Appendix A: Derivation of the non-linear Baarda transformations

We shall derive here the solution to the non-linear Baarda transformations, which are coordinate transformations $\mathbf{x}=S(\mathbf{z})$ from any minimum constraints solution $\mathbf{z}$ to the $\mathbf{x}_0$-nearest element $\mathbf{x}$ of the f induced fiber $F_{f(z)}$. Here

$$\mathbf{x}=\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \vdots\\ \mathbf{x}_N\end{bmatrix}, \qquad \mathbf{x}_0=\begin{bmatrix}\mathbf{x}_{01}\\ \mathbf{x}_{02}\\ \vdots\\ \mathbf{x}_{0N}\end{bmatrix}, \tag{A1}$$

and in the most general case the three-dimensional transformation sought can be expressed point-wise by the ***similarity transformation***

$$\mathbf{x}_i=\lambda\,\mathbf{R}(\boldsymbol{\theta})\mathbf{z}_i+\mathbf{t}=\mathbf{x}_i(\mathbf{z}_i,\mathbf{p}), \qquad \mathbf{p}=\begin{bmatrix}\boldsymbol{\theta}\\ \mathbf{t}\\ \lambda\end{bmatrix} \tag{A2}$$

where the transformation parameters $\mathbf{p}$ consist of 3 rotational parameters $\boldsymbol{\theta}=[\theta_1\theta_2\theta_3]^T$ defining the orthogonal matrix $\mathbf{R}=\mathbf{R}(\boldsymbol{\theta})$, 3 parallel displacement parameters $\mathbf{t}=[t_1t_2t_3]^T$ and a scale parameter λ. The problem is to find the optimal values of $\boldsymbol{\theta}$, $\mathbf{t}$, λ which minimize $\phi=\phi(\mathbf{p})=(\mathbf{x}-\mathbf{x}_0)^T(\mathbf{x}-\mathbf{x}_0)$. We first note that for $\dot{\mathbf{R}}=\frac{\partial}{\partial\alpha}\mathbf{R}$,

$$\mathbf{R}\mathbf{R}^T=\mathbf{I} \quad\Rightarrow\quad \dot{\mathbf{R}}\mathbf{R}^T+\mathbf{R}\dot{\mathbf{R}}^T=\mathbf{0} \quad\Rightarrow\quad \dot{\mathbf{R}}\mathbf{R}^T=-\left(\dot{\mathbf{R}}\mathbf{R}^T\right)^T\equiv[\boldsymbol{\omega}\times] \quad\Rightarrow\quad \dot{\mathbf{R}}=[\boldsymbol{\omega}\times]\mathbf{R}$$

This can be applied for the derivatives with respect to θ_k by setting $\frac{\partial}{\partial\theta_k}\mathbf{R}=[\boldsymbol{\omega}_k\times]\mathbf{R}$.

$$\frac{\partial\mathbf{x}_i}{\partial\theta_k}=\lambda\frac{\partial}{\partial\theta_k}\mathbf{R}\,\mathbf{z}_i=\lambda\,[\boldsymbol{\omega}_k\times]\mathbf{R}\,\mathbf{z}_i=-\lambda\,[(\mathbf{R}\mathbf{z}_i)\times]\boldsymbol{\omega}_k=-\lambda\,\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\,\boldsymbol{\omega}_k \tag{A3}$$

$$\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}=\begin{bmatrix}\frac{\partial\mathbf{x}_i}{\partial\theta_1}\frac{\partial\mathbf{x}_i}{\partial\theta_2}\frac{\partial\mathbf{x}_i}{\partial\theta_3}\end{bmatrix}=-\lambda\,\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T[\boldsymbol{\omega}_1\boldsymbol{\omega}_2\boldsymbol{\omega}_3]=-\lambda\,\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\boldsymbol{\Omega} \tag{A4}$$

while

$$\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}}=\mathbf{I}, \qquad \frac{\partial\mathbf{x}_i}{\partial\lambda}=\mathbf{R}\mathbf{z}_i. \tag{A5}$$

To minimize ϕ we simply set

$$\frac{\partial\phi}{\partial\mathbf{p}}=\frac{\partial\phi}{\partial\mathbf{x}}\frac{\partial\mathbf{x}}{\partial\mathbf{p}}=2(\mathbf{x}-\mathbf{x}_0)^T\begin{bmatrix}\frac{\partial\mathbf{x}}{\partial\boldsymbol{\theta}}\frac{\partial\mathbf{x}}{\partial\mathbf{t}}\frac{\partial\mathbf{x}}{\partial\lambda}\end{bmatrix}=\mathbf{0} \tag{A6}$$

or explicitly

$$\sum_i(\mathbf{x}_i-\mathbf{x}_{0i})^T\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}=\sum_i(\lambda\,\mathbf{R}\mathbf{z}_i+\mathbf{t}-\mathbf{x}_{0i})^T(-\lambda\,\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\boldsymbol{\Omega})=\mathbf{0} \tag{A7}$$

$$\sum_i(\mathbf{x}_i-\mathbf{x}_{0i})^T\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}}=\sum_i(\lambda\,\mathbf{R}\mathbf{z}_i+\mathbf{t}-\mathbf{x}_{0i})^T\,\mathbf{I}=\mathbf{0} \tag{A8}$$

$$\sum_i (\mathbf{x}_i - \mathbf{x}_{0i})^T \frac{\partial \mathbf{x}_i}{\partial \lambda} = \sum_i (\lambda \, \mathbf{R}\mathbf{z}_i + \mathbf{t} - \mathbf{x}_{0i})^T (\mathbf{R}\mathbf{z}_i) = \mathbf{0} \tag{A9}$$

which assuming that $\lambda \neq 0$ and $|\mathbf{\Omega}| \neq 0$, can be rewritten as

$$\lambda \sum_i [\mathbf{z}_i \times]\mathbf{z}_i + \sum_i [\mathbf{z}_i \times]\mathbf{R}^T \mathbf{t} = \sum_i [\mathbf{z}_i \times]\mathbf{R}^T \mathbf{x}_{0i} \tag{A10}$$

$$\lambda \, \mathbf{R} \sum_i \mathbf{z}_i + \sum_i \mathbf{t} = \sum_i \mathbf{x}_{0i} \tag{A11}$$

$$\lambda \sum_i \mathbf{z}_i^T \mathbf{z}_i + \sum_i \mathbf{z}_i^T \mathbf{R}^T \mathbf{t} = \sum_i \mathbf{z}_i^T \mathbf{R}^T \mathbf{x}_{0i} \tag{A12}$$

Taking into account that $[\mathbf{z}_i \times]\mathbf{z}_i = \mathbf{0}$ and setting

$$\bar{\mathbf{z}} \equiv \frac{1}{N} \sum_i \mathbf{z}_i \,, \qquad \bar{\mathbf{x}}_0 \equiv \frac{1}{N} \sum_i \mathbf{x}_{0i} \tag{A13}$$

the above three equations take the form

$$N[\bar{\mathbf{z}} \times]\mathbf{R}^T \mathbf{t} = \sum_i [\mathbf{z}_i \times]\mathbf{R}^T \mathbf{x}_{0i} \tag{A14}$$

$$\lambda \, \mathbf{R}\bar{\mathbf{z}} + \mathbf{t} = \bar{\mathbf{x}}_0 \qquad \text{(A15)}$$

$$\lambda \sum_i \mathbf{z}_i^T \mathbf{z}_i + N \bar{\mathbf{z}}^T \mathbf{R}^T \mathbf{t} = \sum_i \mathbf{z}_i^T \mathbf{R}^T \mathbf{x}_{0i} \tag{A16}$$

Solving (A15) for $\mathbf{t}$

$$\mathbf{t} = \bar{\mathbf{x}}_0 - \lambda \, \mathbf{R}\bar{\mathbf{z}} \tag{A17}$$

and replacing in (A16) gives

$$\lambda = \frac{\sum_i \mathbf{z}_i^T \mathbf{R}^T (\mathbf{x}_{0i} - \bar{\mathbf{x}}_0)}{\sum_i \mathbf{z}_i^T \mathbf{z}_i - N \, \bar{\mathbf{z}}^T \bar{\mathbf{z}}} \tag{A18}$$

which can be also written in the form

$$\lambda = \frac{\sum_i (\mathbf{x}_{0i} - \bar{\mathbf{x}}_0)^T \mathbf{R}(\mathbf{z}_i - \bar{\mathbf{z}})}{\sum_i (\mathbf{z}_i - \bar{\mathbf{z}})^T (\mathbf{z}_i - \bar{\mathbf{z}})} \tag{A19}$$

To determine $\mathbf{R}$ (in fact $\boldsymbol{\theta}$) we replace $\mathbf{t}$ into (A14)

$[\bar{\mathbf{z}} \times]$

$$N[\bar{\mathbf{z}} \times]\mathbf{R}^T \bar{\mathbf{x}}_0 - \lambda \, N[\bar{\mathbf{z}} \times]\bar{\mathbf{z}} = \sum_i [\mathbf{z}_i \times]\mathbf{R}^T \mathbf{x}_{0i} \tag{A20}$$

which in view of $[\bar{\mathbf{z}} \times]\bar{\mathbf{z}} = \mathbf{0}$ can be written as

$$\frac{1}{N}\sum_i [\mathbf{z}_i \times]\mathbf{R}^T \mathbf{x}_{0i} - [\overline{\mathbf{z}}\times]\mathbf{R}^T \overline{\mathbf{x}}_0 = \frac{1}{N}\sum_i [\mathbf{x}_{0i}\times]\mathbf{R}^T \mathbf{z}_i - [\overline{\mathbf{x}}_0\times]\mathbf{R}^T \overline{\mathbf{z}} = \mathbf{0} \tag{A21}$$

or in the equivalent form

$$\frac{1}{N}\sum_i [(\mathbf{x}_{0i} - \overline{\mathbf{x}}_0)\times]\mathbf{R}(\mathbf{z}_i - \overline{\mathbf{z}}) = \mathbf{0}\,. \tag{A22}$$

Equation (A22) is a nonlinear equation which can be solved to obtain the values of the parameters $\boldsymbol{\theta}$ in any particular parameterization of the rotation matrix $\mathbf{R}(\boldsymbol{\theta})$. The obtained values should be next replaced in (A19) in order to determine the value of the scale parameter λ. The similarity transformation can be realized using these values, once $\mathbf{t}$ and λ are replaced from (A17) and (A19), respectively, into (A2) to obtain

$$\mathbf{x}_i = \overline{\mathbf{x}}_0 + \frac{\sum_i (\mathbf{x}_{0i} - \overline{\mathbf{x}}_0)^T \mathbf{R}(\boldsymbol{\theta})(\mathbf{z}_i - \overline{\mathbf{z}})}{\sum_i (\mathbf{z}_i - \overline{\mathbf{z}})^T (\mathbf{z}_i - \overline{\mathbf{z}})} \mathbf{R}(\boldsymbol{\theta})(\mathbf{z}_i - \overline{\mathbf{z}})\,. \tag{A23}$$

In the case of the ***rigid transformation*** $\mathbf{x}_i = \mathbf{R}(\boldsymbol{\theta})\mathbf{z}_i + \mathbf{t}$ we obtain only equations (A7), (A8) with $\lambda=1$ and following the same procedure as before we arrive at the solution

$$\frac{1}{N}\sum_i [(\mathbf{x}_{0i} - \overline{\mathbf{x}}_0)\times]\mathbf{R}(\mathbf{z}_i - \overline{\mathbf{z}}) = \mathbf{0} \tag{A24}$$

$$\mathbf{x}_i = \overline{\mathbf{x}}_0 + \mathbf{R}(\boldsymbol{\theta})(\mathbf{z}_i - \overline{\mathbf{z}})\,. \tag{A25}$$

The planar (two-dimensional) case is somewhat different since the 2×2 rotation matrix $\mathbf{R}(\theta)$ depends only on a single parameter θ. Setting

$$\mathbf{W} \equiv \frac{\partial}{\partial\theta}\mathbf{R}\,\mathbf{R}^T = \dot{\mathbf{R}}\mathbf{R}^T\,, \tag{A26}$$

where $\mathbf{W}$ is a 2×2 antisymmetric matrix, the only difference from the 3-dimensional similarity transformation case (apart from the fact that the vectors $\mathbf{x}_i$, $\mathbf{x}_{0i}$, $\mathbf{z}_i$, $\overline{\mathbf{x}}_0$, $\overline{\mathbf{z}}$ involved are now 2-dimensional) is that (A7) is replaced by

$$\sum_i (\mathbf{x}_i - \mathbf{x}_{0i})^T \frac{\partial \mathbf{x}_i}{\partial\theta} = \sum_i (\lambda\,\mathbf{R}\mathbf{z}_i + \mathbf{t} - \mathbf{x}_{0i})^T (\lambda\,\mathbf{W}\mathbf{R}\mathbf{z}_i) = \mathbf{0} \tag{A27}$$

and the final solution for the ***planar similarity transformation*** takes the form

$$\frac{1}{N}\sum_i (\mathbf{x}_{0i} - \overline{\mathbf{x}}_0)^T \mathbf{W}\mathbf{R}(\mathbf{z}_i - \overline{\mathbf{z}}) = \mathbf{0} \tag{A28}$$

$$\lambda = \frac{\sum_i (\mathbf{x}_{0i} - \overline{\mathbf{x}}_0)^T \mathbf{R}(\mathbf{z}_i - \overline{\mathbf{z}})}{\sum_i (\mathbf{z}_i - \overline{\mathbf{z}})^T (\mathbf{z}_i - \overline{\mathbf{z}})} \tag{A29}$$

$$\mathbf{x}_i = \overline{\mathbf{x}}_0 + \lambda\,\mathbf{R}(\theta)(\mathbf{z}_i - \overline{\mathbf{z}})\,. \tag{A30}$$

We can proceed further with the solution by choosing the usual parameterization

$$\mathbf{R}(\theta)=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \Rightarrow \quad \mathbf{W}=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{A31}$$

With this particular choice equation (A28) takes the form

$$a\cos\theta = b\sin\theta \tag{A32}$$

where

$$a \equiv \frac{1}{N}\sum_i (\mathbf{x}_{0i}-\bar{\mathbf{x}}_0)^T \mathbf{W}(\mathbf{z}_i-\bar{\mathbf{z}}), \qquad b \equiv \frac{1}{N}\sum_i (\mathbf{x}_{0i}-\bar{\mathbf{x}}_0)^T (\mathbf{z}_i-\bar{\mathbf{z}}). \tag{A33}$$

There are two solutions: $\cos\theta = \pm\frac{b}{\sqrt{a^2+b^2}}$, $\sin\theta = \pm\frac{a}{\sqrt{a^2+b^2}}$ (both positive or both negative). Replacing in (A29) we obtain

$$\lambda = \frac{b\cos\theta + a\sin\theta}{\sigma_z^2} = \pm\frac{\sqrt{a^2+b^2}}{\sigma_z^2} \tag{A34}$$

where

$$\sigma_z^2 \equiv \frac{1}{N}\sum_i (\mathbf{z}_i-\bar{\mathbf{z}})^T (\mathbf{z}_i-\bar{\mathbf{z}}). \tag{A35}$$

With the obtained values of $\sin\theta$, $\cos\theta$ and λ, the planar similarity transformation of equation (A30) becomes

$$\mathbf{x}_i = \bar{\mathbf{x}}_0 + \frac{1}{\sigma_z^2}\begin{bmatrix} b & a \\ -a & b \end{bmatrix}(\mathbf{z}_i-\bar{\mathbf{z}}). \tag{A36}$$

In the case of the ***planar rigid transformation***, the solution is given by

$$\frac{1}{N}\sum_i (\mathbf{x}_{0i}-\bar{\mathbf{x}}_0)^T \mathbf{W}\mathbf{R}(\mathbf{z}_i-\bar{\mathbf{z}}) = \mathbf{0} \tag{A37}$$

$$\mathbf{x}_i = \bar{\mathbf{x}}_0 + \mathbf{R}(\theta)(\mathbf{z}_i-\bar{\mathbf{z}}). \tag{A38}$$

and for the above specific parameterization of $\mathbf{R}(\theta)$ we obtain two solutions

$$\mathbf{x}_i = \bar{\mathbf{x}}_0 \pm \frac{1}{\sqrt{a^2+b^2}}\begin{bmatrix} b & a \\ -a & b \end{bmatrix}(\mathbf{z}_i-\bar{\mathbf{z}}) \tag{A39}$$

which correspond to two rotation angles θ and $\theta+\pi$. A further investigation is needed in each particular application, in order to determine which of the two solutions minimizes in fact the target function $\phi=(\mathbf{x}-\mathbf{x}_0)^T(\mathbf{x}-\mathbf{x}_0)$.

Appendix B: Derivation of the metric and the connection coefficients of the solution fiber

$$\mathbf{G}=\left(\frac{\partial\mathbf{x}}{\partial\mathbf{p}}\right)^T\frac{\partial\mathbf{x}}{\partial\mathbf{p}}=\sum_i\left(\frac{\partial\mathbf{x}_i}{\partial\mathbf{p}}\right)^T\frac{\partial\mathbf{x}_i}{\partial\mathbf{p}}=\sum_i\left[\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\,\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}}\,\frac{\partial\mathbf{x}_i}{\partial\lambda}\right]^T\left[\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\,\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}}\,\frac{\partial\mathbf{x}_i}{\partial\lambda}\right]=$$

$$=\sum_i\begin{bmatrix}\left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right)^T\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}} & \left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right)^T\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}} & \left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right)^T\frac{\partial\mathbf{x}_i}{\partial\lambda}\\ & \left(\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}}\right)^T\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}} & \left(\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}}\right)^T\frac{\partial\mathbf{x}_i}{\partial\lambda}\\ \textit{symm.} & & \left(\frac{\partial\mathbf{x}_i}{\partial\lambda}\right)^T\frac{\partial\mathbf{x}_i}{\partial\lambda}\end{bmatrix}=\sum_i\mathbf{G}_i\,. \tag{B1}$$

Using $\frac{\partial}{\partial\theta_k}\mathbf{R}=[\boldsymbol{\omega}_k\times]\mathbf{R}$ and $\boldsymbol{\Omega}=[\boldsymbol{\omega}_1\boldsymbol{\omega}_2\boldsymbol{\omega}_3]$ we get

$$\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}=-\lambda\,\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\boldsymbol{\Omega}\,,\qquad\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}}=\mathbf{I}\,,\qquad\frac{\partial\mathbf{x}_i}{\partial\lambda}=\mathbf{R}\mathbf{z}_i\,. \tag{B2}$$

In view of $[\mathbf{z}_i\times][\mathbf{z}_i\times]=\mathbf{z}_i\mathbf{z}_i^T-(\mathbf{z}_i^T\mathbf{z}_i)\mathbf{I}$ and $[\mathbf{z}_i\times]\mathbf{z}_i=\mathbf{0}$, we get

$$\mathbf{G}_i=\begin{bmatrix}-\lambda^2\boldsymbol{\Omega}^T\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\boldsymbol{\Omega} & \lambda\,\boldsymbol{\Omega}^T\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T & -\lambda^2\boldsymbol{\Omega}^T\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\mathbf{R}\mathbf{z}_i\\ & \mathbf{I} & \mathbf{R}\mathbf{z}_i\\ \textit{symm.} & & \mathbf{z}_i^T\mathbf{R}^T\mathbf{R}\mathbf{z}_i\end{bmatrix}=$$

$$=\begin{bmatrix}-\lambda^2\boldsymbol{\Omega}^T\mathbf{R}[\mathbf{z}_i\times][\mathbf{z}_i\times]\mathbf{R}^T\boldsymbol{\Omega} & \lambda\,\boldsymbol{\Omega}^T\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T & \mathbf{0}\\ & \mathbf{I} & \mathbf{R}\mathbf{z}_i\\ \textit{symm.} & & \mathbf{z}_i^T\mathbf{z}_i\end{bmatrix} \tag{B3}$$

and after summing up the $\mathbf{G}_i$ terms

$$\mathbf{G}=\begin{bmatrix}\lambda^2\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega} & N\lambda\,\boldsymbol{\Omega}^T\mathbf{R}[\bar{\mathbf{z}}\times]\mathbf{R}^T & \mathbf{0}\\ & N\,\mathbf{I} & N\,\mathbf{R}\bar{\mathbf{z}}\\ \textit{symm.} & & \mathbf{z}^T\mathbf{z}\end{bmatrix}. \tag{B4}$$

In order to compute the matrices $\mathbf{K}_m$ (which contain the connection coefficients of the "first kind") we note that

$$\mathbf{K}_m=\left(\frac{\partial\mathbf{x}}{\partial\mathbf{p}}\right)^T\frac{\partial}{\partial p_m}\left(\frac{\partial\mathbf{x}}{\partial\mathbf{p}}\right)=\sum_i\left(\frac{\partial\mathbf{x}_i}{\partial\mathbf{p}}\right)^T\frac{\partial\mathbf{x}_i}{\partial\mathbf{p}}=\sum_i\left[\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\,\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}}\,\frac{\partial\mathbf{x}_i}{\partial\lambda}\right]^T\left[\frac{\partial}{\partial p_m}\left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right)\,\frac{\partial}{\partial p_m}\left(\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}}\right)\frac{\partial}{\partial p_m}\left(\frac{\partial\mathbf{x}_i}{\partial\lambda}\right)\right]=$$

$$=\sum_i\begin{bmatrix}\left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right)^T\frac{\partial}{\partial p_m}\left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right) & \left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right)^T\frac{\partial}{\partial p_m}\left(\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}}\right) & \left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right)^T\frac{\partial}{\partial p_m}\left(\frac{\partial\mathbf{x}_i}{\partial\lambda}\right)\\ \left(\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}}\right)^T\frac{\partial}{\partial p_m}\left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right) & \left(\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}}\right)^T\frac{\partial}{\partial p_m}\left(\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}}\right) & \left(\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}}\right)^T\frac{\partial}{\partial p_m}\left(\frac{\partial\mathbf{x}_i}{\partial\lambda}\right)\\ \left(\frac{\partial\mathbf{x}_i}{\partial\lambda}\right)^T\frac{\partial}{\partial p_m}\left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right) & \left(\frac{\partial\mathbf{x}_i}{\partial\lambda}\right)^T\frac{\partial}{\partial p_m}\left(\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}}\right) & \left(\frac{\partial\mathbf{x}_i}{\partial\lambda}\right)^T\frac{\partial}{\partial p_m}\left(\frac{\partial\mathbf{x}_i}{\partial\lambda}\right)\end{bmatrix}=$$

$$=\sum_i \begin{bmatrix} [\mathbf{K}_m]_{\boldsymbol{\theta\theta}} & [\mathbf{K}_m]_{\boldsymbol{\theta}\mathbf{t}} & [\mathbf{K}_m]_{\boldsymbol{\theta}\lambda} \\ [\mathbf{K}_m]_{\mathbf{t}\boldsymbol{\theta}} & [\mathbf{K}_m]_{\mathbf{tt}} & [\mathbf{K}_m]_{\mathbf{t}\lambda} \\ [\mathbf{K}_m]_{\lambda\boldsymbol{\theta}} & [\mathbf{K}_m]_{\lambda\mathbf{t}} & [\mathbf{K}_m]_{\lambda\lambda} \end{bmatrix} \tag{B5}$$

In order to proceed we need the partial derivatives

$$\frac{\partial}{\partial\theta_m}\left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right)=\frac{\partial}{\partial\theta_m}\left(-\lambda\,\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\boldsymbol{\Omega}\right)=-\lambda\,[\boldsymbol{\omega}_m\times]\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\boldsymbol{\Omega}+\lambda\,\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T[\boldsymbol{\omega}_m\times]\boldsymbol{\Omega}-\lambda\,\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\frac{\partial\boldsymbol{\Omega}}{\partial\theta_m}=$$

$$=\lambda\,[\boldsymbol{\omega}_m\mathbf{z}_i^T\mathbf{R}^T-\mathbf{R}\mathbf{z}_i\boldsymbol{\omega}_m^T]\boldsymbol{\Omega}-\lambda\,\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\frac{\partial\boldsymbol{\Omega}}{\partial\theta_m}=\lambda\,\boldsymbol{\omega}_m\mathbf{z}_i^T\mathbf{R}^T\boldsymbol{\Omega}-\lambda\,\mathbf{R}\mathbf{z}_i\boldsymbol{\omega}_m^T\boldsymbol{\Omega}-\lambda\,\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\frac{\partial\boldsymbol{\Omega}}{\partial\theta_m} \tag{B6}$$

$$\frac{\partial}{\partial\theta_m}\left(\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}}\right)=\frac{\partial}{\partial\theta_m}\mathbf{I}=\mathbf{0} \tag{B7}$$

$$\frac{\partial}{\partial\theta_m}\left(\frac{\partial\mathbf{x}_i}{\partial\lambda}\right)=\frac{\partial}{\partial\theta_m}(\mathbf{R}\mathbf{z}_i)=[\boldsymbol{\omega}_m\times]\mathbf{R}\mathbf{z}_i=-\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\boldsymbol{\omega}_m \tag{B8}$$

$$\frac{\partial}{\partial t_m}\left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right)=\frac{\partial}{\partial t_m}\left(-\lambda\,\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\boldsymbol{\Omega}\right)=\mathbf{0}\,, \tag{B9}$$

$$\frac{\partial}{\partial t_m}\left(\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}}\right)=\frac{\partial}{\partial t_m}\mathbf{I}=\mathbf{0}\,,\qquad \frac{\partial}{\partial t_m}\left(\frac{\partial\mathbf{x}_i}{\partial\lambda}\right)=\frac{\partial}{\partial t_m}(\mathbf{R}\mathbf{z}_i)=\mathbf{0}\,, \tag{B10}$$

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right)=\frac{\partial}{\partial\lambda}\left(-\lambda\,\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\boldsymbol{\Omega}\right)=-\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\boldsymbol{\Omega}\,, \tag{B11}$$

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}}\right)=\frac{\partial}{\partial\lambda}\mathbf{I}=\mathbf{0}\,,\qquad \frac{\partial}{\partial\lambda}\left(\frac{\partial\mathbf{x}_i}{\partial\lambda}\right)=\frac{\partial}{\partial\lambda}(\mathbf{R}\mathbf{z}_i)=\mathbf{0}\,. \tag{B12}$$

The zero partials above lead to the $\mathbf{K}_m$ matrices

$$\mathbf{K}_{\theta_m}=\begin{bmatrix} [\mathbf{K}_{\theta_m}]_{\boldsymbol{\theta\theta}} & \mathbf{0} & [\mathbf{K}_{\theta_m}]_{\boldsymbol{\theta}\lambda} \\ [\mathbf{K}_{\theta_m}]_{\mathbf{t}\boldsymbol{\theta}} & \mathbf{0} & [\mathbf{K}_{\theta_m}]_{\mathbf{t}\lambda} \\ [\mathbf{K}_{\theta_m}]_{\lambda\boldsymbol{\theta}} & \mathbf{0} & [\mathbf{K}_{\theta_m}]_{\lambda\lambda} \end{bmatrix},\qquad \mathbf{K}_{t_m}=\mathbf{0}\,,\qquad \mathbf{K}_{\lambda}=\begin{bmatrix} [\mathbf{K}_{\lambda}]_{\boldsymbol{\theta\theta}} & \mathbf{0} & \mathbf{0} \\ [\mathbf{K}_{\lambda}]_{\mathbf{t}\boldsymbol{\theta}} & \mathbf{0} & \mathbf{0} \\ [\mathbf{K}_{\lambda}]_{\lambda\boldsymbol{\theta}} & \mathbf{0} & 0 \end{bmatrix} \tag{B13}$$

where it remains to calculate the non-zero submatrices

$$[\mathbf{K}_{\theta_m}]_{\boldsymbol{\theta\theta}}=\sum_i\left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right)^T\frac{\partial}{\partial\theta_m}\left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right)=\sum_i\left(\lambda\,\boldsymbol{\Omega}^T\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\right)\left(\lambda\,\boldsymbol{\omega}_m\mathbf{z}_i^T\mathbf{R}^T\boldsymbol{\Omega}-\lambda\,\mathbf{R}\mathbf{z}_i\boldsymbol{\omega}_m^T\boldsymbol{\Omega}-\lambda\,\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\frac{\partial\boldsymbol{\Omega}}{\partial\theta_m}\right)=$$

$$=\lambda^2\,\boldsymbol{\Omega}^T[\boldsymbol{\omega}_m\times]\mathbf{R}\left(-\sum_i\mathbf{z}_i\mathbf{z}_i^T\right)\mathbf{R}^T\boldsymbol{\Omega}-\lambda^2\,\boldsymbol{\Omega}^T\mathbf{R}\left(\sum_i[\mathbf{z}_i\times]\,\mathbf{z}_i\right)\boldsymbol{\omega}_m^T\boldsymbol{\Omega}-\lambda^2\,\boldsymbol{\Omega}^T\mathbf{R}\left(\sum_i[\mathbf{z}_i\times][\mathbf{z}_i\times]\right)\mathbf{R}^T\frac{\partial\boldsymbol{\Omega}}{\partial\theta_m}=$$

$$=\lambda^2\,\boldsymbol{\Omega}^T[\boldsymbol{\omega}_m\times]\mathbf{R}\left(\mathbf{C}_z-(\mathbf{z}^T\mathbf{z})\mathbf{I}\right)\mathbf{R}^T\boldsymbol{\Omega}+\lambda^2\,\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\,\mathbf{R}^T\frac{\partial\boldsymbol{\Omega}}{\partial\theta_m}=$$

$$=\lambda^2\,\boldsymbol{\Omega}^T[\boldsymbol{\omega}_m\times]\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}-\lambda^2(\mathbf{z}^T\mathbf{z})\boldsymbol{\Omega}^T[\boldsymbol{\omega}_m\times]\boldsymbol{\Omega}+\lambda^2\,\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\,\mathbf{R}^T\frac{\partial\boldsymbol{\Omega}}{\partial\theta_m} \tag{B14}$$

$$[\mathbf{K}_{\theta_m}]_{\boldsymbol{\theta}\lambda}=\sum_i\left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right)^T\frac{\partial}{\partial\theta_m}\left(\frac{\partial\mathbf{x}_i}{\partial\lambda}\right)=\sum_i\left(\lambda\,\boldsymbol{\Omega}^T\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\right)\left(-\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\boldsymbol{\omega}_m\right)=$$

$$=-\lambda\,\boldsymbol{\Omega}^T\mathbf{R}\left(\sum_i[\mathbf{z}_i\times][\mathbf{z}_i\times]\right)\mathbf{R}^T\boldsymbol{\omega}_m=\lambda\,\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\omega}_m \tag{B15}$$

$$[\mathbf{K}_{\theta_m}]_{\mathbf{t}\boldsymbol{\theta}}=\sum_i\left(\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}}\right)^T\frac{\partial}{\partial\theta_m}\left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right)=\sum_i\lambda\,\boldsymbol{\omega}_m\mathbf{z}_i^T\mathbf{R}^T\boldsymbol{\Omega}-\lambda\,\mathbf{R}\mathbf{z}_i\boldsymbol{\omega}_m^T\boldsymbol{\Omega}-\lambda\,\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\frac{\partial\boldsymbol{\Omega}}{\partial\theta_m}=$$

$$=N\lambda\,\boldsymbol{\omega}_m\bar{\mathbf{z}}^T\mathbf{R}^T\boldsymbol{\Omega}-N\lambda\,\mathbf{R}\bar{\mathbf{z}}\boldsymbol{\omega}_m^T\boldsymbol{\Omega}-N\lambda\,\mathbf{R}[\bar{\mathbf{z}}\times]\mathbf{R}^T\frac{\partial\boldsymbol{\Omega}}{\partial\theta_m} \tag{B16}$$

$$[\mathbf{K}_{\theta_m}]_{\mathbf{t}\lambda}=\sum_i\left(\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}}\right)^T\frac{\partial}{\partial\theta_m}\left(\frac{\partial\mathbf{x}_i}{\partial\lambda}\right)=\sum_i[\boldsymbol{\omega}_m\times]\mathbf{R}\mathbf{z}_i=N[\boldsymbol{\omega}_m\times]\mathbf{R}\bar{\mathbf{z}}=-N\mathbf{R}[\bar{\mathbf{z}}\times]\mathbf{R}^T\boldsymbol{\omega}_m \tag{B17}$$

$$[\mathbf{K}_{\theta_m}]_{\lambda\boldsymbol{\theta}}=\sum_i\left(\frac{\partial\mathbf{x}_i}{\partial\lambda}\right)^T\frac{\partial}{\partial\theta_m}\left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right)=\sum_i\left(\mathbf{z}_i^T\mathbf{R}^T\right)\left(\lambda\,\boldsymbol{\omega}_m\mathbf{z}_i^T\mathbf{R}^T\boldsymbol{\Omega}-\lambda\,\mathbf{R}\mathbf{z}_i\boldsymbol{\omega}_m^T\boldsymbol{\Omega}-\lambda\,\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\frac{\partial\boldsymbol{\Omega}}{\partial\theta_m}\right)=$$

$$=\lambda\,\boldsymbol{\omega}_m^T\mathbf{R}\left(\sum_i\mathbf{z}_i\mathbf{z}_i^T\right)\mathbf{R}^T\boldsymbol{\Omega}-\lambda\left(\sum_i\mathbf{z}_i^T\mathbf{z}_i\right)\boldsymbol{\omega}_m^T\boldsymbol{\Omega}-\lambda\left(\sum_i\mathbf{z}_i^T[\mathbf{z}_i\times]\right)\mathbf{R}^T\frac{\partial\boldsymbol{\Omega}}{\partial\theta_m}=$$

$$=\lambda\,\boldsymbol{\omega}_m^T\mathbf{R}\left((\mathbf{z}^T\mathbf{z})\mathbf{I}-\mathbf{C}_z\right)\mathbf{R}^T\boldsymbol{\Omega}-\lambda\left(\mathbf{z}^T\mathbf{z}\right)\boldsymbol{\omega}_m^T\boldsymbol{\Omega}=-\lambda\,\boldsymbol{\omega}_m^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega} \tag{B18}$$

$$[\mathbf{K}_{\theta_m}]_{\lambda\lambda}=\sum_i\left(\frac{\partial\mathbf{x}_i}{\partial\lambda}\right)^T\frac{\partial}{\partial\theta_m}\left(\frac{\partial\mathbf{x}_i}{\partial\lambda}\right)=\sum_i\left(\mathbf{z}_i^T\mathbf{R}^T\right)\left(-\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\boldsymbol{\omega}_m\right)=-\sum_i\mathbf{z}_i^T[\mathbf{z}_i\times]\mathbf{R}^T\boldsymbol{\omega}_m=0 \tag{B19}$$

$$[\mathbf{K}_\lambda]_{\boldsymbol{\theta}\boldsymbol{\theta}}=\sum_i\left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right)^T\frac{\partial}{\partial\lambda}\left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right)=\sum_i\left(\lambda\,\boldsymbol{\Omega}^T\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\right)\left(-\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\boldsymbol{\Omega}\right)=-\lambda\sum_i\boldsymbol{\Omega}^T\mathbf{R}[\mathbf{z}_i\times][\mathbf{z}_i\times]\mathbf{R}^T\boldsymbol{\Omega}=$$

$$=-\lambda\,\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega} \tag{B20}$$

$$[\mathbf{K}_\lambda]_{\mathbf{t}\boldsymbol{\theta}}=\sum_i\left(\frac{\partial\mathbf{x}_i}{\partial\mathbf{t}}\right)^T\frac{\partial}{\partial\lambda}\left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right)=\sum_i\left(-\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\boldsymbol{\Omega}\right)=-N\,\mathbf{R}[\bar{\mathbf{z}}\times]\mathbf{R}^T\boldsymbol{\Omega} \tag{B21}$$

$$[\mathbf{K}_\lambda]_{\lambda\boldsymbol{\theta}}=\sum_i\left(\frac{\partial\mathbf{x}_i}{\partial\lambda}\right)^T\frac{\partial}{\partial\lambda}\left(\frac{\partial\mathbf{x}_i}{\partial\boldsymbol{\theta}}\right)=\sum_i\left(\mathbf{z}_i^T\mathbf{R}^T\right)\left(-\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\boldsymbol{\Omega}\right)=-\sum_i\mathbf{z}_i^T[\mathbf{z}_i\times]\boldsymbol{\Omega}=\mathbf{0} \tag{B22}$$

Appendix C: Computation of the metric and connection coefficients of the space-time solution manifold

From

$$\mathbf{x}_i = \lambda\, \mathbf{R}\mathbf{z}_i + \mathbf{t} \tag{C1}$$

we obtain the partial derivatives (see also Appendix B)

$$\frac{\partial \mathbf{x}_i}{\partial t} = \lambda\, \mathbf{R}\dot{\mathbf{z}}_i\,, \qquad \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}_i}{\partial t}\right) = \lambda\, \mathbf{R}\ddot{\mathbf{z}}_i \tag{C2}$$

$$\frac{\partial}{\partial \theta_m}\left(\frac{\partial \mathbf{x}_i}{\partial t}\right) = -\lambda\, \mathbf{R}[\dot{\mathbf{z}}_i \times]\mathbf{R}^T \boldsymbol{\omega}_m \tag{C3}$$

$$\frac{\partial}{\partial \theta_m}\left(\frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}}\right) = \frac{\partial}{\partial \theta_m}\left(-\lambda\, \mathbf{R}[\mathbf{z}_i \times]\mathbf{R}^T \boldsymbol{\Omega}\right) = -\lambda\, \mathbf{R}\mathbf{z}_i \boldsymbol{\omega}_m^T \boldsymbol{\Omega} + \lambda\, \boldsymbol{\omega}_m \mathbf{z}_i^T \mathbf{R}^T \boldsymbol{\Omega} - \lambda\, \mathbf{R}[\mathbf{z}_i \times]\mathbf{R}^T \frac{\partial \boldsymbol{\Omega}}{\partial \theta_m} \tag{C4}$$

$$\frac{\partial}{\partial \theta_m}\left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right) = \mathbf{0}\,, \qquad \frac{\partial}{\partial \theta_m}\left(\frac{\partial \mathbf{x}_i}{\partial \lambda}\right) = \frac{\partial}{\partial \theta_m}(\mathbf{R}\mathbf{z}_i) = -\mathbf{R}[\mathbf{z}_i \times]\mathbf{R}^T \boldsymbol{\omega}_m \tag{C5}$$

$$\frac{\partial}{\partial t_m}\left(\frac{\partial \mathbf{x}_i}{\partial t}\right) = \frac{\partial}{\partial t_m}\left(\frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}}\right) = \frac{\partial}{\partial t_m}\left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right) = \frac{\partial}{\partial t_m}\left(\frac{\partial \mathbf{x}_i}{\partial \lambda}\right) = \mathbf{0} \tag{C6}$$

$$\frac{\partial}{\partial \lambda}\left(\frac{\partial \mathbf{x}_i}{\partial t}\right) = \mathbf{R}\dot{\mathbf{z}}_i\,, \qquad \frac{\partial}{\partial \lambda}\left(\frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}}\right) = -\mathbf{R}[\mathbf{z}_i \times]\mathbf{R}^T \boldsymbol{\Omega}\,, \qquad \frac{\partial}{\partial \lambda}\left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right) = \mathbf{0}\,, \qquad \frac{\partial}{\partial \lambda}\left(\frac{\partial \mathbf{x}_i}{\partial \lambda}\right) = \mathbf{0} \tag{C7}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}}\right) = -\lambda\, \mathbf{R}[\dot{\mathbf{z}}_i \times]\mathbf{R}^T \boldsymbol{\Omega}\,, \qquad \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}_i}{\partial \lambda}\right) = \mathbf{R}\dot{\mathbf{z}}_i\,, \qquad \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right) = \mathbf{0} \tag{C8}$$

We will also make use of the auxiliary quantities

$$\mathbf{h} = \sum_i [\mathbf{z}_i \times]\dot{\mathbf{z}}_i\,, \qquad \dot{\mathbf{h}} = \sum_i [\mathbf{z}_i \times]\ddot{\mathbf{z}}_i\,, \qquad \mathbf{C}_z = -\sum_i [\mathbf{z}_i \times][\mathbf{z}_i \times] = (\mathbf{z}^T \mathbf{z})\mathbf{I} - \sum_i \mathbf{z}_i \mathbf{z}_i^T\,, \tag{C9}$$

$$\mathbf{W}_z = -\sum_i [\dot{\mathbf{z}}_i \times][\mathbf{z}_i \times] = (\mathbf{z}^T \dot{\mathbf{z}})\mathbf{I} - \sum_i \mathbf{z}_i \dot{\mathbf{z}}_i^T\,, \qquad \dot{\mathbf{C}}_z = \mathbf{W}_z + \mathbf{W}_z^T\,. \tag{C10}$$

Using the above relations in the definition of the metric and the connection matrices (of the first kind) and the fact that $\mathbf{x} = [\mathbf{x}_1^T \mathbf{x}_2^T \ldots \mathbf{x}_N^T]^T$, it follows that their submatrices are

$$\mathbf{g} = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)^T \frac{\partial \mathbf{x}}{\partial t} = \sum_i \begin{bmatrix} \left(\frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}}\right)^T \frac{\partial \mathbf{x}_i}{\partial t} \\ \left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right)^T \frac{\partial \mathbf{x}_i}{\partial t} \\ \left(\frac{\partial \mathbf{x}_i}{\partial \lambda}\right)^T \frac{\partial \mathbf{x}_i}{\partial t} \end{bmatrix} = \begin{bmatrix} \lambda^2 \boldsymbol{\Omega}^T \mathbf{R}\mathbf{h} \\ N\lambda\, \mathbf{R}\dot{\bar{\mathbf{z}}} \\ \lambda\, \mathbf{z}^T \dot{\mathbf{z}} \end{bmatrix} \tag{C11}$$

$$\gamma=\left(\frac{\partial \mathbf{x}}{\partial t}\right)^T \frac{\partial \mathbf{x}}{\partial t}=\sum_i \left(\frac{\partial \mathbf{x}_i}{\partial t}\right)^T \frac{\partial \mathbf{x}_i}{\partial t}=\lambda^2 \dot{\mathbf{z}}^T \dot{\mathbf{z}} \tag{C12}$$

$$\mathbf{h}_{\theta_m}=\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)^T \frac{\partial}{\partial \theta_m}\left(\frac{\partial \mathbf{x}}{\partial t}\right)=\sum_i \begin{bmatrix} \left(\frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}}\right)^T \frac{\partial}{\partial \theta_m}\left(\frac{\partial \mathbf{x}_i}{\partial t}\right) \\ \left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right)^T \frac{\partial}{\partial \theta_m}\left(\frac{\partial \mathbf{x}_i}{\partial t}\right) \\ \left(\frac{\partial \mathbf{x}_i}{\partial \lambda}\right)^T \frac{\partial}{\partial \theta_m}\left(\frac{\partial \mathbf{x}_i}{\partial t}\right) \end{bmatrix}=\begin{bmatrix} \lambda^2 \boldsymbol{\Omega}^T \mathbf{R} \mathbf{W}_z^T \mathbf{R}^T \boldsymbol{\omega}_m \\ -N\lambda\, \mathbf{R}[\dot{\bar{\mathbf{z}}}\times]\mathbf{R}^T \boldsymbol{\omega}_m \\ -\lambda\, \mathbf{h}^T \mathbf{R}^T \boldsymbol{\omega}_m \end{bmatrix} \tag{C13}$$

$$\mathbf{h}_{t_m}=\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)^T \frac{\partial}{\partial t_m}\left(\frac{\partial \mathbf{x}}{\partial t}\right)=\sum_i \begin{bmatrix} \left(\frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}}\right)^T \frac{\partial}{\partial t_m}\left(\frac{\partial \mathbf{x}_i}{\partial t}\right) \\ \left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right)^T \frac{\partial}{\partial t_m}\left(\frac{\partial \mathbf{x}_i}{\partial t}\right) \\ \left(\frac{\partial \mathbf{x}_i}{\partial \lambda}\right)^T \frac{\partial}{\partial t_m}\left(\frac{\partial \mathbf{x}_i}{\partial t}\right) \end{bmatrix}=\mathbf{0}, \tag{C14}$$

$$\mathbf{h}_{\lambda}=\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)^T \frac{\partial}{\partial \lambda}\left(\frac{\partial \mathbf{x}}{\partial t}\right)=\sum_i \begin{bmatrix} \left(\frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}}\right)^T \frac{\partial}{\partial \lambda}\left(\frac{\partial \mathbf{x}_i}{\partial t}\right) \\ \left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right)^T \frac{\partial}{\partial \lambda}\left(\frac{\partial \mathbf{x}_i}{\partial t}\right) \\ \left(\frac{\partial \mathbf{x}_i}{\partial \lambda}\right)^T \frac{\partial}{\partial \lambda}\left(\frac{\partial \mathbf{x}_i}{\partial t}\right) \end{bmatrix}=\begin{bmatrix} \lambda\, \boldsymbol{\Omega}^T \mathbf{R}\mathbf{h} \\ N\, \mathbf{R}\dot{\bar{\mathbf{z}}} \\ \mathbf{z}^T \dot{\mathbf{z}} \end{bmatrix}, \tag{C15}$$

$$\tilde{\mathbf{h}}_{\theta_m}=\frac{\partial}{\partial \theta_m}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)^T \frac{\partial \mathbf{x}}{\partial t}=\sum_i \begin{bmatrix} \frac{\partial}{\partial \theta_m}\left(\frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}}\right)^T \frac{\partial \mathbf{x}_i}{\partial t} \\ \frac{\partial}{\partial \theta_m}\left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right)^T \frac{\partial \mathbf{x}_i}{\partial t} \\ \frac{\partial}{\partial \theta_m}\left(\frac{\partial \mathbf{x}_i}{\partial \lambda}\right)^T \frac{\partial \mathbf{x}_i}{\partial t} \end{bmatrix}=\begin{bmatrix} -\lambda^2 \boldsymbol{\Omega}^T \mathbf{R} \mathbf{W}_z \mathbf{R}^T \boldsymbol{\omega}_m + \lambda^2 \frac{\partial \boldsymbol{\Omega}^T}{\partial \theta_m} \mathbf{R}\mathbf{h} \\ \mathbf{0} \\ \lambda\, \mathbf{h}^T \mathbf{R}^T \boldsymbol{\omega}_m \end{bmatrix} \tag{C16}$$

$$\tilde{\mathbf{h}}_{t_m}=\frac{\partial}{\partial t_m}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)^T \frac{\partial \mathbf{x}}{\partial t}=\sum_i \begin{bmatrix} \frac{\partial}{\partial t_m}\left(\frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}}\right)^T \frac{\partial \mathbf{x}_i}{\partial t} \\ \frac{\partial}{\partial t_m}\left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right)^T \frac{\partial \mathbf{x}_i}{\partial t} \\ \frac{\partial}{\partial t_m}\left(\frac{\partial \mathbf{x}_i}{\partial \lambda}\right)^T \frac{\partial \mathbf{x}_i}{\partial t} \end{bmatrix}=\mathbf{0}, \tag{C17}$$

$$\tilde{\mathbf{h}}_{\lambda}=\frac{\partial}{\partial \lambda}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)^T \frac{\partial \mathbf{x}}{\partial t}=\sum_i \begin{bmatrix} \frac{\partial}{\partial \lambda}\left(\frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}}\right)^T \frac{\partial \mathbf{x}_i}{\partial t} \\ \frac{\partial}{\partial \lambda}\left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right)^T \frac{\partial \mathbf{x}_i}{\partial t} \\ \frac{\partial}{\partial \lambda}\left(\frac{\partial \mathbf{x}_i}{\partial \lambda}\right)^T \frac{\partial \mathbf{x}_i}{\partial t} \end{bmatrix}=\begin{bmatrix} \lambda\, \boldsymbol{\Omega}^T \mathbf{R}\mathbf{h} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{C18}$$

$$\kappa_{\theta_m} = \left(\frac{\partial \mathbf{x}}{\partial t}\right)^T \frac{\partial}{\partial \theta_m}\left(\frac{\partial \mathbf{x}}{\partial t}\right) = \sum_i \left(\frac{\partial \mathbf{x}_i}{\partial t}\right)^T \frac{\partial}{\partial \theta_m}\left(\frac{\partial \mathbf{x}_i}{\partial t}\right) = 0\,, \tag{C19}$$

$$\kappa_{t_m} = \left(\frac{\partial \mathbf{x}}{\partial t}\right)^T \frac{\partial}{\partial t_m}\left(\frac{\partial \mathbf{x}}{\partial t}\right) = \sum_i \left(\frac{\partial \mathbf{x}_i}{\partial t}\right)^T \frac{\partial}{\partial t_m}\left(\frac{\partial \mathbf{x}_i}{\partial t}\right) = 0\,, \tag{C20}$$

$$\kappa_{\lambda} = \left(\frac{\partial \mathbf{x}}{\partial t}\right)^T \frac{\partial}{\partial \lambda}\left(\frac{\partial \mathbf{x}}{\partial t}\right) = \sum_i \left(\frac{\partial \mathbf{x}_i}{\partial t}\right)^T \frac{\partial}{\partial \lambda}\left(\frac{\partial \mathbf{x}_i}{\partial t}\right) = \lambda\, \dot{\mathbf{z}}^T \dot{\mathbf{z}}\,, \tag{C21}$$

$$\mathbf{K}_t = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)^T \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right) = \sum_i \begin{bmatrix} \left(\frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}}\right)^T \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}}\right) & \left(\frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}}\right)^T \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right) & \left(\frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}}\right)^T \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}_i}{\partial \lambda}\right) \\ \left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right)^T \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}}\right) & \left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right)^T \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right) & \left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right)^T \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}_i}{\partial \lambda}\right) \\ \left(\frac{\partial \mathbf{x}_i}{\partial \lambda}\right)^T \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}}\right) & \left(\frac{\partial \mathbf{x}_i}{\partial \lambda}\right)^T \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right) & \left(\frac{\partial \mathbf{x}_i}{\partial \lambda}\right)^T \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}_i}{\partial \lambda}\right) \end{bmatrix} =$$

$$= \begin{bmatrix} \lambda^2 \boldsymbol{\Omega}^T \mathbf{R} \mathbf{W}_z^T \mathbf{R}^T \boldsymbol{\Omega} & \mathbf{0} & \lambda\, \boldsymbol{\Omega}^T \mathbf{R} \mathbf{h} \\ -N\lambda\, \mathbf{R}[\dot{\bar{\mathbf{z}}}\times]\mathbf{R}^T \boldsymbol{\Omega} & \mathbf{0} & N\,\mathbf{R}\dot{\bar{\mathbf{z}}} \\ -\lambda\, \mathbf{h}^T \mathbf{R}^T \boldsymbol{\Omega} & \mathbf{0} & \mathbf{z}^T \dot{\mathbf{z}} \end{bmatrix}, \tag{C22}$$

$$\mathbf{h}_t = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)^T \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}}{\partial t}\right) = \sum_i \begin{bmatrix} \left(\frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}}\right)^T \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}_i}{\partial t}\right) \\ \left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right)^T \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}_i}{\partial t}\right) \\ \left(\frac{\partial \mathbf{x}_i}{\partial \lambda}\right)^T \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}_i}{\partial t}\right) \end{bmatrix} = \begin{bmatrix} \lambda^2 \boldsymbol{\Omega}^T \mathbf{R} \dot{\mathbf{h}} \\ N\lambda\, \mathbf{R}\ddot{\bar{\mathbf{z}}} \\ \lambda\, \mathbf{z}^T \ddot{\mathbf{z}} \end{bmatrix}, \tag{C23}$$

$$\tilde{\mathbf{h}}_t = \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)^T \left(\frac{\partial \mathbf{x}}{\partial t}\right) = \sum_i \begin{bmatrix} \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}}\right)^T \frac{\partial \mathbf{x}_i}{\partial t} \\ \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right)^T \frac{\partial \mathbf{x}_i}{\partial t} \\ \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}_i}{\partial \lambda}\right)^T \frac{\partial \mathbf{x}_i}{\partial t} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \lambda\, \dot{\mathbf{z}}^T \dot{\mathbf{z}} \end{bmatrix}, \tag{C24}$$

$$\kappa_t = \left(\frac{\partial \mathbf{x}}{\partial t}\right)^T \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}}{\partial t}\right) = \sum_i \left(\frac{\partial \mathbf{x}_i}{\partial t}\right)^T \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}_i}{\partial t}\right) = \lambda^2 \dot{\mathbf{z}}^T \ddot{\mathbf{z}}\,. \tag{C25}$$

Appendix D: Derivation of the differential equations of geodesics on the space-time manifold

The derivation is based on the geodesic equations (99) and (100)

$$\mathbf{G}\ddot{\mathbf{p}}-\frac{\ddot{s}}{\dot{s}}(\mathbf{G}\dot{\mathbf{p}}+\mathbf{g})+\sum_{m=1}^{7}\dot{p}_m\mathbf{K}_m\dot{\mathbf{p}}+\mathbf{H}\dot{\mathbf{p}}+\mathbf{K}_t\dot{\mathbf{p}}+\mathbf{h}_t=\mathbf{0} \tag{D1}$$

$$\mathbf{g}^T\ddot{\mathbf{p}}-\frac{\ddot{s}}{\dot{s}}(\mathbf{g}^T\dot{\mathbf{p}}+\gamma)+\dot{\mathbf{p}}^T\tilde{\mathbf{H}}\dot{\mathbf{p}}+\boldsymbol{\kappa}^T\dot{\mathbf{p}}+\tilde{\mathbf{h}}_t^T\dot{\mathbf{p}}+\kappa_t=0 \tag{D2}$$

where

$$\frac{\ddot{s}}{\dot{s}}=\frac{\dot{s}\ddot{s}}{\dot{s}^2}=\frac{\frac{1}{2}\frac{d(\dot{s}^2)}{dt}}{\dot{s}^2}=\frac{\frac{1}{2}\frac{d(\dot{\mathbf{x}}^T\dot{\mathbf{x}})}{dt}}{\dot{\mathbf{x}}^T\dot{\mathbf{x}}}=\frac{\dot{\mathbf{x}}^T\ddot{\mathbf{x}}}{\dot{\mathbf{x}}^T\dot{\mathbf{x}}} \tag{D3}$$

The derivation is based on the values of the submatrices $\mathbf{G}$ and $\mathbf{K}_m$, derived in section 5 for a single fiber and the values of the submatrices $\mathbf{g}$, γ and $\mathbf{K}_t$, $\mathbf{h}_m$, $\mathbf{h}_t$, $\tilde{\mathbf{h}}_m$, $\tilde{\mathbf{h}}_t$, κ_m, κ_t, derived in section 7, which extend the metric and connections to the space-time solution manifold. To keep the computations handy we restrict ourselves to the simpler case where the reference motion is a *principal motion*. This causes no loss of generality since any available reference motion can be easily converted to a principal motion. For this simpler case where

$$\bar{\mathbf{z}}=\mathbf{0}, \qquad \mathbf{z}^T\dot{\mathbf{z}}=0, \qquad \mathbf{z}^T\ddot{\mathbf{z}}=-\dot{\mathbf{z}}^T\dot{\mathbf{z}}, \qquad \mathbf{W}_z=\mathbf{W}_z^T=\dot{\mathbf{C}}_z, \tag{D4}$$

the relevant submatrices become

$$\mathbf{G}=\begin{bmatrix}\lambda^2\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & N\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{z}^T\mathbf{z}\end{bmatrix}, \tag{D5}$$

$$[\mathbf{K}_{\theta_m}]_{\boldsymbol{\theta\theta}}=\lambda^2\,\boldsymbol{\Omega}^T[\boldsymbol{\omega}_m\times]\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}-\lambda^2(\mathbf{z}^T\mathbf{z})\boldsymbol{\Omega}^T[\boldsymbol{\omega}_m\times]\boldsymbol{\Omega}+\lambda^2\,\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\,\mathbf{R}^T\frac{\partial\boldsymbol{\Omega}}{\partial\theta_m} \tag{D6}$$

$$\mathbf{K}_{\theta_m}=\begin{bmatrix}[\mathbf{K}_{\theta_m}]_{\boldsymbol{\theta\theta}} & \mathbf{0} & \lambda\,\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\omega}_m\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ -\lambda\,\boldsymbol{\omega}_m^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega} & \mathbf{0} & 0\end{bmatrix} \tag{D7}$$

$$\mathbf{K}_{t_m}=\mathbf{0}, \qquad \mathbf{K}_\lambda=\begin{bmatrix}\lambda\,\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & 0\end{bmatrix}. \tag{D8}$$

$$\mathbf{G}^{-1}=\begin{bmatrix}\frac{1}{\lambda^2}\boldsymbol{\Omega}^{-1}\mathbf{R}\mathbf{C}_z^{-1}\mathbf{R}^T\boldsymbol{\Omega}^{-T} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \frac{1}{N}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \frac{1}{\mathbf{z}^T\mathbf{z}}\end{bmatrix}, \qquad \mathbf{g}=\begin{bmatrix}\lambda^2\boldsymbol{\Omega}^T\mathbf{R}\mathbf{h}\\ \mathbf{0}\\ 0\end{bmatrix}, \qquad \gamma=\lambda^2\dot{\mathbf{z}}^T\dot{\mathbf{z}} \tag{D9}$$

$$\mathbf{h}_{\theta_m} = \begin{bmatrix} \lambda^2 \mathbf{\Omega}^T \mathbf{R}\dot{\mathbf{C}}_z \mathbf{R}^T \boldsymbol{\omega}_m \\ \mathbf{0} \\ -\lambda\, \mathbf{h}^T \mathbf{R}^T \boldsymbol{\omega}_m \end{bmatrix}, \qquad \mathbf{h}_{t_m} = \mathbf{0}, \qquad \mathbf{h}_\lambda = \begin{bmatrix} \lambda\, \mathbf{\Omega}^T \mathbf{R}\mathbf{h} \\ \mathbf{0} \\ 0 \end{bmatrix}, \tag{D10}$$

$$\tilde{\mathbf{h}}_{\theta_m} = \begin{bmatrix} -\lambda^2 \mathbf{\Omega}^T \mathbf{R}\dot{\mathbf{C}}_z \mathbf{R}^T \boldsymbol{\omega}_m + \lambda^2 \dfrac{\partial \mathbf{\Omega}^T}{\partial \theta_m} \mathbf{R}\mathbf{h} \\ \mathbf{0} \\ \lambda\, \mathbf{h}^T \mathbf{R}^T \boldsymbol{\omega}_m \end{bmatrix}, \qquad \tilde{\mathbf{h}}_{t_m} = \mathbf{0}, \qquad \tilde{\mathbf{h}}_\lambda = \begin{bmatrix} \lambda\, \mathbf{\Omega}^T \mathbf{R}\mathbf{h} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{D11}$$

$$\kappa_{\theta_m} = 0, \qquad \kappa_{t_m} = 0, \qquad \kappa_\lambda = \lambda\, \dot{\mathbf{z}}^T \dot{\mathbf{z}}, \qquad \kappa_t = \lambda^2 \dot{\mathbf{z}}^T \ddot{\mathbf{z}}, \tag{D12}$$

$$\mathbf{K}_t = \begin{bmatrix} \lambda^2 \mathbf{\Omega}^T \mathbf{R}\dot{\mathbf{C}}_z \mathbf{R}^T \mathbf{\Omega} & \mathbf{0} & \lambda\, \mathbf{\Omega}^T \mathbf{R}\mathbf{h} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\lambda\, \mathbf{h}^T \mathbf{R}^T \mathbf{\Omega} & \mathbf{0} & 0 \end{bmatrix}, \qquad \mathbf{h}_t = \begin{bmatrix} \lambda^2 \mathbf{\Omega}^T \mathbf{R}\dot{\mathbf{h}} \\ \mathbf{0} \\ -\lambda\, \dot{\mathbf{z}}^T \dot{\mathbf{z}} \end{bmatrix}, \qquad \tilde{\mathbf{h}}_t = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \lambda\, \dot{\mathbf{z}}^T \dot{\mathbf{z}} \end{bmatrix}, \tag{D13}$$

$$\mathbf{H} = [\mathbf{h}_1 \cdots \mathbf{h}_7] = \left[\mathbf{h}_{\theta_1} \mathbf{h}_{\theta_2} \mathbf{h}_{\theta_3} \quad \mathbf{h}_{t_1} \mathbf{h}_{t_2} \mathbf{h}_{t_3} \quad \mathbf{h}_\lambda \right] = [\mathbf{H}_{\boldsymbol{\theta}}\ \mathbf{0}\ \mathbf{h}_\lambda] = \begin{bmatrix} \lambda^2 \mathbf{\Omega}^T \mathbf{R}\dot{\mathbf{C}}_z \mathbf{R}^T \mathbf{\Omega} & \mathbf{0} & \lambda\, \mathbf{\Omega}^T \mathbf{R}\mathbf{h} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\lambda\, \mathbf{h}^T \mathbf{R}^T \mathbf{\Omega} & \mathbf{0} & 0 \end{bmatrix} = \mathbf{K}_t \tag{D14}$$

$$\tilde{\mathbf{H}} = [\tilde{\mathbf{h}}_1 \cdots \tilde{\mathbf{h}}_7] = \left[\tilde{\mathbf{h}}_{\theta_1} \tilde{\mathbf{h}}_{\theta_2} \tilde{\mathbf{h}}_{\theta_3} \quad \tilde{\mathbf{h}}_{t_1} \tilde{\mathbf{h}}_{t_2} \tilde{\mathbf{h}}_{t_3} \quad \tilde{\mathbf{h}}_\lambda \right] = [\tilde{\mathbf{H}}_{\boldsymbol{\theta}}\ \mathbf{0}\ \tilde{\mathbf{h}}_\lambda] =$$

$$= \begin{bmatrix} -\lambda^2 \mathbf{\Omega}^T \mathbf{R}\dot{\mathbf{C}}_z \mathbf{R}^T \mathbf{\Omega} + \lambda^2 \left\{ \dfrac{\partial \mathbf{\Omega}^T}{\partial \theta_m} \mathbf{R}\mathbf{h} \right\} & \mathbf{0} & \lambda\, \mathbf{\Omega}^T \mathbf{R}\mathbf{h} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \lambda\, \mathbf{h}^T \mathbf{R}^T \mathbf{\Omega} & \mathbf{0} & 0 \end{bmatrix} \tag{D15}$$

where

$$\left\{ \frac{\partial \mathbf{\Omega}^T}{\partial \theta_m} \mathbf{R}\mathbf{h} \right\} \equiv \left[\frac{\partial \mathbf{\Omega}^T}{\partial \theta_1} \mathbf{R}\mathbf{h} \quad \frac{\partial \mathbf{\Omega}^T}{\partial \theta_2} \mathbf{R}\mathbf{h} \quad \frac{\partial \mathbf{\Omega}^T}{\partial \theta_3} \mathbf{R}\mathbf{h} \right] = \begin{bmatrix} \dfrac{\partial \boldsymbol{\omega}_1^T}{\partial \theta_1} \mathbf{R}\mathbf{h} & \dfrac{\partial \boldsymbol{\omega}_1^T}{\partial \theta_2} \mathbf{R}\mathbf{h} & \dfrac{\partial \boldsymbol{\omega}_1^T}{\partial \theta_3} \mathbf{R}\mathbf{h} \\ \dfrac{\partial \boldsymbol{\omega}_2^T}{\partial \theta_1} \mathbf{R}\mathbf{h} & \dfrac{\partial \boldsymbol{\omega}_2^T}{\partial \theta_2} \mathbf{R}\mathbf{h} & \dfrac{\partial \boldsymbol{\omega}_2^T}{\partial \theta_3} \mathbf{R}\mathbf{h} \\ \dfrac{\partial \boldsymbol{\omega}_3^T}{\partial \theta_1} \mathbf{R}\mathbf{h} & \dfrac{\partial \boldsymbol{\omega}_3^T}{\partial \theta_2} \mathbf{R}\mathbf{h} & \dfrac{\partial \boldsymbol{\omega}_3^T}{\partial \theta_3} \mathbf{R}\mathbf{h} \end{bmatrix} = \begin{bmatrix} \mathbf{h}^T \mathbf{R}^T \dfrac{\partial \boldsymbol{\omega}_1}{\partial \boldsymbol{\theta}} \\ \mathbf{h}^T \mathbf{R}^T \dfrac{\partial \boldsymbol{\omega}_2}{\partial \boldsymbol{\theta}} \\ \mathbf{h}^T \mathbf{R}^T \dfrac{\partial \boldsymbol{\omega}_3}{\partial \boldsymbol{\theta}} \end{bmatrix} \tag{D16}$$

$$\boldsymbol{\kappa}^T = [\kappa_1 \cdots \kappa_7] = \left[\kappa_{\theta_1} \kappa_{\theta_2} \kappa_{\theta_3} \quad \kappa_{t_1} \kappa_{t_2} \kappa_{t_3} \quad \kappa_\lambda \right] = [\mathbf{0}\ \ \mathbf{0}\ \ \kappa_\lambda] = [\mathbf{0}\ \ \mathbf{0}\ \ \lambda\, \dot{\mathbf{z}}^T \dot{\mathbf{z}}]. \tag{D17}$$

Carrying out the computations for the terms appearing in the first of the geodesic equations ((D1) and (D2) we obtain

$$\mathbf{G}\ddot{\mathbf{p}} = \begin{bmatrix} \lambda^2 \mathbf{\Omega}^T \mathbf{R}\dot{\mathbf{C}}_z \mathbf{R}^T \mathbf{\Omega} \ddot{\boldsymbol{\theta}} \\ N \ddot{\mathbf{t}} \\ \ddot{\lambda}\, \mathbf{z}^T \mathbf{z} \end{bmatrix}, \qquad \mathbf{G}\,\dot{\mathbf{p}} + \mathbf{g} = \begin{bmatrix} \lambda^2 \mathbf{\Omega}^T \mathbf{R}\mathbf{C}_z \mathbf{R}^T \mathbf{\Omega}\dot{\boldsymbol{\theta}} + \lambda^2 \mathbf{\Omega}^T \mathbf{R}\mathbf{h} \\ N \dot{\mathbf{t}} \\ \dot{\lambda}\, \mathbf{z}^T \mathbf{z} \end{bmatrix}, \tag{D18}$$

$$\mathbf{g}^T \dot{\mathbf{p}} + \gamma = \lambda^2 \mathbf{h}^T \mathbf{R}^T \mathbf{\Omega}\dot{\boldsymbol{\theta}} + \lambda^2 \dot{\mathbf{z}}^T \dot{\mathbf{z}}. \tag{D19}$$

To evaluate the term $\sum_m \dot{p}_m \mathbf{K}_m \dot{\mathbf{p}}$ we shall first need the basic relations

$$\sum_{m=1}^{3} \dot{\theta}_m \boldsymbol{\omega}_m = \boldsymbol{\Omega}\dot{\boldsymbol{\theta}}\,, \qquad \sum_{m=1}^{3} \dot{\theta}_m [\boldsymbol{\omega}_m \times] = [(\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})\times]\,, \qquad \sum_{m=1}^{3} \dot{\theta}_m \frac{\partial \boldsymbol{\Omega}}{\partial \theta_m} = \dot{\boldsymbol{\Omega}}\,, \tag{D20}$$

which will be used to evaluate the terms in the sum. For the 3 terms related to the rotational parameters we have

$$\mathbf{K}_{\theta_m} \dot{\mathbf{p}} = \begin{bmatrix} [\mathbf{K}_{\theta_m}]_{\boldsymbol{\theta\theta}} \dot{\boldsymbol{\theta}} + \dot{\lambda}\lambda\, \boldsymbol{\Omega}^T \mathbf{R}\mathbf{C}_z \mathbf{R}^T \boldsymbol{\omega}_m \\ \mathbf{0} \\ -\lambda\, \boldsymbol{\omega}_m^T \mathbf{R}\mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} \end{bmatrix}, \tag{D21}$$

where

$$\sum_{m=1}^{3} \dot{\theta}_m [\mathbf{K}_{\theta_m}]_{\boldsymbol{\theta\theta}} \dot{\boldsymbol{\theta}} =$$

$$= \lambda^2\, \boldsymbol{\Omega}^T \left(\sum_{m=1}^{3} \dot{\theta}_m [\boldsymbol{\omega}_m \times] \right) \mathbf{R}\mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} - \lambda^2 (\mathbf{z}^T \mathbf{z}) \boldsymbol{\Omega}^T \left(\sum_{m=1}^{3} \dot{\theta}_m [\boldsymbol{\omega}_m \times] \right) \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + \lambda^2\, \boldsymbol{\Omega}^T \mathbf{R}\mathbf{C}_z\, \mathbf{R}^T \left(\sum_{m=1}^{3} \dot{\theta}_m \frac{\partial \boldsymbol{\Omega}}{\partial \theta_m} \right) \dot{\boldsymbol{\theta}} =$$

$$= \lambda^2\, \boldsymbol{\Omega}^T [(\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})\times] \mathbf{R}\mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} - \lambda^2 (\mathbf{z}^T \mathbf{z}) \boldsymbol{\Omega}^T [(\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})\times] \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + \lambda^2\, \boldsymbol{\Omega}^T \mathbf{R}\mathbf{C}_z\, \mathbf{R}^T \dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}} =$$

$$= \lambda^2\, \boldsymbol{\Omega}^T [(\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})\times] \mathbf{R}\mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + \lambda^2\, \boldsymbol{\Omega}^T \mathbf{R}\mathbf{C}_z\, \mathbf{R}^T \dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}}\,, \tag{D22}$$

and therefore

$$\sum_{m=1}^{3} \dot{\theta}_m \mathbf{K}_{\theta_m} \dot{\mathbf{p}} = \sum_{m=1}^{3} \dot{\theta}_m \begin{bmatrix} [\mathbf{K}_{\theta_m}]_{\boldsymbol{\theta\theta}} \dot{\boldsymbol{\theta}} + \dot{\lambda}\lambda\, \boldsymbol{\Omega}^T \mathbf{R}\mathbf{C}_z \mathbf{R}^T \boldsymbol{\omega}_m \\ \mathbf{0} \\ -\lambda\, \boldsymbol{\omega}_m^T \mathbf{R}\mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} \end{bmatrix} = \begin{bmatrix} \sum_{m=1}^{3} \dot{\theta}_m [\mathbf{K}_{\theta_m}]_{\boldsymbol{\theta\theta}} \dot{\boldsymbol{\theta}} + \dot{\lambda}\lambda\, \boldsymbol{\Omega}^T \mathbf{R}\mathbf{C}_z \mathbf{R}^T \sum_{m=1}^{3} \dot{\theta}_m \boldsymbol{\omega}_m \\ \mathbf{0} \\ -\lambda \sum_{m=1}^{3} \dot{\theta}_m \boldsymbol{\omega}_m^T \mathbf{R}\mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} \end{bmatrix} =$$

$$= \begin{bmatrix} \lambda^2 \boldsymbol{\Omega}^T [(\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})\times] \mathbf{R}\mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + \lambda^2 \boldsymbol{\Omega}^T \mathbf{R}\mathbf{C}_z \mathbf{R}^T \dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}} + \dot{\lambda}\lambda\, \boldsymbol{\Omega}^T \mathbf{R}\mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} \\ \mathbf{0} \\ -\lambda\, \dot{\boldsymbol{\theta}}^T \boldsymbol{\Omega}^T \mathbf{R}\mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} \end{bmatrix}. \tag{D23}$$

The terms related to the displacement parameters vanish and the remaining scale term is

$$\mathbf{K}_\lambda \dot{\mathbf{p}} = \begin{bmatrix} \lambda\, \boldsymbol{\Omega}^T \mathbf{R}\mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} \\ \mathbf{0} \\ 0 \end{bmatrix} \tag{D24}$$

Adding the above terms we finally get

$$\sum_{m=1}^{7} \dot{p}_m \mathbf{K}_m \dot{\mathbf{p}} = \sum \dot{\theta}_m \mathbf{K}_{\theta_m} \dot{\mathbf{p}} + \dot{\lambda}\, \mathbf{K}_\lambda \dot{\mathbf{p}} =$$

$$=\begin{bmatrix} \lambda^2\boldsymbol{\Omega}^T[(\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})\times]\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\lambda^2\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}}+2\dot{\lambda}\lambda\,\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}} \\ \mathbf{0} \\ -\lambda\,\dot{\boldsymbol{\theta}}^T\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}} \end{bmatrix}. \tag{D25}$$

The remaining terms are

$$\mathbf{H}\dot{\mathbf{p}}=\begin{bmatrix} \lambda^2\boldsymbol{\Omega}^T\mathbf{R}\dot{\mathbf{C}}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\dot{\lambda}\lambda\,\boldsymbol{\Omega}^T\mathbf{R}\mathbf{h} \\ \mathbf{0} \\ -\lambda\,\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}} \end{bmatrix}, \tag{D26}$$

$$\mathbf{K}_t\dot{\mathbf{p}}+\mathbf{h}_t=\begin{bmatrix} \lambda^2\boldsymbol{\Omega}^T\mathbf{R}\dot{\mathbf{C}}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\dot{\lambda}\lambda\,\boldsymbol{\Omega}^T\mathbf{R}\mathbf{h}+\lambda^2\boldsymbol{\Omega}^T\mathbf{R}\dot{\mathbf{h}} \\ \mathbf{0} \\ -\lambda\,\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}-\lambda\,\dot{\mathbf{z}}^T\dot{\mathbf{z}} \end{bmatrix}. \tag{D27}$$

Replacing the evaluated terms in the first geodesic equation (D1) we obtain after splitting it in three parts (rotational, displacement, scale)

$$\begin{aligned}&\lambda^2\boldsymbol{\Omega}^T\mathbf{R}\dot{\mathbf{C}}_z\mathbf{R}^T\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}}+\lambda^2\boldsymbol{\Omega}^T[(\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})\times]\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\lambda^2\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}}+2\dot{\lambda}\lambda\,\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\\ &\quad+\lambda^2\boldsymbol{\Omega}^T\mathbf{R}\dot{\mathbf{C}}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\dot{\lambda}\lambda\,\boldsymbol{\Omega}^T\mathbf{R}\mathbf{h}+\lambda^2\boldsymbol{\Omega}^T\mathbf{R}\dot{\mathbf{C}}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\dot{\lambda}\lambda\,\boldsymbol{\Omega}^T\mathbf{R}\mathbf{h}+\lambda^2\boldsymbol{\Omega}^T\mathbf{R}\dot{\mathbf{h}}-\\ &\quad-\frac{\ddot{s}}{\dot{s}}(\lambda^2\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\lambda^2\boldsymbol{\Omega}^T\mathbf{R}\mathbf{h})=\mathbf{0}\end{aligned} \tag{D28}$$

$$N\,\ddot{\mathbf{t}}-\frac{\ddot{s}}{\dot{s}}N\,\dot{\mathbf{t}}=\mathbf{0} \tag{D29}$$

$$\ddot{\lambda}\,\mathbf{z}^T\mathbf{z}-\lambda\,\dot{\boldsymbol{\theta}}^T\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}-2\lambda\,\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}-\lambda\,\dot{\mathbf{z}}^T\dot{\mathbf{z}}-\frac{\ddot{s}}{\dot{s}}(\dot{\lambda}\,\mathbf{z}^T\mathbf{z})=0\,. \tag{D30}$$

For the sake of completeness we evaluate the second geodesic equation (D2) corresponding to the time coordinate t, although it will not be used. The relevant remaining terms are

$$\mathbf{g}^T\ddot{\mathbf{p}}=\lambda^2\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}} \tag{D31}$$

$$\mathbf{g}^T\dot{\mathbf{p}}+\gamma=\lambda^2\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\lambda^2\dot{\mathbf{z}}^T\dot{\mathbf{z}} \tag{D32}$$

$$\left\{\frac{\partial\boldsymbol{\Omega}^T}{\partial\theta_m}\mathbf{R}\mathbf{h}\right\}\dot{\boldsymbol{\theta}}=\sum_m\dot{\theta}_m\frac{\partial\boldsymbol{\Omega}^T}{\partial\theta_m}\mathbf{R}\mathbf{h}=\dot{\boldsymbol{\Omega}}^T\mathbf{R}\mathbf{h} \tag{D33}$$

$$\widetilde{\mathbf{H}}\dot{\mathbf{p}}=\begin{bmatrix} -\lambda^2\boldsymbol{\Omega}^T\mathbf{R}\dot{\mathbf{C}}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\lambda^2\dot{\boldsymbol{\Omega}}^T\mathbf{R}\mathbf{h}+\dot{\lambda}\lambda\,\boldsymbol{\Omega}^T\mathbf{R}\mathbf{h} \\ \mathbf{0} \\ \lambda\,\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}} \end{bmatrix} \tag{D34}$$

$$\begin{aligned}\dot{\mathbf{p}}^T\widetilde{\mathbf{H}}\dot{\mathbf{p}}&=-\lambda^2\dot{\boldsymbol{\theta}}^T\boldsymbol{\Omega}^T\mathbf{R}\dot{\mathbf{C}}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\lambda^2\dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\Omega}}^T\mathbf{R}\mathbf{h}+\dot{\lambda}\lambda\,\dot{\boldsymbol{\theta}}^T\boldsymbol{\Omega}^T\mathbf{R}\mathbf{h}+\dot{\lambda}\lambda\,\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}=\\ &=-\lambda^2\dot{\boldsymbol{\theta}}^T\boldsymbol{\Omega}^T\mathbf{R}\dot{\mathbf{C}}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\lambda^2\mathbf{h}^T\mathbf{R}^T\dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}}+2\dot{\lambda}\lambda\,\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}\end{aligned} \tag{D35}$$

$$\boldsymbol{\kappa}^T\dot{\mathbf{p}}=\dot{\lambda}\lambda\,\dot{\mathbf{z}}^T\dot{\mathbf{z}} \tag{D36}$$

$$\tilde{\mathbf{h}}_t^T \dot{\mathbf{p}} + \kappa_t = \dot{\lambda}\lambda\, \dot{\mathbf{z}}^T \dot{\mathbf{z}} + \lambda^2 \dot{\mathbf{z}}^T \ddot{\mathbf{z}} \tag{D37}$$

which replaced in equation (D2) give

$$\lambda^2 \mathbf{h}^T \mathbf{R}^T \boldsymbol{\Omega}\ddot{\boldsymbol{\theta}} - \lambda^2 \dot{\boldsymbol{\theta}}^T \boldsymbol{\Omega}^T \mathbf{R}\dot{\mathbf{C}}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + \lambda^2 \mathbf{h}^T \mathbf{R}^T \dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}} + 2\dot{\lambda}\lambda\, \mathbf{h}^T \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + 2\dot{\lambda}\lambda\, \dot{\mathbf{z}}^T \dot{\mathbf{z}} + \lambda^2 \dot{\mathbf{z}}^T \ddot{\mathbf{z}} -$$

$$- \frac{\ddot{s}}{\dot{s}} (\lambda^2 \mathbf{h}^T \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + \lambda^2 \dot{\mathbf{z}}^T \dot{\mathbf{z}}) = 0 \tag{D38}$$

It remains to compute the factor $\frac{\ddot{s}}{\dot{s}} = \frac{\dot{\mathbf{x}}^T \ddot{\mathbf{x}}}{\dot{\mathbf{x}}^T \dot{\mathbf{x}}}$ for which we need

$$\dot{\mathbf{x}}_i = \frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}} \dot{\boldsymbol{\theta}} + \frac{\partial \mathbf{x}_i}{\partial \mathbf{t}} \dot{\mathbf{t}} + \frac{\partial \mathbf{x}_i}{\partial \lambda} \dot{\lambda} + \frac{\partial \mathbf{x}_i}{\partial t} = -\lambda\, \mathbf{R}[\mathbf{z}_i \times] \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + \dot{\mathbf{t}} + \dot{\lambda}\, \mathbf{R}\mathbf{z}_i + \lambda\, \mathbf{R}\dot{\mathbf{z}}_i \tag{D39}$$

so that

$$\dot{\mathbf{x}}^T \dot{\mathbf{x}} = \sum_i \dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_i = \lambda^2 \mathbf{z}^T \mathbf{z} + \lambda^2 \dot{\mathbf{z}}^T \dot{\mathbf{z}} + 2\lambda^2 \mathbf{h}^T \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + \lambda^2 \dot{\boldsymbol{\theta}}^T \boldsymbol{\Omega}^T \mathbf{R}\mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + N \dot{\mathbf{t}}^T \dot{\mathbf{t}} \tag{D40}$$

and differentiating

$$\dot{\mathbf{x}}^T \ddot{\mathbf{x}} = \frac{1}{2} \frac{d(\dot{\mathbf{x}}^T \dot{\mathbf{x}})}{dt} = \ddot{\lambda}\dot{\lambda}\, \mathbf{z}^T \mathbf{z} + \dot{\lambda}\lambda\, \dot{\mathbf{z}}^T \dot{\mathbf{z}} + \lambda^2 \dot{\mathbf{z}}^T \ddot{\mathbf{z}} + 2\dot{\lambda}\lambda\, \mathbf{h}^T \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + \lambda^2 \mathbf{h}^T \mathbf{R}^T (\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}} + \dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}}) + \lambda^2 \dot{\mathbf{h}}^T \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} +$$

$$+ \dot{\lambda}\lambda\, \dot{\boldsymbol{\theta}}^T \boldsymbol{\Omega}^T \mathbf{R}\mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + \frac{1}{2} \lambda^2 \dot{\boldsymbol{\theta}}^T \boldsymbol{\Omega}^T \mathbf{R}\dot{\mathbf{C}}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + \lambda^2 \dot{\boldsymbol{\theta}}^T \boldsymbol{\Omega}^T \mathbf{R}\mathbf{C}_z \mathbf{R}^T (\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}} + \dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}}) + N\, \dot{\mathbf{t}}^T \ddot{\mathbf{t}} \tag{D41}$$

Appendix E: Derivation of geodesics on the space-time manifold as minimum energy curves

We shall derive the differential equations of the curves on the space-time manifold $\mathcal{M}$ which minimize

$$\int_0^t L\, d\tau = \text{minimum} \tag{E1}$$

where

$$L = \frac{ds^2}{d\tau^2} = \dot{\mathbf{x}}^T \dot{\mathbf{x}} = 2T \tag{E2}$$

i.e. a factor 2 has been included for convenience in the kinetic energy $T = \frac{1}{2} v^2 = \frac{1}{2} |\dot{\mathbf{x}}|^2$. When the curve sought is described with respect to the arc length s, the well known Euler-Lagrange equations from the calculus of variations are

$$\frac{\partial L}{\partial u_i} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{u}_i}\right) = 0 \tag{E3}$$

or in more convenient matrix form

$$\left(\frac{\partial L}{\partial \mathbf{u}}\right)^T - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\mathbf{u}}}\right)^T = \mathbf{0}\,. \tag{E4}$$

The curve is intrinsically described by $\mathbf{u}(s) = [\boldsymbol{\theta}(s)^T\ \mathbf{t}(s)^T\ \lambda(s)\ t(s)]^T$ and the above equations can be separated into the following sets

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\boldsymbol{\theta}}}\right)^T - \left(\frac{\partial L}{\partial \boldsymbol{\theta}}\right)^T = \mathbf{0} \qquad (\boldsymbol{\theta}\text{ - equation}) \tag{E5}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\mathbf{t}}}\right)^T - \left(\frac{\partial L}{\partial \mathbf{t}}\right)^T = \mathbf{0} \qquad (\mathbf{t}\text{ - equation}) \tag{E6}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\lambda}}\right)^T - \left(\frac{\partial L}{\partial \lambda}\right)^T = 0 \qquad (\lambda\text{ - equation}) \tag{E7}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{t}}\right)^T - \left(\frac{\partial L}{\partial t}\right)^T = 0\,. \qquad (t\text{ - equation}) \tag{E8}$$

Using the notation $\dot{q} = \frac{dq}{ds}$ and $q' = \frac{dq}{dt}$ for the derivatives with respect to arc length and time, respectively, the function $L(\mathbf{u}, \dot{\mathbf{u}}) = L(\boldsymbol{\theta}, \mathbf{t}, \lambda, t, \dot{\boldsymbol{\theta}}, \dot{\mathbf{t}}, \dot{\lambda})$ is given by

$$L = \dot{\mathbf{u}}^T \overline{\mathbf{G}} \dot{\mathbf{u}} = [\dot{\mathbf{p}}^T \dot{t}] \begin{bmatrix} \mathbf{G} & \mathbf{g} \\ \mathbf{g}^T & \gamma \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}} \\ \dot{t} \end{bmatrix} = \dot{\mathbf{p}}^T \mathbf{G} \dot{\mathbf{p}} + 2\dot{t}\, \mathbf{g}^T \dot{\mathbf{p}} + \dot{t}^2 \gamma =$$

$$= \dot{\lambda}^2 \mathbf{z}^T \mathbf{z} + \dot{t}^2 \lambda^2 (\mathbf{z}'^T \mathbf{z}') + \lambda^2 \dot{\boldsymbol{\theta}}^T \boldsymbol{\Omega}^T \mathbf{R} \mathbf{C}_z \mathbf{R} \boldsymbol{\Omega} \dot{\boldsymbol{\theta}} + 2\dot{t} \lambda^2 \mathbf{h}^T \mathbf{R}^T \boldsymbol{\Omega} \dot{\boldsymbol{\theta}} + N\, \dot{\mathbf{t}}^T \dot{\mathbf{t}} \tag{E9}$$

where

$$\mathbf{z}'_i = \frac{d\mathbf{z}_i}{dt}, \qquad \boldsymbol{\omega}_s = \boldsymbol{\Omega}\dot{\boldsymbol{\theta}}. \tag{E10}$$

For the $\boldsymbol{\theta}$ - equation we have

$$\frac{\partial L}{\partial \dot{\boldsymbol{\theta}}} = 2\lambda^2 \dot{\boldsymbol{\theta}}^T \boldsymbol{\Omega}^T \mathbf{R}\mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega} + 2\dot{t}\lambda^2 \mathbf{h}^T \mathbf{R}^T \boldsymbol{\Omega} \tag{E11}$$

$$\left(\frac{\partial L}{\partial \dot{\boldsymbol{\theta}}}\right)^T = 2\lambda^2 \boldsymbol{\Omega}^T \mathbf{R}\mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + 2\dot{t}\lambda^2 \boldsymbol{\Omega}^T \mathbf{R}\mathbf{h} = 2\lambda^2 \boldsymbol{\Omega}^T \mathbf{R}(\dot{t}\mathbf{h} + \mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}}) \tag{E12}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\boldsymbol{\theta}}}\right)^T = 4\dot{\lambda}\lambda\, \boldsymbol{\Omega}^T \mathbf{R}(\dot{t}\,\mathbf{h} + \mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}}) + 2\lambda^2 \dot{\boldsymbol{\Omega}}^T \mathbf{R}(\dot{t}\,\mathbf{h} + \mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}}) + 2\lambda^2 \boldsymbol{\Omega}^T [\boldsymbol{\omega}_s \times] \mathbf{R}(\dot{t}\,\mathbf{h} + \mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}}) +$$

$$+ 2\lambda^2 \boldsymbol{\Omega}^T \left(\ddot{t}\,\mathbf{h} + \dot{t}^2 \mathbf{h}' + \dot{t}\,\mathbf{C}'_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} - \mathbf{C}_z \mathbf{R}^T [\boldsymbol{\omega}_s \times] \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + \mathbf{C}_z \mathbf{R}^T (\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}} + \dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}})\right) =$$

$$= 2\lambda \left(2\dot{\lambda}\, \boldsymbol{\Omega}^T + \lambda\, \dot{\boldsymbol{\Omega}}^T + \lambda\, \boldsymbol{\Omega}^T [\boldsymbol{\omega}_s \times]\right) \mathbf{R} \left(\dot{t}\,\mathbf{h} + \mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}}\right) +$$

$$+ 2\lambda^2 \boldsymbol{\Omega}^T \mathbf{R}\left(\ddot{t}\,\mathbf{h} + \dot{t}^2 \mathbf{h}' + \dot{t}\,\mathbf{C}'_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + \mathbf{C}_z \mathbf{R}^T (\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}} + \dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}})\right) \tag{E13}$$

$$\frac{\partial L}{\partial \theta_m} = 2\lambda^2 \boldsymbol{\omega}_s^T \mathbf{R}\mathbf{C}_z \mathbf{R}^T \frac{\partial \boldsymbol{\omega}_s}{\partial \theta_m} + \lambda^2 \boldsymbol{\omega}_s^T [\boldsymbol{\omega}_m \times] \mathbf{R}\mathbf{C}_z \mathbf{R}^T \boldsymbol{\omega}_s - \lambda^2 \boldsymbol{\omega}_s^T \mathbf{R}\mathbf{C}_z \mathbf{R}^T [\boldsymbol{\omega}_m \times] \boldsymbol{\omega}_s -$$

$$- 2\dot{t}\lambda^2 \mathbf{h}^T \mathbf{R}^T [\boldsymbol{\omega}_m \times] \boldsymbol{\omega}_s + 2\dot{t}\lambda^2 \mathbf{h}^T \mathbf{R}^T \frac{\partial \boldsymbol{\omega}_s}{\partial \theta_m} =$$

$$= 2\lambda^2 \boldsymbol{\omega}_s^T \mathbf{R}\mathbf{C}_z \mathbf{R}^T \frac{\partial \boldsymbol{\omega}_s}{\partial \theta_m} + 2\lambda^2 \boldsymbol{\omega}_s^T \mathbf{R}\mathbf{C}_z \mathbf{R}^T [\boldsymbol{\omega}_s \times] \boldsymbol{\omega}_m + 2\dot{t}\lambda^2 \mathbf{h}^T \mathbf{R}^T [\boldsymbol{\omega}_s \times] \boldsymbol{\omega}_m + 2\dot{t}\lambda^2 \mathbf{h}^T \mathbf{R}^T \frac{\partial \boldsymbol{\omega}_s}{\partial \theta_m} =$$

$$= 2\lambda^2 \left(\dot{t}\,\mathbf{h} + \mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}}\right)^T \mathbf{R}^T \frac{\partial \boldsymbol{\omega}_s}{\partial \theta_m} + 2\lambda^2 \left(\dot{t}\,\mathbf{h} + \mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}}\right)^T \mathbf{R}^T [\boldsymbol{\omega}_s \times] \boldsymbol{\omega}_m \tag{E14}$$

$$\frac{\partial L}{\partial \boldsymbol{\theta}} = 2\lambda^2 (\dot{t}\,\mathbf{h} + \mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}})^T \mathbf{R}^T \frac{\partial \boldsymbol{\omega}_s}{\partial \boldsymbol{\theta}} + 2\lambda^2 (\dot{t}\,\mathbf{h} + \mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}})^T \mathbf{R}^T [\boldsymbol{\omega}_s \times] \boldsymbol{\Omega} \tag{E15}$$

$$\left(\frac{\partial L}{\partial \boldsymbol{\theta}}\right)^T = 2\lambda^2 \left(\frac{\partial \boldsymbol{\omega}_s}{\partial \boldsymbol{\theta}}\right)^T \mathbf{R}(\dot{t}\,\mathbf{h} + \mathbf{C}_z \mathbf{R}^T \boldsymbol{\omega}_s) - 2\lambda^2 \boldsymbol{\Omega}^T [\boldsymbol{\omega}_s \times] \mathbf{R}(\dot{t}\,\mathbf{h} + \mathbf{C}_z \mathbf{R}^T \boldsymbol{\omega}_s) \tag{E16}$$

We need an expression for $\frac{\partial \boldsymbol{\omega}_s}{\partial \boldsymbol{\theta}}$. Since $\frac{\partial^2}{\partial \theta_m \partial s}\mathbf{R} = \frac{\partial^2}{\partial s \partial \theta_m}\mathbf{R}$ we have

$$\dot{\mathbf{R}} = [\boldsymbol{\omega}_s \times] \mathbf{R}, \qquad \frac{\partial \mathbf{R}}{\partial \theta_m} = [\boldsymbol{\omega}_m \times] \mathbf{R}, \tag{E17}$$

$$\frac{\partial^2}{\partial \theta_m \partial s}\mathbf{R} = \frac{\partial}{\partial \theta_m}\dot{\mathbf{R}} = [\frac{\partial \boldsymbol{\omega}_s}{\partial \theta_m} \times] \mathbf{R} + [\boldsymbol{\omega}_s \times] \frac{\partial \mathbf{R}}{\partial \theta_m} = [\frac{\partial \boldsymbol{\omega}_s}{\partial \theta_m} \times] \mathbf{R} + [\boldsymbol{\omega}_s \times][\boldsymbol{\omega}_m \times] \mathbf{R} \tag{E18}$$

$$\frac{\partial^2}{\partial s \partial \theta_m}\mathbf{R} = \frac{\partial}{\partial s}\left([\boldsymbol{\omega}_m \times] \mathbf{R}\right) = [\dot{\boldsymbol{\omega}}_m \times] \mathbf{R} + [\boldsymbol{\omega}_m \times] \dot{\mathbf{R}} = [\dot{\boldsymbol{\omega}}_m \times] \mathbf{R} + [\boldsymbol{\omega}_m \times][\boldsymbol{\omega}_s \times] \mathbf{R} \tag{E19}$$

The left sides of the above two equations are equal and so will be the right sides

$$[\frac{\partial \boldsymbol{\omega}_s}{\partial \theta_m}\times]\mathbf{R}+[\boldsymbol{\omega}_s\times][\boldsymbol{\omega}_m\times]\mathbf{R}=[\dot{\boldsymbol{\omega}}_m\times]\mathbf{R}+[\boldsymbol{\omega}_m\times][\boldsymbol{\omega}_s\times]\mathbf{R} \tag{E20}$$

$$[\frac{\partial \boldsymbol{\omega}_s}{\partial \theta_m}\times]+[\boldsymbol{\omega}_s\times][\boldsymbol{\omega}_m\times]=[\dot{\boldsymbol{\omega}}_m\times]+[\boldsymbol{\omega}_m\times][\boldsymbol{\omega}_s\times] \tag{E21}$$

$$[\frac{\partial \boldsymbol{\omega}_s}{\partial \theta_m}\times]=[\dot{\boldsymbol{\omega}}_m\times]+[\boldsymbol{\omega}_m\times][\boldsymbol{\omega}_s\times]-[\boldsymbol{\omega}_s\times][\boldsymbol{\omega}_m\times]=[\dot{\boldsymbol{\omega}}_m\times]+\boldsymbol{\omega}_s\boldsymbol{\omega}_m^T-\boldsymbol{\omega}_m\boldsymbol{\omega}_s^T=[\dot{\boldsymbol{\omega}}_m\times]+[([\boldsymbol{\omega}_m\times]\boldsymbol{\omega}_s)\times] \tag{E22}$$

where we have used the identity $\mathbf{b}\mathbf{a}^T-\mathbf{a}\mathbf{b}^T=[([\mathbf{a}\times]\mathbf{b})\times]$. Therefore

$$\frac{\partial \boldsymbol{\omega}_s}{\partial \theta_m}=\dot{\boldsymbol{\omega}}_m+[\boldsymbol{\omega}_m\times]\boldsymbol{\omega}_s=\dot{\boldsymbol{\omega}}_m-[\boldsymbol{\omega}_s\times]\boldsymbol{\omega}_m \qquad \Rightarrow \tag{E23}$$

$$\frac{\partial \boldsymbol{\omega}_s}{\partial \boldsymbol{\theta}}=\dot{\boldsymbol{\Omega}}-[\boldsymbol{\omega}_s\times]\boldsymbol{\Omega}, \qquad \left(\frac{\partial \boldsymbol{\omega}_s}{\partial \boldsymbol{\theta}}\right)^T=\dot{\boldsymbol{\Omega}}^T+\boldsymbol{\Omega}^T[\boldsymbol{\omega}_s\times], \tag{E24}$$

$$\left(\frac{\partial L}{\partial \boldsymbol{\theta}}\right)^T=2\lambda^2\left(\dot{\boldsymbol{\Omega}}^T+\boldsymbol{\Omega}^T[\boldsymbol{\omega}_s\times]\right)\mathbf{R}(\dot{t}\,\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\boldsymbol{\omega}_s)-2\lambda^2\boldsymbol{\Omega}^T[(\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})\times]\,\mathbf{R}(\dot{t}\,\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\boldsymbol{\omega}_s)=$$

$$=2\lambda^2\dot{\boldsymbol{\Omega}}^T\mathbf{R}(\dot{t}\,\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}) \tag{E25}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\boldsymbol{\theta}}}\right)^T-\left(\frac{\partial L}{\partial \boldsymbol{\theta}}\right)^T=2\lambda\left(2\dot{\lambda}\,\boldsymbol{\Omega}^T+\lambda\,\dot{\boldsymbol{\Omega}}^T+\lambda\,\boldsymbol{\Omega}^T[(\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})\times]\right)\mathbf{R}\left(\dot{t}\,\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}\right)+$$
$$+2\lambda^2\boldsymbol{\Omega}^T\mathbf{R}\left(\ddot{t}\,\mathbf{h}+\dot{t}^2\mathbf{h}'+\dot{t}\,\mathbf{C}_z'\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\mathbf{C}_z\mathbf{R}^T(\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}}+\dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}})\right)-2\lambda^2\dot{\boldsymbol{\Omega}}^T\mathbf{R}(\dot{t}\,\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})=$$

$$=2\lambda\left(2\dot{\lambda}\,\boldsymbol{\Omega}^T+\lambda\,\boldsymbol{\Omega}^T[(\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})\times]\right)\mathbf{R}\left(\dot{t}\,\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}\right)+$$
$$+2\lambda^2\boldsymbol{\Omega}^T\mathbf{R}\left(\ddot{t}\,\mathbf{h}+\dot{t}^2\mathbf{h}'+\dot{t}\,\mathbf{C}_z'\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\mathbf{C}_z\mathbf{R}^T(\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}}+\dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}})\right)=\mathbf{0} \tag{E26}$$

$$\left(2\dot{\lambda}\,\boldsymbol{\Omega}^T+\lambda\,\boldsymbol{\Omega}^T[(\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})\times]\right)\mathbf{R}\left(\dot{t}\,\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}\right)+$$
$$+\lambda\,\boldsymbol{\Omega}^T\mathbf{R}\left(\ddot{t}\,\mathbf{h}+\dot{t}^2\mathbf{h}'+\dot{t}\,\mathbf{C}_z'\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\mathbf{C}_z\mathbf{R}^T(\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}}+\dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}})\right)=\mathbf{0} \tag{E27}$$

It is possible to switch from the arc length s to any other parameter τ, using the relations

$$\dot{t}=t'\dot{\tau}, \qquad \dot{\boldsymbol{\theta}}=\boldsymbol{\theta}'\dot{\tau}, \qquad \dot{\boldsymbol{\Omega}}=\boldsymbol{\Omega}'\dot{\tau}, \qquad \dot{\lambda}=\lambda'\dot{\tau}, \tag{E28}$$

$$\ddot{t}=\ddot{\tau}\,t'+\dot{\tau}^2t''=\dot{\tau}^2\left(\frac{\ddot{\tau}}{\dot{\tau}^2}t'+t''\right)=\dot{\tau}^2\left(t''-\frac{s''}{s'}t'\right) \tag{E29}$$

$$\ddot{\boldsymbol{\theta}}=\dot{\tau}^2\left(\boldsymbol{\theta}''-\frac{s''}{s'}\boldsymbol{\theta}'t'\right), \qquad \ddot{\lambda}=\dot{\tau}^2\left(\lambda''-\frac{s''}{s'}\lambda'\right). \tag{E30}$$

In our very special case where $\tau=t$, we have $t'=1$, $t''=0$ and we simply must replace

$$\dot{t}=\dot{\tau}, \qquad \dot{\boldsymbol{\theta}}=\boldsymbol{\theta}'\dot{\tau}, \qquad \dot{\boldsymbol{\Omega}}=\boldsymbol{\Omega}'\dot{\tau}, \qquad \dot{\lambda}=\lambda'\dot{\tau}, \tag{E31}$$

$$\ddot{t}=-\dot{\tau}^2\frac{s''}{s'}, \qquad \ddot{\boldsymbol{\theta}}=\dot{\tau}^2\left(\boldsymbol{\theta}''-\frac{s''}{s'}\boldsymbol{\theta}'\right), \qquad \ddot{\lambda}=\dot{\tau}^2\left(\lambda''-\frac{s''}{s'}\lambda'\right) \tag{E32}$$

to obtain after dividing with $\dot{\tau}^2$

$$\left(2\lambda'\,\mathbf{\Omega}^T+\lambda\,\mathbf{\Omega}^T[(\mathbf{\Omega\theta}')\times]\right)\mathbf{R}\left(\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\mathbf{\Omega\theta}'\right)+\lambda\,\mathbf{\Omega}^T\mathbf{R}\left(\mathbf{h}'+\mathbf{C}_z'\mathbf{R}^T\mathbf{\Omega\theta}'+\mathbf{C}_z\mathbf{R}^T(\mathbf{\Omega\theta}''+\mathbf{\Omega}'\mathbf{\theta}')\right)-$$
$$-\frac{s''}{s'}\lambda\,\mathbf{\Omega}^T\mathbf{R}\left(\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\mathbf{\Omega\theta}'\right)=\mathbf{0} \qquad \text{(E33)}$$

Assuming further that $|\mathbf{\Omega}|\neq 0$ the above equation reduces further to

$$\left(2\lambda'+\lambda[(\mathbf{R}^T\mathbf{\Omega\theta}')\times]\right)\left(\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\mathbf{\Omega\theta}'\right)+\lambda\left(\mathbf{h}'+\mathbf{C}_z'\mathbf{R}^T\mathbf{\Omega\theta}'+\mathbf{C}_z\mathbf{R}^T(\mathbf{\Omega\theta}''+\mathbf{\Omega}'\mathbf{\theta}')\right)-$$
$$-\frac{s''}{s'}\lambda\left(\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\mathbf{\Omega\theta}'\right)=\mathbf{0} \qquad \text{(E34)}$$

For the **t** - equation we have:

$$\frac{\partial L}{\partial\dot{\mathbf{t}}}=2N\,\dot{\mathbf{t}}^T \qquad \text{(E35)}$$

$$\left(\frac{\partial L}{\partial\dot{\mathbf{t}}}\right)^T=2N\,\dot{\mathbf{t}} \qquad \text{(E36)}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial\dot{\mathbf{t}}}\right)^T=2N\,\ddot{\mathbf{t}} \qquad \text{(E37)}$$

$$\frac{\partial L}{\partial t_m}=0\,, \qquad \left(\frac{\partial L}{\partial\mathbf{t}}\right)^T=\mathbf{0} \qquad \text{(E38)}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial\dot{\mathbf{t}}}\right)^T-\left(\frac{\partial L}{\partial\mathbf{t}}\right)^T=2N\,\ddot{\mathbf{t}}=\mathbf{0} \qquad \text{(E39)}$$

$$\ddot{\mathbf{t}}=\mathbf{0}\,. \qquad \text{(E40)}$$

In order to switch from s to t we must use

$$\ddot{\mathbf{t}}=\dot{\tau}^2\left(\mathbf{t}''-\frac{s''}{s'}\mathbf{t}'\right) \qquad \text{(E41)}$$

which gives

$$\mathbf{t}''-\frac{s''}{s'}\mathbf{t}'=\mathbf{0}\,. \qquad \text{(E42)}$$

For the λ - equation we have

$$\frac{\partial L}{\partial\dot{\lambda}}=2\dot{\lambda}\,\mathbf{z}^T\mathbf{z} \qquad \text{(E43)}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial\dot{\lambda}}\right)=2\ddot{\lambda}\,\mathbf{z}^T\mathbf{z}+4\dot{t}\dot{\lambda}\,\mathbf{z}^T\mathbf{z}'=2\ddot{\lambda}\,\mathbf{z}^T\mathbf{z} \qquad \text{(E44)}$$

$$\frac{\partial L}{\partial \lambda}=2\dot{t}^2\lambda\,\mathbf{z}'^T\mathbf{z}'+2\lambda\,\dot{\boldsymbol{\theta}}^T\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+4\dot{t}\lambda\,\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}} \tag{E45}$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\lambda}}\right)-\frac{\partial L}{\partial \lambda}=2\ddot{\lambda}\,\mathbf{z}^T\mathbf{z}-2\dot{t}^2\lambda\,\mathbf{z}'^T\mathbf{z}'-2\lambda\,\dot{\boldsymbol{\theta}}^T\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}-4\dot{t}\lambda\,\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}=0 \tag{E46}$$

$$\ddot{\lambda}\,\mathbf{z}^T\mathbf{z}-\dot{t}^2\lambda\,\mathbf{z}'^T\mathbf{z}'-\lambda\,\dot{\boldsymbol{\theta}}^T\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}-2\dot{t}\lambda\,\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}=0 \tag{E47}$$

Switching from s to t we obtain as before

$$\lambda''\,\mathbf{z}^T\mathbf{z}-\lambda\,\mathbf{z}'^T\mathbf{z}'-\lambda\,\boldsymbol{\theta}'^T\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\boldsymbol{\theta}'-2\lambda\,\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\boldsymbol{\theta}'-\frac{s''}{s'}\lambda'\,\mathbf{z}^T\mathbf{z}=0 \tag{E48}$$

For the t - equation we have:

$$\frac{\partial L}{\partial \dot{t}}=2\dot{t}\lambda^2\,\mathbf{z}'^T\mathbf{z}'+2\lambda^2\,\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}} \tag{E49}$$

$$\begin{aligned}\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{t}}\right)&=2\ddot{t}\lambda^2\,\mathbf{z}'^T\mathbf{z}'+4\dot{t}\dot{\lambda}\lambda\,\mathbf{z}'^T\mathbf{z}'+4\dot{t}^2\lambda^2\mathbf{z}'^T\mathbf{z}''+4\dot{\lambda}\lambda\,\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+2\dot{t}\lambda^2\mathbf{h}'^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}-\\&\quad-2\lambda^2\mathbf{h}^T\mathbf{R}^T[\boldsymbol{\omega}_s\times]\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+2\lambda^2\mathbf{h}^T\mathbf{R}^T(\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}}+\dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}})=\\&=2\ddot{t}\lambda^2\,\mathbf{z}'^T\mathbf{z}'+4\dot{t}\dot{\lambda}\lambda\,\mathbf{z}'^T\mathbf{z}'+4\dot{t}^2\lambda^2\mathbf{z}'^T\mathbf{z}''+4\dot{\lambda}\lambda\,\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+2\dot{t}\lambda^2\mathbf{h}'^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\\&\quad+2\lambda^2\mathbf{h}^T\mathbf{R}^T(\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}}+\dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}})\end{aligned} \tag{E50}$$

$$\begin{aligned}\frac{\partial L}{\partial t}&=2\dot{\lambda}^2\mathbf{z}^T\mathbf{z}'+2\dot{t}^2\lambda^2\mathbf{z}'^T\mathbf{z}''+\lambda^2\boldsymbol{\omega}_s^T\mathbf{R}\mathbf{C}'_z\mathbf{R}^T\boldsymbol{\omega}_s+2\dot{t}\lambda^2\mathbf{h}'^T\mathbf{R}^T\boldsymbol{\omega}_s=\\&=2\dot{t}^2\lambda^2\mathbf{z}'^T\mathbf{z}''+\lambda^2\dot{\boldsymbol{\theta}}^T\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}'_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+2\dot{t}\lambda^2\mathbf{h}'^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}\end{aligned} \tag{E51}$$

$$\begin{aligned}\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{t}}\right)-\frac{\partial L}{\partial t}&=2\ddot{t}\lambda^2\,\mathbf{z}'^T\mathbf{z}'+4\dot{t}\dot{\lambda}\lambda\,\mathbf{z}'^T\mathbf{z}'+4\dot{t}^2\lambda^2\mathbf{z}'^T\mathbf{z}''+4\dot{\lambda}\lambda\,\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+2\dot{t}\lambda^2\mathbf{h}'^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\\&\quad+2\lambda^2\mathbf{h}^T\mathbf{R}^T(\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}}+\dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}})-2\dot{t}^2\lambda^2\mathbf{z}'^T\mathbf{z}''-\lambda^2\dot{\boldsymbol{\theta}}^T\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}'_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}-2\dot{t}\lambda^2\mathbf{h}'^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}=\\&=2\ddot{t}\lambda^2\,\mathbf{z}'^T\mathbf{z}'+4\dot{t}\dot{\lambda}\lambda\,\mathbf{z}'^T\mathbf{z}'+2\dot{t}^2\lambda^2\mathbf{z}'^T\mathbf{z}''+4\dot{\lambda}\lambda\,\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+2\lambda^2\mathbf{h}^T\mathbf{R}^T(\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}}+\dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}})-\\&\quad-\lambda^2\dot{\boldsymbol{\theta}}^T\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}'_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}=0\end{aligned} \tag{E52}$$

$$\begin{aligned}&\ddot{t}\lambda^2\,\mathbf{z}'^T\mathbf{z}'+2\dot{t}\dot{\lambda}\lambda\,\mathbf{z}'^T\mathbf{z}'+\dot{t}^2\lambda^2\mathbf{z}'^T\mathbf{z}''+2\dot{\lambda}\lambda\,\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\lambda^2\mathbf{h}^T\mathbf{R}^T(\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}}+\dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}})-\\&\quad-\frac{1}{2}\lambda^2\dot{\boldsymbol{\theta}}^T\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}'_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}=0\end{aligned} \tag{E53}$$

Switching from s to t we obtain as before

$$\begin{aligned}&-\dot{\tau}^2\frac{s''}{s'}\lambda^2\,\mathbf{z}'^T\mathbf{z}'+2\dot{\tau}\lambda'\dot{\tau}\lambda\,\mathbf{z}'^T\mathbf{z}'+\dot{\tau}^2\lambda^2\mathbf{z}'^T\mathbf{z}''+2\lambda'\dot{\tau}\lambda\,\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\boldsymbol{\theta}'\dot{\tau}+\lambda^2\mathbf{h}^T\mathbf{R}^T\left(\boldsymbol{\Omega}\dot{\tau}^2(\boldsymbol{\theta}''-\frac{s''}{s'}\boldsymbol{\theta}')+\boldsymbol{\Omega}'\dot{\tau}\,\boldsymbol{\theta}'\dot{\tau}\right)-\\&\quad-\frac{1}{2}\lambda^2(\boldsymbol{\theta}'^T\dot{\tau})\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}'_z\mathbf{R}^T\boldsymbol{\Omega}\boldsymbol{\theta}'\dot{\tau}=0\end{aligned} \tag{E54}$$

$$2\lambda'\lambda\,\mathbf{z}'^T\mathbf{z}'+\lambda^2\mathbf{z}'^T\mathbf{z}''+2\lambda'\lambda\,\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\boldsymbol{\theta}'+\lambda^2\mathbf{h}^T\mathbf{R}^T(\boldsymbol{\Omega}\boldsymbol{\theta}''+\boldsymbol{\Omega}'\boldsymbol{\theta}')-\frac{1}{2}\lambda^2\boldsymbol{\theta}'^T\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}'_z\mathbf{R}^T\boldsymbol{\Omega}\boldsymbol{\theta}'-$$

$$-\frac{s''}{s'}(\lambda^2\,\mathbf{z}'^T\mathbf{z}'+\lambda^2\mathbf{h}^T\mathbf{R}^T\mathbf{\Omega}\boldsymbol{\theta}')=0\,. \qquad \text{(E55)}$$

If dots are used instead to denote differentiation with respect to time (as everywhere else in this work) then the derived equations can be summarized into

$\boldsymbol{\theta}$ - equation:

$$\left(2\dot{\lambda}+\lambda\,[(\mathbf{R}^T\mathbf{\Omega}\dot{\boldsymbol{\theta}})\times]\right)\left(\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\mathbf{\Omega}\dot{\boldsymbol{\theta}}\right)+\lambda\left(\dot{\mathbf{h}}+\dot{\mathbf{C}}_z\mathbf{R}^T\mathbf{\Omega}\dot{\boldsymbol{\theta}}+\mathbf{C}_z\mathbf{R}^T(\mathbf{\Omega}\ddot{\boldsymbol{\theta}}+\dot{\mathbf{\Omega}}\dot{\boldsymbol{\theta}})\right)-$$
$$-\frac{\ddot{s}}{\dot{s}}\lambda\left(\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\mathbf{\Omega}\dot{\boldsymbol{\theta}}\right)=\mathbf{0} \qquad \text{(E56)}$$

t – equation:

$$\ddot{\mathbf{t}}-\frac{\ddot{s}}{\dot{s}}\dot{\mathbf{t}}=\mathbf{0} \qquad \text{(E57)}$$

λ - equation:

$$\ddot{\lambda}\,\mathbf{z}^T\mathbf{z}-\lambda\,\dot{\mathbf{z}}^T\dot{\mathbf{z}}-\lambda\,\dot{\boldsymbol{\theta}}^T\mathbf{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\mathbf{\Omega}\dot{\boldsymbol{\theta}}-2\lambda\,\mathbf{h}^T\mathbf{R}^T\mathbf{\Omega}\dot{\boldsymbol{\theta}}-\frac{\ddot{s}}{\dot{s}}\dot{\lambda}\,\mathbf{z}^T\mathbf{z}=0 \qquad \text{(E58)}$$

t – equation:

$$2\dot{\lambda}\lambda\,\dot{\mathbf{z}}^T\dot{\mathbf{z}}+\lambda^2\dot{\mathbf{z}}^T\ddot{\mathbf{z}}+2\dot{\lambda}\lambda\,\mathbf{h}^T\mathbf{R}^T\mathbf{\Omega}\dot{\boldsymbol{\theta}}+\lambda^2\mathbf{h}^T\mathbf{R}^T(\mathbf{\Omega}\ddot{\boldsymbol{\theta}}+\dot{\mathbf{\Omega}}\,\dot{\boldsymbol{\theta}})-\frac{1}{2}\lambda^2\dot{\boldsymbol{\theta}}^T\mathbf{\Omega}^T\mathbf{R}\dot{\mathbf{C}}_z\mathbf{R}^T\mathbf{\Omega}\dot{\boldsymbol{\theta}}-$$
$$-\frac{\ddot{s}}{\dot{s}}(\lambda^2\,\dot{\mathbf{z}}^T\dot{\mathbf{z}}+\lambda^2\mathbf{h}^T\mathbf{R}^T\mathbf{\Omega}\dot{\boldsymbol{\theta}})=0\,. \qquad \text{(E59)}$$