mu^2 (c^2 E_{xx} + 2\,c\,s\,E_{xy} + s^2 E_{yy}) + \frac{\mu^2 - 1}{2} \tag{24}$$

$$\tilde{E}_{xy} = \mu^2 [-cs\,E_{xx} + (c^2 - s^2) E_{xy} + cs\,E_{yy}] \tag{25}$$

$$\tilde{E}_{yy} = \mu^2 (s^2 E_{xx} - 2\,c\,s\,E_{xy} + c^2 E_{yy}) + \frac{\mu^2 - 1}{2} \tag{26}$$

For the other deformation parameters we have

$$\tilde{\Delta} = \mu^2 \Delta + \mu^2 - 1 \tag{27}$$

or

$$1 + \tilde{\Delta} = \mu^2 (1 + \Delta) \tag{28}$$

$$\tilde{\gamma}_1 = \mu^2 (\cos 2\theta\, \gamma_1 + \sin 2\theta\, \gamma_2) \tag{29}$$

$$\tilde{\gamma}_2 = \mu^2 (-\sin 2\theta\, \gamma_1 + \cos 2\theta\, \gamma_2) \tag{30}$$

$$\tilde{\gamma} = \mu^2 \gamma \tag{31}$$

$$\tilde{e}_{max} = \mu^2 e_{max} + \frac{\mu^2 - 1}{2} \tag{32}$$

$$\tilde{e}_{min} = \mu^2 e_{min} + \frac{\mu^2 - 1}{2} \tag{33}$$

$$\tilde{\varphi} = \varphi + \theta \tag{34}$$

In the special case that $\lambda' = \lambda$, i.e., when both frames have known scale provided by distance measurements, the quantities

$$\Delta,\ \gamma,\ e_{max},\ e_{min}$$

are numerical invariants independent of frame definition.

3. Invariance characteristics when crustal deformation parameters are obtained from the interpolation of discrete information

3.1. General considerations

In practice, continuous information concerning the variation of coordinates of all points in a certain area is never available from observations. Observations and analysis of plane geodetic networks lead to discrete coordinate information referring to the network points. With this in mind the results of the previous section must be seen as limiting conditions in the case where network points become denser and denser.

The continuous information necessary for the computation of strain parameters is explicitly or implicitly obtained by means of a chosen interpolation technique. Such a technique is essentially a mapping

$$\mathbf{r} = [x \; y]^T \rightarrow \mathbf{r}' = [x' \; y']^T \tag{35}$$

assigning "new" coordinates (x', y') at epoch t' to any point of the area defined through its "old" coordinates (x, y) at epoch t.

The interpolation mapping must also depend on the coordinates of the network points

$$\mathbf{r}_i = [x_i \; y_i]^T \tag{36}$$

and

$$\mathbf{r}_i' = [x_i' \; y_i']^T \tag{37}$$

at epochs t and t' respectively.

Denoting by $\{\mathbf{r}_i\}$, $\{\mathbf{r}_i'\}$ the sets of all coordinates of network points $\mathbf{r}_i$ and $\mathbf{r}_i'$ respectively, the general interpolation mapping can be expressed as a function

$$\mathbf{r}' = f(\mathbf{r}, \{\mathbf{r}_i\}, \{\mathbf{r}_i'\}) . \tag{38}$$

Let U be an element of a (not necessarily proper) subgroup Γ of the group of similarity coordinate transformations, e.g., rigid transformations, parallel translations, etc. Denote by

$$\tilde{\mathbf{r}} = U(\mathbf{r}) \tag{39}$$

the result of the transformation $U \in \Gamma$ on the coordinate pair (x, y) and by

$$\tilde{\mathbf{r}}_i = U(\mathbf{r}_i) \tag{40}$$

the result of the same transformation applied pointwise on all network coordinates. Similar notation $U' \in \Gamma'$ is used for transformations of the new coordinates $\mathbf{r}'$ and $\mathbf{r}_i'$.

Definition 1.

A coordinate interpolation

$$\mathbf{r}' = f(\mathbf{r}, \{\mathbf{r}_i\}, \{\mathbf{r}_i'\})$$

is said to be invariant under independent transformations $U \in \Gamma$ and $U' \in \Gamma'$ of the old and new coordinates respectively, for given groups Γ and Γ', when

$$f[U(\mathbf{r}), \{U(\mathbf{r}_i)\}, \{U'(\mathbf{r}_i')\}] = U'(\mathbf{r}') = U'[f(\mathbf{r}, \{\mathbf{r}_i\}, \{\mathbf{r}_i'\})] . \tag{41}$$

for any $U \in \Gamma$ and any $U' \in \Gamma'$.

This means that when the original data are transformed under independent transformations, the results of the interpolation are accordingly transformed.

Definition 2.

A coordinate interpolation

$$r' = f(r, \{r_i\}, \{r'_i\})$$

is said to be invariant under a common transformation $U \in \Gamma$ of both old and new coordinates, for a given transformation group Γ, when

$$f[U(r), \{U(r_i)\}, \{U(r'_i)\}] = U(r') = U[f(r, \{r_i\}, \{r'_i\})] \tag{42}$$

for any $U \in \Gamma$.

Note that when an interpolation is invariant under independent transformations (according to definition 1), it is also invariant under common transformations (according to definition 2). The second property is a particular case of the first one.

Lemma 1.

Necessary and sufficient conditions for the interpolation

$$r' = f(r, \{r_i\}, \{r'_i\})$$

to be invariant under independent transformations $U \in \Gamma$ of old and $U' \in \Gamma'$ of new coordinates when $\Gamma' \subset \Gamma$, are

i. Invariance under common transformations

$$f[U'(r), \{U'(r_i)\}, \{U'(r'_i)\}] = U'(r') = U'[f(r, \{r_i\}, \{r'_i\})] \tag{43}$$

for any $U' \in \Gamma'$.

ii. Invariance under transformations of the old coordinates only

$$f[U(r), \{U(r_i)\}, \{r'_i\}] = r' = f(r, \{r_i\}, \{r'_i\}) \tag{44}$$

for any $U \in \Gamma$.

Proof :

Necessity : For condition i set $U = U'$ in equation (41). This is allowed since $U' \in \Gamma' \subset \Gamma$ implies that $U' \in \Gamma$ also. For condition ii set $U' = I$ (identity transformation) in equation (41).

Sufficiency : Apply in succession equation (43) for the transformation U' and equation (44) for the transformation $U \circ (U')^{-1}$. This is allowed since $U' \in \Gamma$ implies that $(U')^{-1} \in \Gamma$ and $U \circ (U')^{-1} \in \Gamma$ also. Indeed the first step leads to

$$f[U'(r), \{U'(r_i)\}, \{U'(r'_i)\}] = U'(r') \tag{45}$$

and the second one to

$$f[U \circ (U')^{-1} \circ U'(r), \{U \circ (U')^{-1} \circ U'(r_i)\}, \{U'(r'_i)\}] = U'(r') \tag{46}$$

which is equivalent to equation (41). □

Lemma 2.

Necessary and sufficient conditions for the interpolation

$$r' = f\,(r, \{r_i\}, \{r'_i\})$$

to be invariant under independent transformations $U \in \Gamma$ of old and $U' \in \Gamma'$ of new coordinates when $\Gamma \subset \Gamma'$, are

i. Invariance under common transformations

$$f\,[U(r), \{U(r_i)\}, \{U(r'_i)\}] = U(r') = U[f(r, \{r_i\}, \{r'_i\})] \tag{43*}$$

for any $U \in \Gamma$.

ii. Invariance under transformations of the new coordinates only

$$f\,[r, \{r_i\}, \{U'(r'_i)\}] = U'(r') \tag{47}$$

for any $U' \in \Gamma'$.

Proof :

Necessity : For condition i set $U' = U$ in equation (41). This is allowed since $U \in \Gamma \subset \Gamma'$ implies that $U \in \Gamma'$ also. For condition ii set $U = I$ in equation (41).

Sufficiency : Apply in succession equation (43*) for the transformation U and equation (47) for the transformation $U' o U^{-1}$. This is allowed since $U \in \Gamma'$ implies that $U^{-1} \in \Gamma'$ and $U' o U^{-1} \in \Gamma'$ also. The first step leads to equation (43*) and the second to

$$f\,[U(r), \{U(r_i)\}, \{U' o U^{-1} o U(r'_i)\}] = U' o U^{-1} o U(r') \tag{48}$$

which is equivalent to (41). □

The invariance of the interpolation under independent transformations of old and new coordinates as defined by equation (41), is a sufficient condition for the estimates of the strain parameters Δ , γ_1 , γ_2 , γ , e_{max} , e_{min} , φ to transform according to equations (27) through (34) and in particular for Δ , γ , e_{max} , e_{min} to be independent of frame definitions when only rigid transformations are involved.

When only one of the above strain parameters is required to follow the transformation rules of the continuous case (section 2), a weaker condition than equation (41) can be derived. However, there seems to be no need for such weaker conditions, since all strain parameters are of interest.

When all strain parameters Δ , γ_1 , γ_2 , γ , e_{max} , e_{min} φ are required to conform with the transformation rules derived in section 2 for the continuous case, equation (41) becomes not only a sufficient but also a necessary condition.

Given a particular interpolation, its appropriateness for the determination of strain parameters from discrete coordinates can be investigated by simply checking whether equation (41) is fulfilled, i.e., whether the interpolation is invariant under independent transformations of old and new coordinates. For this purpose, lemmata 1 and 2 are indispensable tools.

Recalling that estimability is equivalent to invariance under coordinate transformations which leave the observations invariant (Grafarend and Schaffrin, 1976), the groups Γ and Γ' of transformations U and U' to be used in equation (41) depend on the type of geodetic observations performed at the respective epochs t and t'. In pure triangulation networks where no distance observations are performed, the transformations leaving observations invariant are the similarity transformations. In trilateration or combined triangulation–trilateration networks where at least one distance is observed, scale is defined and the transformations leaving all observations invariant are the rigid transformations.

Therefore the transformation U to be used in equation (41) will be a rigid or a similarity transformation, depending on whether the observations performed at epoch t include distance observations or not respectively. The same holds for transformation U' and epoch t'.

In the previous discussion we have assumed that the two configurations of the network at the two epochs are "unconnected," in the sense that one configuration can move with respect to the other, in a rigid or similarity manner, with no detection of this motion by observational or any other means.

However, in many cases a number of network stations (at least two of them) are known, or believed, to remain fixed and this constraint is taken into account when the network is adjusted. In this case we say that the two network configurations at the two epochs are "connected." This intrinsic connection does not allow independent coordinate transformations at the two epochs and the interpolation should be checked for invariance under common coordinate transformations only, according to definition 2.

Connected network configurations result in practice when there is geophysical evidence that some stations remain practically motionless. A good example is the study of deformations of technical constructions with networks including fixed points outside the construction. Fixed stations may be also identified by statistical techniques; see, e.g., Koch and Fritsch, 1981.

A quite different type of connection results when the two originally unconnected configurations are tied together by means of a rigid or similarity transformation of one of them in a way that the sum of the squares of their coordinate differences is minimized; see, e.g., Brunner et al., 1981, Dermanis, 1981.

In the case of connected networks the interpolation must be checked for invariance under common coordinate transformations for both epochs. The trasnformations to be considered are the similarity transformations when none of the two epochs includes distance observations or the rigid transformations when distances are observed at least in one of the two epochs.

3.2. Linear interpolation of coordinates

More specific conditions for invariance of the interpolation under independent or common coordinate transformations can be obtained in the particular case where the interpolating function $f(r, \{r_i\}, \{r'_i\})$ is linear with respect to the data $\{r'_i\}$. This type of interpolation is the one most commonly used in practice. In our case it has the form

$$x' = \sum_{i=1}^{n} a_i(r, \{r_i\}) x'_i + \sum_{i=1}^{n} b_i(r, \{r_i\}) y'_i + g_x(r, \{r_i\}) \tag{49}$$

$$y' = \sum_{i=1}^{n} c_i(r, \{r_i\}) x'_i + \sum_{i=1}^{n} d_i(r, \{r_i\}) y'_i + g_y(r, \{r_i\}) \ . \tag{50}$$

Setting

$$\mathbf{g} = \mathbf{g}(\mathbf{r}, \{\mathbf{r}_i\}) = [g_x \ g_y]^T \tag{51}$$

and

$$\mathbf{A}_i = \mathbf{A}_i(\mathbf{r}, \{\mathbf{r}_i\}) = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \tag{52}$$

equations (49) and (50) can be written in the matrix form

$$\mathbf{r}' = \sum_{i=1}^{n} \mathbf{A}_i \mathbf{r}'_i + \mathbf{g} \ . \tag{53}$$

For this linear interpolation the condition (41) for invariance under independent coordinate transformations U and U′ takes the form

$$\begin{aligned} &\sum_{i=1}^{n} \mathbf{A}_i [U(\mathbf{r}), \{U(\mathbf{r}_i)\}]\, U'(\mathbf{r}'_i) + \mathbf{g}[U(\mathbf{r}), \{U(\mathbf{r}_i)\}] = \\ &\quad = U'(\mathbf{r}') = U'\left(\sum_{i=1}^{n} \mathbf{A}_i \mathbf{r}'_i + \mathbf{g}\right) \end{aligned} \tag{54}$$

Setting

$$\tilde{\mathbf{r}}' = U'(\mathbf{r}') \tag{55}$$

$$\tilde{\mathbf{r}}'_i = U'(\mathbf{r}'_i) \tag{56}$$

$$\tilde{\mathbf{A}}_i = \begin{bmatrix} \tilde{a}_i & \tilde{b}_i \\ \tilde{c}_i & \tilde{d}_i \end{bmatrix} = \mathbf{A}_i(\tilde{\mathbf{r}}, \{\tilde{\mathbf{r}}_i\}) \tag{57}$$

$$\tilde{\mathbf{g}} = \mathbf{g}(\tilde{\mathbf{r}}, \{\tilde{\mathbf{r}}_i\}) \tag{58}$$

condition (54) can be simply written

$$\tilde{r}' = \sum_{i=1}^{n} \tilde{A}_i \tilde{r}_i' + \tilde{g} \tag{59}$$

For invariance under common transformations, condition (42) takes the form

$$\begin{aligned} \sum_{i=1}^{n} A_i [U(r), \{U(r_i)\}] \, U(r_i') + g [U(r), \{U(r_i)\}] = \\ = U(r') = U(\sum_{i=1}^{n} A_i r_i' + g) \end{aligned} \tag{60}$$

or the same form of equation (59) if U' is replaced by U in equations (55) and (56).

In order to study the invariance properties of linear interpolation, U and U' are explicitly given by equations (13) and (14) respectively, when they are similarity transformations. In case they are rigid transformations, the same equations are used with $\lambda = 1$ and $\lambda' = 1$.

Lemma 3.

For the interpolation of equations (49) and (50) to be invariant under independent coordinate transformations, both similarity or both rigid, or one of them similarity and the other rigid, the following conditions are necessary and sufficient :

$$g_x = g_y = 0 \tag{61}$$

$$c_i = -b_i \tag{62}$$

$$d_i = a_i \tag{63}$$

$$\sum_{i=1}^{n} a_i = 1 \tag{64}$$

$$\sum_{i=1}^{n} b_i = 0 \tag{65}$$

$$\tilde{a}_i = a_i \tag{66}$$

$$\tilde{b}_i = b_i \tag{67}$$

Proof :

Lemma 3 can be proved on the basis of either the general condition (54), or lemma 1 when $\Gamma' \subset \Gamma$, or lemma 2 when $\Gamma \subset \Gamma'$. Independent investigations for similarity or rigid transformations, or combinations of one similarity and one rigid transformation, lead to exactly the same results. We shall prove here only the case of rigid transformations utilizing lemma 2. The proofs for similarity transformations or combinations of one similarity and one rigid transformation are completely analogous.

Invariance under $U = U'$:

The interpolation of transformed coordinates gives

$$\tilde{\mathbf{r}}' = \sum_i \tilde{\mathbf{A}}_i (\mathbf{R}\, \mathbf{r}_i' + \mathbf{b}) + \tilde{\mathbf{g}} = \sum_i (\tilde{\mathbf{A}}_i \mathbf{R})\, \mathbf{r}_i + (\sum_i \tilde{\mathbf{A}}_i)\mathbf{b} + \tilde{\mathbf{g}} \ . \tag{68}$$

The transformation of interpolated coordinates gives

$$\tilde{\mathbf{r}}' = \mathbf{R}\,(\sum_i \mathbf{A}_i \mathbf{r}_i' + \mathbf{g}) + \mathbf{b} = \sum_i (\mathbf{R}\,\mathbf{A}_i)\,\mathbf{r}_i' + \mathbf{R}\,\mathbf{g} + \mathbf{b} \tag{69}$$

Equating the right–hand sides of (68) and (69) we came up with a Diophantine equation with respect to the arbitrary $\mathbf{r}_i'$. Comparing all coefficients of the $\mathbf{r}_i'$ terms we obtain

$$\tilde{\mathbf{A}}_i = \mathbf{R}\,\mathbf{A}_i\,\mathbf{R}^T \ . \tag{70}$$

Equating the constant terms we obtain

$$(\sum_i \tilde{\mathbf{A}}_i)\, \mathbf{b} + \tilde{\mathbf{g}} = \mathbf{R}\,\mathbf{g} + \mathbf{b} \tag{71}$$

Invariance under $U = I$ and $U' \neq I$:

Taking into account that in this case $\tilde{\mathbf{A}}_i = \mathbf{A}_i$ and $\tilde{\mathbf{g}} = \mathbf{g}$, the interpolation of transformed coordinates gives

$$\tilde{\mathbf{r}}' = \sum_i \tilde{\mathbf{A}}_i (\mathbf{R}'\, \mathbf{r}_i' + \mathbf{b}') + \tilde{\mathbf{g}} = \sum_i (\mathbf{A}_i \mathbf{R}')\, \mathbf{r}_i' + (\sum_i \mathbf{A}_i)\,\mathbf{b}' + \mathbf{g} \ . \tag{72}$$

The transformation of interpolated coordinates gives

$$\tilde{\mathbf{r}}' = \mathbf{R}'\,(\sum_i \mathbf{A}_i \mathbf{r}_i' + \mathbf{g}) + \mathbf{b}' = \sum_i (\mathbf{R}'\,\mathbf{A}_i)\,\mathbf{r}_i' + \mathbf{R}'\,\mathbf{g} + \mathbf{b}' \ . \tag{73}$$

Comparison of the right–hand sides of (72) and (73) gives

$$\mathbf{A}_i = \mathbf{R}'\,\mathbf{A}_i\,(\mathbf{R}')^T \tag{74}$$

and

$$(\sum_i \mathbf{A}_i)\, \mathbf{b}' + \mathbf{g} = \mathbf{R}'\,\mathbf{g} + \mathbf{b}' \ . \tag{75}$$

Since both $\mathbf{R}'$ and $\mathbf{b}'$ are arbitrary, equation (75) holds if and only if

$$\sum_i \mathbf{A}_i = \mathbf{I} \tag{76}$$

and

$$\tilde{\mathbf{g}} = \mathbf{g} = 0 \tag{77}$$

since when $\mathbf{g}$ vanishes, $\tilde{\mathbf{g}}$ also vanishes.

The next step is to combine conditions (70), (71), (74), (76) and (77). Since (74) holds for any $\mathbf{R}'$ it also holds for $\mathbf{R}' = \mathbf{R}$, in which case comparison with (70) gives

$$\widetilde{\mathbf{A}}_i = \mathbf{A}_i \ . \tag{78}$$

Combination of (78) with (70) gives

$$\mathbf{A}_i = \mathbf{R}\,\mathbf{A}_i\,\mathbf{R}^T \ . \tag{79}$$

Equations (76), (77), (78) and (79) are the necessary and sufficient conditions, expressed in matrix form. Necessity has already been proved. For sufficiency it is easy to first show that when these conditions are satisfied equations (68) and (69) give identical results, next show that the same holds for equations (72) and (73) and finally apply lemma 2. Equation (77) is equivalent to equation (61). Equation (79) holds for any rotation matrix $\mathbf{R}$ and in particular for the matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{80}$$

of rotation with an angle of $\pi/2$. Therefore

$$\mathbf{A}_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} = \mathbf{P}\,\mathbf{A}_i\,\mathbf{P}^T = \begin{bmatrix} d_i & -c_i \\ -b_i & a_i \end{bmatrix} \tag{81}$$

which is equivalent to equations (62) and (63). Taking this into account equation (76) is equivalent to equations (64), (65) and equation (78) is equivalent to equations (66) and (67).

This completes the proof for the case of rigid transformations. The proof for similarity transformations is completely analogous. □

Lemma 3 may be more clearly expressed as follows :

Given the coordinates $(x_i\ , y_i)$ and $(x_i',\ y_i')$ of n points in a geodetic network at two different epochs t and t$'$, an interpolation $(x\ ,y\ ) \rightarrow (x',\ y')$ providing new coordinates (x',y') for any other point in the area identified by its old coordinates $(x\ ,y)$, which is linear with respect to the (x_i', y_i') coordinates and is furthermore independent of the choice of coordinate frames at both epochs t and t$'$, must be of the form

$$x' = \sum_{i=1}^{n} (a_i x_i' + b_i y_i') \tag{82}$$

$$y' = \sum_{i=1}^{n} (-b_i x_i' + a_i y_i') \ . \tag{83}$$

The coefficients a_i, b_i are functions of (x, y) and (x_i, y_i), $i = 1, 2, \ldots, n$, and must remain invariant

$$a_i\,[x, y, \{(x_i, y_i)\}] = a_i\,[\tilde{x}, \tilde{y}, \{(\tilde{x}_i, \tilde{y}_i)\}]$$

according to condition (78) and definition (57), where $(\tilde{x}, \tilde{y})$ and $(\tilde{x}_i, \tilde{y}_i)$ are the coordinates resulting from a rigid or similarity transformation of the original coordinates (x, y) and (x_i, y_i) respectively.

When the network configurations at the two epochs are connected, in the sense explained above, only common transformations should be allowed and the following two lemmata hold instead of lemma 3.

Lemma 4.

For the linear interpolation of equation (53) to be invariant under common rigid transformations

$$\tilde{\mathbf{r}} = \mathbf{R}\,\mathbf{r} + \mathbf{b} \tag{84}$$

$$\tilde{\mathbf{r}}' = \mathbf{R}\,\mathbf{r}' + \mathbf{b} \tag{85}$$

tne following conditions are necessary and sufficient :

$$\tilde{\mathbf{A}}_i = \mathbf{R}\,\mathbf{A}_i\,\mathbf{R}^T \tag{86}$$

$$\tilde{\mathbf{g}} = \mathbf{R}\mathbf{g} + \mathbf{R}(\mathbf{I} - \sum_i \mathbf{A}_i)\,\mathbf{R}^T\,\mathbf{b} \quad . \tag{87}$$

Proof :

The interpolation of transformed coordinates gives

$$\tilde{\mathbf{r}}' = \sum_i \tilde{\mathbf{A}}_i(\mathbf{R}\,\mathbf{r}_i' + \mathbf{b}) + \tilde{\mathbf{g}} = \sum_i (\tilde{\mathbf{A}}_i\,\mathbf{R})\mathbf{r}_i' + (\sum_i \tilde{\mathbf{A}}_i)\,\mathbf{b} + \tilde{\mathbf{g}} \quad . \tag{88}$$

The transformation of interpolated coordinates gives

$$\tilde{\mathbf{r}}' = \mathbf{R}\,(\sum_i \mathbf{A}_i\,\mathbf{r}_i' + \mathbf{g}) + \mathbf{b} = \sum_i (\mathbf{R}\mathbf{A}_i)\,\mathbf{r}_i' + \mathbf{R}\,\mathbf{g} + \mathbf{b} \quad . \tag{89}$$

Equating the right–hand sides of equations (88) and (89) a Diophantine equation with respect to $\mathbf{r}_i'$ results. The coefficients of $\mathbf{r}_i'$ give directly equation (86). The remaining constant terms give

$$(\sum_i \tilde{\mathbf{A}}_i)\,\mathbf{b} + \tilde{\mathbf{g}} = \mathbf{R}\,\mathbf{g} + \mathbf{b}\,. \tag{90}$$

Taking (86) into account, equation (90) becomes equation (87). This completes the proof for necessity. For sufficiency simply use equation (86) and (87) to prove equation (60). □

Lemma 5.

For the linear interpolation of equation (53) to be invariant under common similarity transformations

$$\tilde{\mathbf{r}} = \lambda \mathbf{R} \mathbf{r} + \mathbf{b} \tag{91}$$

$$\tilde{\mathbf{r}}' = \lambda \mathbf{R} \mathbf{r}' + \mathbf{b} \tag{92}$$

the following conditions are necessary and sufficient :

$$\tilde{\mathbf{A}}_i = \mathbf{R} \mathbf{A}_i \mathbf{R}^T \tag{93}$$

$$\tilde{\mathbf{g}} = \lambda \mathbf{R} \mathbf{g} + \mathbf{R} (\mathbf{I} - \sum_i \mathbf{A}_i) \mathbf{R}^T \mathbf{b} \ . \tag{94}$$

Proof :

The proof is completely analogous to the proof of lemma 4. □

When interpolations where a priori $\mathbf{g} = \mathbf{0}$ are checked for invariance under common transformations the following lemma applies.

Lemma 6.

For an interpolation of the form

$$x' = \sum_i (a_i x_i' + b_i y_i') \tag{95}$$

$$y' = \sum_i (c_i x_i' + d_i y_i') \tag{96}$$

to be invariant under common similarity or rigid transformations, the following conditions are sufficient and necessary :

$$\sum_i a_i = \sum_i d_i = 1 \tag{97}$$

$$\sum_i b_i = \sum_i c_i = 0 \tag{98}$$

$$\tilde{a}_i = \cos^2 \theta \ a_i + \cos \theta \sin \theta \ (b_i + c_i) + \sin^2 \theta \ d_i \tag{99}$$

$$\tilde{b}_i = \cos^2 \theta \ b_i + \cos \theta \sin \theta \ (d_i - a_i) - \sin^2 \theta \ c_i \tag{100}$$

$$\tilde{c}_i = \cos^2 \theta \ c_i + \cos \theta \sin \theta \ (d_i - a_i) - \sin^2 \theta \ b_i \tag{101}$$

$$\tilde{d}_i = \cos^2 \theta \ d_i - \cos \theta \sin \theta \ (b_i + c_i) + \sin^2 \theta \ a_i \tag{102}$$

Proof :

Comparison of the interpolation of transformed coordinates with the transformation of interpolated ones results in equations (93) and in

$$(\sum_i \tilde{\mathbf{A}}_i)\,\mathbf{b} = \mathbf{b}\ . \tag{103}$$

Taking (93) into account this becomes

$$\mathbf{R}\,(\sum_i \mathbf{A}_i)\,\mathbf{R}^T\,\mathbf{b} = \mathbf{b}\ . \tag{104}$$

Since b is arbitrary, the above equation holds only when

$$\sum_i \mathbf{A}_i = \mathbf{I}\ . \tag{105}$$

Equation (105) is equivalent to equations (97) and (98). Equation (93) is equivalent to equations (99), (100), (101) and (102). □

3.3. Linear interpolation of displacements

In most cases, one interpolates displacements, i.e., coordinate differences, rather than coordinates. Setting

$$\mathbf{p} = \begin{bmatrix} u \\ \\ v \end{bmatrix} = \begin{bmatrix} x' - x \\ \\ y' - y \end{bmatrix} = \mathbf{r}' - \mathbf{r} \tag{106}$$

for the displacement between the two epochs t and t′, the linear interpolation of displacements has the general form

$$\mathbf{p} = \sum_i \mathbf{A}_i\,\mathbf{p}_i + \mathbf{h} \tag{107}$$

Recalling (106) and setting similarily

$$\mathbf{p}_i = \mathbf{r}'_i - \mathbf{r}_i = \begin{bmatrix} x'_i - x_i \\ \\ y'_i - y_i \end{bmatrix} = \begin{bmatrix} u_i \\ \\ v_i \end{bmatrix} \tag{108}$$

equation (107) takes the form

$$\mathbf{r}' = \sum_i \mathbf{A}_i\,\mathbf{r}'_i + \mathbf{h} - \sum_i \mathbf{A}_i\,\mathbf{r}_i + \mathbf{r}\ . \tag{109}$$

Consequently, the interpolation of displacements is equivalent to an interpolation of coordinates

$$\mathbf{r}' = \sum_i \mathbf{A}_i\,\mathbf{r}'_i + \mathbf{g} \tag{110}$$

where in particular $\mathbf{g}$ has the form

$$\mathbf{g} = \mathbf{h} - \sum_i \mathbf{A}_i \mathbf{r}_i + \mathbf{r} \quad . \tag{111}$$

All the results of the previous section can now be applied to the interpolation of displacements by simply taking equation (111) into account.

Lemma 7.

For a linear interpolation of displacements to be invariant under independent coordinate transformations, both similarity or both rigid, or one similarity and the other rigid, it is necessary and sufficient that the interpolation has the form

$$u = \sum_i (a_i u_i + b_i v_i) + \sum_i (a_i x_i + b_i y_i) - x \tag{112}$$

$$v = \sum_i (-b_i u_i + a_i v_i) + \sum_i (-b_i x_i + a_i y_i) - y \tag{113}$$

and furthermore the following conditions hold

$$\sum_i a_i = 1 \tag{114}$$

$$\sum_i b_i = 0 \tag{115}$$

$$\tilde{a}_i = a_i \tag{116}$$

$$\tilde{b}_i = b_i \tag{117}$$

Proof :

The conditions of lemma 3 apply also in this case. In particular equation (61) in combination with equation (111) gives

$$\mathbf{g} = \mathbf{h} - \sum_i \mathbf{A}_i \mathbf{r}_i + \mathbf{r} = \mathbf{0} \quad . \tag{118}$$

From equation (118) the two components h_x and h_y of $\mathbf{h}$ are computed and replaced in equation (107) to finally give equations (112) and (113). □

Lemma 8.

For the linear interpolation of displacements of equation (107) to be invariant under common rigid transformations, the following conditions are necessary and sufficient

$$\tilde{\mathbf{A}}_i = \mathbf{R}\, \mathbf{A}_i\, \mathbf{R}^T \tag{119}$$

$$\tilde{\mathbf{h}} = \mathbf{R}\, \mathbf{h} \quad . \tag{120}$$

Proof :

According to lemma 4, equations (86) and (87) apply in this case too. Equation (86) is identical with equation (119). Taking into account equation (111) and its transformed version

$$\tilde{\mathbf{g}} = \tilde{\mathbf{h}} - \sum_i \tilde{\mathbf{A}}_i \tilde{\mathbf{r}}_i + \tilde{\mathbf{r}} \tag{121}$$

in equation (87) we have

$$\tilde{\mathbf{h}} - \sum_i \tilde{\mathbf{A}}_i \tilde{\mathbf{r}}_i + \tilde{\mathbf{r}} = \mathbf{R}\,(\mathbf{h} - \sum_i \mathbf{A}_i \mathbf{r}_i + \mathbf{r}) + \mathbf{R}\,(\mathbf{I} - \sum_i \mathbf{A}_i)\,\mathbf{R}^T \mathbf{b} \tag{122}$$

Replacing $\tilde{\mathbf{r}}$, $\tilde{\mathbf{r}}_i$ from equation (84) and $\tilde{\mathbf{A}}_i$ from equation (119), equation (122) becomes equation (120). □

Lemma 9.

For the linear interpolation of displacements of equation (107) to be invariant under common similarity transformations the following conditions are necessary and sufficient

$$\tilde{\mathbf{A}}_i = \mathbf{R}\,\mathbf{A}_i\,\mathbf{R}^T \tag{123}$$

$$\tilde{\mathbf{h}} = \lambda\,\mathbf{R}\,\mathbf{h} \ . \tag{124}$$

Proof :

Completely analogous to the proof of lemma 8, utilizing lemma 5 instead of lemma 4. □

In the particular situation where all given displacements $\mathbf{p}_i$ vanish at all network points, one would naturally like to obtain zero displacements $\mathbf{p}$ at any other point. Setting $\mathbf{p}_i = \mathbf{0}$ and $\mathbf{p} = \mathbf{0}$ in equation (107) it follows that this is possible only when $\mathbf{h} = \mathbf{0}$.

For the linear interpolation of displacements where a priori $\mathbf{h} = \mathbf{0}$, the following two lemmata hold.

Lemma 10.

For a linear interpolation of displacements of the form

$$u = \sum_i (a_i u_i + b_i v_i) \tag{125}$$

$$v = \sum_i (c_i u_i + d_i v_i) \tag{126}$$

to be invariant under independent transformations, both similarity or both rigid, or one similarity and the other rigid, the following conditions are sufficient and necessary

$$c_i = -b_i \tag{127}$$

$$d_i = a_i \quad (128)$$

$$\sum_i a_i = 1 \quad (129)$$

$$\sum_i b_i = 0 \quad (130)$$

$$\tilde{a}_i = a_i \quad (131)$$

$$\tilde{b}_i = b_i \quad (132)$$

$$\sum_i (a_i x_i + b_i y_i) = x \quad (133)$$

$$\sum_i (-b_i x_i + a_i y_i) = y \ . \quad (134)$$

Proof :

The proof follows the proof of lemma 7. The conditions of lemma 3 apply also here and thus equations (127) through (132) are identical with equations (62) through (67) respectively. In particular equation (61) in combination with equation (111) and taking into account that in this case $\mathbf{h} = \mathbf{0}$, gives

$$\mathbf{g} = -\sum_i \mathbf{A}_i \mathbf{r}_i + \mathbf{r} = \mathbf{0} \ . \quad (135)$$

Equation (135) in combination with equations (127) and (128) leads to equations (133) and (134). □

Lemma 11.

For an interpolation of displacements of the form of equations (125) and (126) to be invariant under common similarity or rigid transformations, the conditions of equations (99), (100), (101) and (102) are sufficient and necessary.

Proof :

When $\mathbf{h} = \mathbf{0}$, then $\tilde{\mathbf{h}} = \mathbf{0}$ and condition (120) of lemma 8, as well as, condition (124) of lemma 9, are trivially satisfied. The remaining condition (119), which is the same as condition (123), is equivalent to conditions (99) through (102). □

Interpolation usually means "exact interpolation" in the sense that the interpolation equations (53) or (107) give back the known coordinates $\mathbf{r}_i'$ or displacements $\mathbf{p}_i$ respectively, at the network points $\mathbf{r}_i$. However, no such restriction has been imposed in the previous discussion. All results hold for the most general case of nonexact interpolation (smoothing interpolation).

The conditions for the interpolation of displacements to be exact follow by setting $\mathbf{r} = \mathbf{r}_k$ for all network points in equation (107)

$$\mathbf{p}_k = \sum_i \mathbf{A}_i (\mathbf{r}_k, \{\mathbf{r}_j\}) \mathbf{p}_i + \mathbf{h}(\mathbf{r}_k) \ . \quad (136)$$

Since the displacements $\mathbf{p}_i$ are independent, equation (136) is satisfied only when

$$\mathbf{A}_i(\mathbf{r}_k, \{\mathbf{r}_j\}) = \delta_{ik}\,\mathbf{I} \tag{137}$$

where δ_{ik} is the Kronecker delta and

$$\mathbf{h}(\mathbf{r}_k) = \mathbf{0} \tag{138}$$

for all networks points k.

Completely analogous results hold for the exact interpolation of coordinates according to equation (53). In addition to equation (137) it must also hold

$$\mathbf{g}(\mathbf{r}_k) = \mathbf{0} \tag{139}$$

for all network points k.

Equation (111) establishes a relation between an interpolation of displacements and its equivalent interpolation of coordinates. Condition (137) holds for both exact interpolation of displacements and exact interpolation of coordinates. Applying equation (137) at all network points equation (111) becomes for exact interpolation

$$\mathbf{g}(\mathbf{r}_k) = \mathbf{h}(\mathbf{r}_k) - \sum_i \delta_{ik}\,\mathbf{r}_i + \mathbf{r}_k = \mathbf{h}(\mathbf{r}_k) \ . \tag{111*}$$

Each one of equations (138) and (139) implies the other in view of equation (111*). Therefore, when an interpolation of coordinates is exact, its equivalent interpolation of coordinates is also exact and vice versa.

In the case where $\mathbf{h} = \mathbf{0}$, the conditions (129) and (130) have the following significance. When the given set of displacements is the same $\mathbf{p}_i = \mathbf{p}_0$ at all network points, equations (129) and (130) assert that the same constant vector $\mathbf{p}_0$ is interpolated at any other point.

The results obtained in this section and the previous one are summarized in *Table 1.*

4. Applications

In the previous sections the necessary tools have been developed for the study of the invariance characteristics of any particular interpolation method used for the estimation of crustal deformation parameters from the discrete results of plane triangulations and trilaterations. The use of these tools will be illustrated with two examples related to the most common interpolation methods used in practice.

4.1. Piecewise linear interpolation with triangular elements

In this widely used technique, (Rikitake, 1976, Livieratos and Vlachos, 1981, Dermanis, 1981), the area of interest is divided into triangular elements with the stations of the geodetic network as vertices. Within each triangular element a field of displacements linear with respect to the coordinates is sought

$$u = \lambda_1 x + \lambda_2 y + \lambda_3 \tag{140}$$

$$v = \mu_1 x + \mu_2 y + \mu_3 \tag{141}$$

Table 1

Summary of results

A. *Interpolation of coordinates :* $\mathbf{r} = \sum_i \mathbf{A}_i \mathbf{r}_i + \mathbf{g}$

Conditions for invariance under :

A1. Independent similarity or rigid transformations

$$\mathbf{A}_i = \begin{bmatrix} a_i & b_i \\ -b_i & a_i \end{bmatrix}, \quad \sum_i \mathbf{A}_i = \mathbf{I}, \quad \widetilde{\mathbf{A}}_i = \mathbf{A}_i, \quad \widetilde{\mathbf{g}} = \mathbf{g} = \mathbf{0} \qquad \text{(lemma 3)}$$

A2. Common rigid transformations

$$\widetilde{\mathbf{A}}_i = \mathbf{R}\,\mathbf{A}_i\,\mathbf{R}^T \qquad \widetilde{\mathbf{g}} = \mathbf{R}\,\mathbf{g} + \mathbf{R}\,(\mathbf{I} - \sum_i \mathbf{A}_i)\mathbf{R}^T\mathbf{b} \qquad \text{(lemma 4)}$$

A3. Common similarity transformations

$$\widetilde{\mathbf{A}}_i = \mathbf{R}\,\mathbf{A}_i\,\mathbf{R}^T \qquad \widetilde{\mathbf{g}} = \lambda\,\mathbf{R}\,\mathbf{g} + \mathbf{R}\,(\mathbf{I} - \sum_i \mathbf{A}_i)\,\mathbf{R}^T\mathbf{b} \qquad \text{(lemma 5)}$$

A4. Common similarity or rigid transformations when a priori $\mathbf{g} = \mathbf{0}$.

$$\sum_i \mathbf{A}_i = \mathbf{I}, \qquad \widetilde{\mathbf{A}}_i = \mathbf{R}\,\mathbf{A}_i\,\mathbf{R}^T \qquad \text{(lemma 6)}$$

B. *Interpolation of displacements :* $\mathbf{p} = \sum_i \mathbf{A}_i \mathbf{p}_i + \mathbf{h}$

Conditions for invariance under :

B1. Independent similarity or rigid transformations

$$\mathbf{A}_i = \begin{bmatrix} a_i & b_i \\ -b_i & a_i \end{bmatrix}, \quad \sum_i \mathbf{A}_i = \mathbf{I}, \quad \widetilde{\mathbf{A}}_i = \mathbf{A}_i \quad \mathbf{h} = \sum_i \mathbf{A}_i \mathbf{r}_i - \mathbf{r} \qquad \text{(lemma 7)}$$

B2. Independent similarity or rigid transformations when a priori $\mathbf{h} = \mathbf{0}$

$$\mathbf{A}_i = \begin{bmatrix} a_i & b_i \\ -b_i & a_i \end{bmatrix}, \quad \sum_i \mathbf{A}_i = \mathbf{I}, \quad \widetilde{\mathbf{A}}_i = \mathbf{A}_i, \quad \mathbf{r} = \sum_i \mathbf{A}_i \mathbf{r}_i \qquad \text{(lemma 10)}$$

B3. Common rigid transformations

$$\widetilde{\mathbf{A}}_i = \mathbf{R}\,\mathbf{A}_i\,\mathbf{R}^T \qquad \widetilde{\mathbf{h}} = \mathbf{R}\,\mathbf{h} \qquad \text{(lemma 8)}$$

B4. Common similarity transformations

$$\widetilde{\mathbf{A}}_i = \mathbf{R}\,\mathbf{A}_i\,\mathbf{R}^T \qquad \widetilde{\mathbf{h}} = \lambda\,\mathbf{R}\,\mathbf{h} \qquad \text{(lemma 9)}$$

B5. Common similarity or rigid transformations when a priori $\mathbf{h} = \mathbf{0}$

$$\widetilde{\mathbf{A}}_i = \mathbf{R}\,\mathbf{A}_i\,\mathbf{R}^T \qquad \text{(lemma 11)}$$

with the requirement that the given displacements (u_i, v_i) result at the triangle vertices (x_i, y_i), $i = 1, 2, 3$. Introducing the 3×1 matrices $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$, $\mathbf{v}$, λ, μ, δ with elements $x_i, y_i, u_i, v_i, \lambda_i, \mu_i, \delta_i = 1$ respectively, the last requirement is written

$$\mathbf{u} = [\mathbf{x}\ \mathbf{y}\ \delta]\ \lambda \tag{142}$$

$$\mathbf{v} = [\mathbf{x}\ \mathbf{y}\ \delta]\ \mu \tag{143}$$

and the interpolation takes the form

$$u = [x\ y\ 1]\ \lambda = [x\ y\ 1]\ [\mathbf{x}\ \mathbf{y}\ \delta]^{-1}\ \mathbf{u} = \mathbf{a}^T\ \mathbf{u} = \sum_i a_i\ u_i \tag{144}$$

$$v = [x\ y\ 1]\ \mu = [x\ y\ 1]\ [\mathbf{x}\ \mathbf{y}\ \delta]^{-1}\ \mathbf{v} = \mathbf{a}^T\mathbf{v} = \sum_i a_i\ v_i \tag{145}$$

where

$$\mathbf{a}^T = [a_1\ a_2\ a_3] = [x\ y\ 1]\ [\mathbf{x}\ \mathbf{y}\ \delta]^{-1}\ . \tag{146}$$

This interpolation of displacements is of the form of equations (125), (126) with equations (127), (128), (130), (132) immediately satisfied since $b_i = 0$ in this case. We shall show that equations (129), (131), (133), (134) also hold and therefore, according to lemma 10, this interpolation is invariant under independent similarity or rigid transformations. From equation (13) it is easily verified that

$$[\tilde{x}\ \tilde{y}\ 1] = [x\ y\ 1] \begin{bmatrix} \lambda\ \mathbf{R}^T & \mathbf{0} \\ & \\ \mathbf{b}^T & 1 \end{bmatrix} = [x\ y\ 1]\ \mathbf{G} \tag{147}$$

and accordingly

$$[\tilde{\mathbf{x}}\ \tilde{\mathbf{y}}\ \delta] = [\mathbf{x}\ \mathbf{y}\ \delta]\ \mathbf{G}\ . \tag{148}$$

From (146) we have

$$\tilde{\mathbf{a}}^T = [\tilde{x}\ \tilde{y}\ 1]\ [\tilde{\mathbf{x}}\ \tilde{\mathbf{y}}\ \delta]^{-1} = [x\ y\ 1]\ \mathbf{G}\left\{[\mathbf{x}\ \mathbf{y}\ \delta]\ \mathbf{G}\right\}^{-1} = \mathbf{a}^T \tag{149}$$

and therefore equation (131) holds. Equation (146) also gives

$$\mathbf{a}^T\ [\mathbf{x}\ \mathbf{y}\ \delta] = [\mathbf{a}^T\mathbf{x}\ \ \mathbf{a}^T\mathbf{y}\ \ \mathbf{a}^T\delta] = [x\ y\ 1] \tag{150}$$

or

$$\mathbf{a}^T\ \mathbf{x} = \sum_i a_i\ x_i = x \tag{151}$$

$$\mathbf{a}^T\ \mathbf{y} = \sum_i a_i\ y_i = y \tag{152}$$

$$\mathbf{a}^T \delta = \sum_i a_i = 1 \tag{153}$$

which are equivalent to equations (133), (134), (129) respectively.

As a consequence of the invariance under independent similarity or rigid transformations, the resulting crustal deformation parameters transform according to equations (27) through (34).

This result does not agree at first sight with the results of (Dermanis, 1981) (see his equations (23) through (26)). The reason for this disagreement is that here exact equations are used, while as already stated in the introduction, the transformation equations of (Dermanis, 1981) refer to the common practice of computing "infinitesimal" deformation parameters, ignoring second–order terms in displacement gradients.

Equations (140) and (141) can be rewritten

$$\mathbf{p} = \mathbf{r}' - \mathbf{r} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{bmatrix} \mathbf{r} + \begin{bmatrix} \lambda_3 \\ \mu_3 \end{bmatrix} \tag{154}$$

and consequently

$$\frac{\partial \mathbf{r}'}{\partial \mathbf{r}} = \mathbf{I} + \begin{bmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{bmatrix} . \tag{155}$$

Inserting equation (155) into equation (5) the strain tensor becomes

$$\mathbf{E} = \begin{bmatrix} \lambda_1 & \frac{1}{2}(\mu_1 + \lambda_2) \\ \frac{1}{2}(\mu_1 + \lambda_2) & \mu_1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \lambda_1^2 + \lambda_2^2 & \lambda_1 \mu_1 + \lambda_2 \mu_2 \\ \lambda_1 \mu_1 + \lambda_2 \mu_2 & \mu_1^2 + \mu_2^2 \end{bmatrix} . \tag{156}$$

The infinitesimal strain tensor in (Dermanis, 1981) (where the notation $\lambda_1 = e_{xx}$, $\lambda_2 = e_{xy}$, $\mu_1 = e_{yx}$, $\mu_2 = e_{yy}$ is used) is given by

$$\mathbf{E}_{inf} = \begin{bmatrix} \lambda_1 & \frac{1}{2}(\mu_1 + \lambda_2) \\ \frac{1}{2}(\mu_1 + \lambda_2) & \mu_1 \end{bmatrix} . \tag{157}$$

For the sake of completeness we give the formulae for the computation of the coefficients λ_i, μ_i .

$$\lambda_1 = \frac{1}{D} \left\{ u_1 (y_2 - y_3) + u_2 (y_3 - y_1) + u_3 (y_1 - y_2) \right\} \tag{158}$$

$$\lambda_2 = \frac{1}{D} \left\{ u_1 (x_3 - x_2) + u_2 (x_1 - x_3) + u_3 (x_2 - x_1) \right\} \tag{159}$$

$$\mu_1 = \frac{1}{D} \left\{ \nu_1 (y_2 - y_3) + \nu_2 (y_3 - y_1) + \nu_3 (y_1 - y_2) \right\} \tag{160}$$

$$\mu_2 = \frac{1}{D} \left\{ \nu_1 (x_3 - x_2) + \nu_2 (x_1 - x_3) + \nu_3 (x_2 - x_1) \right\} \tag{161}$$

where

$$D = x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2) \quad . \tag{162}$$

4.2. Linear minimum variance prediction of displacements

One way of interpolating displacements is by means of an algorithm which is an imitation of the well–known technique of predicting values of a stochastic process from known point values of a realization of the same process. This linear minimum variance prediction is also called mean square error linear prediction, or simply least squares prediction. Geodesists prefer the somewhat abused term collocation.

The stochastic assumptions behind this algorithm, which by the way are invariant under rotations, are

$$E\left\{ u(\mathbf{r}) \right\} = E\left\{ \nu(\mathbf{r}) \right\} = 0 \tag{163}$$

$$E\left\{ u(\mathbf{r})\, u(\mathbf{r}') \right\} = E\left\{ \nu(\mathbf{r})\, \nu(\mathbf{r}') \right\} = \sigma(\mathbf{r}, \mathbf{r}') \tag{164}$$

$$E\left\{ u(\mathbf{r})\, \nu(\mathbf{r}') \right\} = 0 \tag{165}$$

where $\sigma(\mathbf{r}, \mathbf{r}')$ is the covariance function. A usual further assumption is that the covariance function is homogeneous and isotropic. The interpolation algorithm is (Dermanis et al. 1981)

$$u(\mathbf{r}) = \mathbf{c}^T \mathbf{C}^{-1} \mathbf{u} \tag{166}$$

$$\nu(\mathbf{r}) = \mathbf{c}^T \mathbf{C}^{-1} \mathbf{v} \tag{167}$$

where the matrices **C** and **c** have elements

$$\mathbf{C}_{ij} = \sigma(\mathbf{r}_i, \mathbf{r}_j) \tag{168}$$

$$\mathbf{c}_i = \sigma(\mathbf{r}, \mathbf{r}_i) \tag{169}$$

and **u** , **v** are $n \times 1$ matrices with elements the given displacements u_i , ν_i respectively.

This interpolation is of the type of equation (107) with

$$\mathbf{h} = \mathbf{0} \tag{170}$$

$$a_i = d_i \tag{171}$$

$$b_i = c_i = 0 \quad . \tag{172}$$

In fact

$$\mathbf{a} = \mathbf{C}^{-1}\,\mathbf{c} \tag{173}$$

and assuming that the covariance function $\sigma(\mathbf{r}_i, \mathbf{r}_j)$ is homogeneous and isotropic, the coefficients a_i are invariant under rigid transformations

$$\tilde{a}_i = a_i \ . \tag{174}$$

Since $\mathbf{h} = \mathbf{0}$, lemma 10 and lemma 11 must be taken into account. For the invariance of this type of interpolation under independent rigid transformations, the following three conditions must also hold

$$\sum_i a_i = \mathbf{c}^T\,\mathbf{C}^{-1}\,\delta = 0 \tag{175}$$

$$x = \sum_i a_i x_i = \mathbf{c}^T\,\mathbf{C}^{-1}\,\mathbf{x} \tag{176}$$

$$y = \sum_i a_i y_i = \mathbf{c}^T\,\mathbf{C}^{-1}\,\mathbf{y} \tag{177}$$

according to lemma 10. Since the three conditions above are unlikely to hold for the usual choices of covariance functions, we can hope at most for invariance under common rigid transformations. Conditions (99) through (102) trivially hold in this case, in view of equations (171), (172) and (173). Therefore, the displacement interpolation of equations (166) and (167) is invariant under common rigid transformations of coordinates when the covariance function $\sigma(\mathbf{r}, \mathbf{r}')$ is invariant under rigid transformations. Such a covariance must be a function of the distance d between the points $\mathbf{r}$ and $\mathbf{r}'$.

Invariance under common similarity transformations is also possible if the covariance function is a function of the ratio d/D, where D is a linear combination of distances between network points, e.g., a type of mean distance or the distance for a selected pair of points. In this case equation (174) holds for similarity transformations also.

This type of interpolation or prediction should be used only for connected type network configurations, as explained in section 3.1. In particular the assumption of equation (163) requires that the trend has already been removed from both displacement components. For unconnected configurations this is achieved when the new coordinates are transformed from an original arbitrary frame, to another one, in a way that a best fit to the old coordinates is attained, by minimizing the sum of the displacement components at the network points.

5. Final remarks

In the approach taken here the coordinates of the stations at two epochs, or equivalently the coordinates at the first epoch and the displacements between the two epochs, are treated as independent unknown parameters in the adjustment of the relevant geodetic observations. As a consequence, displacements are first estimated in the adjustment and crustal deformation parameters are next obtained by means of an interpolation technique.

This approach, in its more general context, has been characterized as "parameter–like treatment of signals" in (Dermanis, 1979), the signals being the displacements at network points, and as "deterministic approach" in (Dermanis and Grafarend, 1981 ; Dermanis, 1984).

The results obtained must be further extended for two other approaches to the same problem, which are based on the obvious fact that displacements in neighbouring points are likely to be similar. In this case the independence assumption of the previous approach is somewhat superfluous and must be relaxed.

One approach in this direction is to introduce specific models for the displacements, either one model for the whole area, or piecewise different ones for parts of the area, especially when the area is separated by faults. This approach has been characterized as "model function approach" in (Dermanis and Grafarend, 1981 ; Dermanis, 1984) and has been proposed in the specific context of the study of crustal deformations by (Chrzanowski et al., 1983). The displacements at network points, as well as any other point, are expressed as functions of a set of unknown model parameters, generally less than the original network displacement components, which participate together with the coordinates of the first epoch as unknowns in the adjustment of the geodetic observations.

The other approach in the same direction is one that we might call the "stochastic approach." In this case the vector displacement function is treated as a vector–valued stochastic process with known mean and covariance function. Displacements at network points are thus correlated and this correlation is taken into account in the adjustment of the geodetic observations. This approach has been characterized as "observation–like treatment of signals" in (Dermanis, 1979) and is no other than the familiar least squares collocation. Such an approach is proposed for the study of crustal motions in (Hein and Kistermann, 1981 ; Bock, 1983).

Finally a hybrid approach is also possible where the vector displacement function is separated into a deterministic model–function part containing unknown parameters, the so–called trend in time series analysis, and a remaining stochastic part with zero mean and known covariance function.

The extension of the results of this work to the other possible approaches for the determination of crustal deformation parameters is by no means straightforward and deserves a study of its own.

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REFERENCES

W. BAARDA (1962) : "A generalization of the concept of strength of figure". Report of the Geodetic Institute, Delft.

W. BAARDA (1967) : "Statistical concepts in geodesy". Netherlands Geodetic Commission, Publ. on Geodesy, New Series, Vol. 2, No. 5, Delft.

W. BAARDA (1971) : "Criteria for precision of geodetic networks". Computing Center of the Delft Geodetic Institute.

W. BAARDA (1973) : "S–transformations and criterion matrices". Netherlands Geod. Comm., Publ. on Geodesy, New Series, Vol. 5, No. 2, Delft.

Y. BOCK (1983) : "Estimating crustal deformations from a combination of baseline measurements and geophysical models". Bull. Géod., No. 57, pp. 294–311.

J. BOSSLER (1973) : "A note on the meaning of generalized inverse solutions in Geodesy". J. of Geoph. Res., 78, pp. 2616–2617.

F.K. BRUNNER, R. COLEMAN and B. HIRSCH (1981) : "A comparison of computation methods for crustal strains from geodetic measurements". Tectonophysics, 71, pp. 281–298.

A. CHRZANOWSKI, Y.Q. CHEN and J.M. SECORD (1983) : "On the strain analysis of tectonic movements using fault crossing geodetic surveys". Tectonophysics, 97, pp. 297–315.

A. DERMANIS (1979) : "Adjustment of geodetic observations in the presence of signals". Boll. Geod. Sci. Aff., 38, 4, pp. 513–539.

A. DERMANIS (1981) : "Geodetic estimability of crustal deformation parameters". Quaterniones Geodaesiae, 2, 2, pp. 159–169.

A. DERMANIS and E. GRAFAREND (1981) : "Estimability analysis of geodetic, astrometric and geodynamical quantities in very long baseline interferometry". Geophys. J. R. astr. Soc., 64, pp. 31–56.

A. DERMANIS, E. LIVIERATOS, D. ROSSIKOPOULOS and D. VLACHOS (1981) : "Geodetic prediction of crustal deformations at the seismic area of Volvi". Int. Symp. on Geodetic Networks and Computations, Munich, 1981. In : Deutsche Geod. Komm ., Reihe B , Heft Nr. 258/V, pp. 234–248.

A. DERMANIS (1984) : "Optimization problems in geodetic networks with signals". Int. School of Geodesy, 3rd Course : "Optimization and design of geodetic networks", Springer Verlag (in press).

A. DERMANIS and E. LIVIERATOS (1983) : "Applications of deformation analysis in geodesy and geodynamics". Reviews of Geophysics and Space Physics, 21, pp. 41–50.

E. GRAFAREND and B. SCHAFFRIN (1974) : "Unbiased free net adjustment". Survey Review, 22, 171, pp. 200–218.

E. GRAFAREND and B. SCHAFFRIN (1976) : "Equivalence of estimable quantities and invariants in geodetic networks". Zeitschrift f. Vermessungswesen, 101, 11, pp. 485–491.

G. HEIN and R. KISTERMANN (1981) : "Mathematical foundation of non–tectonic effects in geodetic recent crustal movement models". Tectonophysics, 71, pp. 315–334.

K.R. KOCH (1980) : "Parameterschätzung und Hypothesentests in linearen Modellen". Dümmlers Verlag, Bonn.

K.R. KOCH and D. FRITSCH (1981) : "Multivariate hypothesis tests for detecting recent crustal movements". Tectonophysics, 71, pp. 301–313.

E. LIVIERATOS and D. VLACHOS (1981) : "The influence of correlations in computing crustal strains from trigonometric network results". Tectonophysics, 77, pp. 323–332.

U.A. RAUHALA (1980) : "General theory of linear estimators and matrix inverses". Geodetic Services, Inc., Technical Report, 80–015.

T. RIKITAKE (1976) : "Earthquake prediction". Elsevier Publ. Co., Amsterdam.

U.S. National Research Council (1981) : "Geodetic monitoring of tectonic deformation - - Toward a strategy". National Academy Press, Washington, D.C.

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