On the Feasibility of Biased Kriging

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The problem:
For a homogeneous constand-mean random field $u(P)$

$E\{u(P)\} = \mu$, $E\{[u(P)-\mu][u(Q)-\mu]\} = C(P,Q) = C(|PQ|)$

Given noisy observations $y_i = L_i(u)+v_i$, $i = 1,\ldots,n$,
L_i = linear functionals
Predict the value of another linear functional $L_p(u)$.

Usual case: $L_i(u) = u_i = u(P_i)$ & $L_p(u) = u(P)$

Observed: $\mathbf{y} = \mathbf{u} + \mathbf{v}$

Known:
$\boldsymbol{\mu} = E\{\mathbf{u}\} = \mu\mathbf{s}$, $\mathbf{s} = [\,1\ 1\ \ldots\ 1]^T$, $\mathbf{C} = E\{(\mathbf{u}-\boldsymbol{\mu})(\mathbf{u}-\boldsymbol{\mu})^T\}$,
$E\{\mathbf{v}\} = \mathbf{0}$, $\mathbf{C_v} = E\{\mathbf{v}\mathbf{v}^T\}$, $\mathbf{C_{uv}} = E\{(\mathbf{u}-\boldsymbol{\mu})\mathbf{v}^T\} = \mathbf{0}$
$E\{u(P)\} = \mu$, $\mathbf{c}_P = E\{(\mathbf{u}-\boldsymbol{\mu})(\underline{u}(P)-\mu)\}$

Required:
Linear prediction $\hat{u}(P) = \widehat{u(P)}$ of u(P)
with minimum Mean Square Prediction Error

$$m_P^2 = E\{\varepsilon^2\} = E\{[\hat{u}(P) - u(P)]^2\} = \min$$
$$\hat{m}_P^2 = \min m_P^2$$

Reduction to a finite-dimensional model: *Linear Random Effects Model*

In terms of the ***covariance function*** $C(P,Q) = E\{[u(P)-\mu][u(Q)-\mu]\}$ $\quad C_0 = C(P,P) = C(0)$

	Unbiased $E\{\hat{u}(P)\} = E\{u(P)\} = \mu$	Biased Bias $\beta = E\{\hat{u}(P)\} - E\{u(P)\}$
Linear inhomogeneous $\hat{u}(P) = \mathbf{d}^T\mathbf{y} + k$	*inhomBLUP* (collocation) $\hat{u}(P) = \mu + \mathbf{c}_P^T(\mathbf{C}+\mathbf{C_v})^{-1}(\mathbf{y}-\boldsymbol{\mu})$ $\beta = 0$ $\quad \hat{m}_{\hat{u}(P)}^2 = \sigma_\varepsilon^2 = C_0 - \mathbf{c}_P^T(\mathbf{C}+\mathbf{C_v})^{-1}\mathbf{c}_P$	*inhomBLIP* ← reduces to inhomBLUP
Linear homogeneous $\hat{u}(P) = \mathbf{d}^T\mathbf{y}$ (Shaffrin's robust predictors)	*homBLUP* $\hat{u}(P) = \alpha_1 + \mathbf{c}_P^T(\mathbf{C}+\mathbf{C_v})^{-1}(\mathbf{y}-\alpha_1\mathbf{s})$ $\alpha_1 = \dfrac{\mathbf{s}^T(\mathbf{C}+\mathbf{C_v})^{-1}\mathbf{y}}{\mathbf{s}^T(\mathbf{C}+\mathbf{C_v})^{-1}\mathbf{s}}$ $\beta = 0$ $\hat{m}_P^2 = C_0 - \mathbf{c}_P^T(\mathbf{C}+\mathbf{C_v})^{-1}\mathbf{c}_P + \dfrac{[1-\mathbf{c}_P^T(\mathbf{C}+\mathbf{C_v})^{-1}\mathbf{s}]^2}{\mathbf{s}^T(\mathbf{C}+\mathbf{C_v})^{-1}\mathbf{s}}$	*homBLIP* $\hat{u}(P) = \alpha_2\mu + \mathbf{c}_P^T(\mathbf{C}+\mathbf{C_v})^{-1}(\mathbf{y}-\alpha_2\mu\mathbf{s})$ $\alpha_2 = \dfrac{\mu\mathbf{s}^T(\mathbf{C}+\mathbf{C_v})^{-1}\mathbf{y}}{1+\mu^2\mathbf{s}^T(\mathbf{C}+\mathbf{C_v})^{-1}\mathbf{s}}$ $\beta = -\dfrac{\mu - \mathbf{c}_P^T(\mathbf{C}+\mathbf{C_v})^{-1}\boldsymbol{\mu}}{1+\boldsymbol{\mu}^T(\mathbf{C}+\mathbf{C_v})^{-1}\boldsymbol{\mu}}$ $\hat{m}_P^2 = C_0 - \mathbf{c}_P^T(\mathbf{C}+\mathbf{C_v})^{-1}\mathbf{c}_P + \mu^2\dfrac{[1-\mathbf{c}_P^T(\mathbf{C}+\mathbf{C_v})^{-1}\mathbf{s}]^2}{1+\mu^2\mathbf{s}^T(\mathbf{C}+\mathbf{C_v})^{-1}\mathbf{s}}$

In terms of the ***variogram*** $\gamma(P,Q) = \frac{1}{2}E\{[u(P)-u(Q)]^2\}$ $\quad \Gamma_{ik} = \gamma(P_i,P_k)$ $\quad (\boldsymbol{\gamma}_P)_i = \gamma(P,P_i)$

	Unbiased	Biased
Linear homogeneous $\hat{u}(P) = \mathbf{d}^T\mathbf{y}$	***(Ordinary) Kriging*** $\hat{u}(P) = \alpha_K + \boldsymbol{\gamma}_P^T(\boldsymbol{\Gamma}-\mathbf{C_v})^{-1}(\mathbf{y}-\alpha_K\mathbf{s})$ $\alpha_K = \dfrac{\mathbf{s}^T(\boldsymbol{\Gamma}-\mathbf{C_v})^{-1}\mathbf{y}}{\mathbf{s}^T(\boldsymbol{\Gamma}-\mathbf{C_v})^{-1}\mathbf{s}}$ $\beta = 0$ $\hat{m}_P^2 = H + \boldsymbol{\gamma}_P^T(\boldsymbol{\Gamma}-\mathbf{C_v})^{-1}\boldsymbol{\gamma}_P - \dfrac{[\boldsymbol{\gamma}_P^T(\boldsymbol{\Gamma}-\mathbf{C_v})^{-1}\mathbf{s}]^2}{\mathbf{s}^T(\boldsymbol{\Gamma}-\mathbf{C_v})^{-1}\mathbf{s}}$	***Biased Kriging*** **NEW !** $\hat{u}(P) = \alpha_B + \boldsymbol{\gamma}_P^T(\boldsymbol{\Gamma}-\mathbf{C}_v)^{-1}(\mathbf{y}-\alpha_B\mathbf{s})$ $\alpha_B = \dfrac{H\mathbf{s}^T(\boldsymbol{\Gamma}-\mathbf{C}_v)^{-1}\mathbf{y}}{H\mathbf{s}^T(\boldsymbol{\Gamma}-\mathbf{C}_v)^{-1}\mathbf{s}-1}$ $\beta = \mu\dfrac{1-\boldsymbol{\gamma}^T(\boldsymbol{\Gamma}-\mathbf{C}_v)^{-1}\mathbf{s}}{H\,\mathbf{s}^T(\boldsymbol{\Gamma}-\mathbf{C}_v)^{-1}\mathbf{s}-1}$ $\hat{m}_P^2 = \boldsymbol{\gamma}_P^T(\boldsymbol{\Gamma}-\mathbf{C}_v)^{-1}\boldsymbol{\gamma} - \dfrac{H(\mathbf{s}^T(\boldsymbol{\Gamma}-\mathbf{C}_v)^{-1}\boldsymbol{\gamma}-1)^2}{H\mathbf{s}^T(\boldsymbol{\Gamma}-\mathbf{C}_v)^{-1}\mathbf{s}-1}$

Unknown mean μ not present !

Unknown mean μ present only through estimable $H = C_0 + \mu^2$

Biased Kriging is feasible !