B. The Space-Time Datum problem

6. Introduction

The study of deformable networks has introduced one more dimension, that of time, in geodetic theory and applications. The typical approach is the repetition of observations at different time epochs leading to a discrete-time situation, where the datum problem must not only be solved at every single epoch but the solutions must be furthermore interrelated in a reasonable way. Although coordinate variations (displacements) are not the appropriate means for describing deformation and frame-invariant quantities should rather be used (see Dermanis & Grafarend, 1992), in practice they are a convenient means of description, in the same way that single epoch coordinates are a means of describing the shape of the network in the place of more appropriate frame invariant parameters. It is essential to solve the datum problem in a way that displacement between the different epochs are minimized thus avoiding to introduce additional "deformation" coming from the frame definition and not from the actual shape alteration.

Two obvious ways to treat the discrete time non linear datum problem are the following:

a. Solve the datum problem in some way at the initial epoch t_0 (or any other particular epoch) and use the solution $\mathbf{x}_0 = \mathbf{x}(t_0)$ to obtain a $\mathbf{x}_0$-nearest solution at all the remaining epochs. In this way the frames introduced at any epoch are as close as possible to the frame introduced for the epoch t_0.
b. Solve the datum problem at the initial epoch t_0 and for any next epoch t_k use the $\mathbf{x}_0$-nearest solution where $\mathbf{x}_0$ is the solution obtained at the previous epoch $\mathbf{x}_0 = \mathbf{x}(t_{k-1})$.

We shall study next the situation where instead of observations repeated at discrete epochs we have a continuous monitoring of a deformable network. This continuous-time situation is not just a theoretical abstraction. For example the Geographical Survey Institute of Japan has established such networks where electronic distance measurements are carried out continuously and transmitted to a computer center for further analysis.

The available observations are functions of time $y'(t)$ which due to errors do not belong to the model manifold $R(f)$. We do not treat here the adjustment problem which is an important problem by itself. We remark only that this cannot be based on the separate adjustment problem solutions for each epoch t. The reason is that if the white-noise model is adopted for the instantaneous observations, then $y'(t)$ is a white noise stochastic process (though with non zero mean function) with an erratic behavior. As a consequence the epoch-wise application of an adjustment principle like the least squares one will produce a function of adjusted observations $y(t)$ on $R(f)$ with the same erratic behavior. This is in contrast to a fundamental modeling requirement that the true function of the observables $y^*(t)$ (of which $y(t)$ is an estimate) should be continuous, or piecewise continuous to allow for discontinuities at the known epochs of earthquake occurrence. The solution to this adjustment problem needs a separate study, perhaps along the lines of the approach developed in Sanso & Sona (1995).

In any way, we will assume here that the adjustment problem has been solved in a way that produces a continuous function $y(t)$ such that $y(t) \in R(f)$ for every t and we shall concentrate on the solution of the space-time nonlinear datum problem in a situation that can be described as time-continuous and space-discrete, due to the discrete nature of any geodetic network. We seek a continuous curve $x(t)$ in X such that for every epoch t the point $x(t)$ belongs to the solution fiber of $y(t)$, i.e., $x(t) \in F_{y(t)}$.

The space-time datum problem for discrete-time has been first investigated by Cannon (1979) and Dermanis (1980) in connection to global VLBI networks.

7. Differential Geometry of the Space-Time Solution Manifold

The subspace $\mathcal{M}$ of the coordinate space X, formulated by the solution fibers $F_{y(t)}$ as the time parameters t runs in an open interval $I = (0,T) \subset R$, i.e.,

$$\mathcal{M} = \bigcup_t F_{y(t)} \tag{71}$$

is a submanifold of X which we will call the ***space-time solution manifold***. (We assume that the mapping $y:I\rightarrow R(f):t\rightarrow y(t)$ is a smooth one.) We shall view here X as the Euclidean space $E^m=E^{3N}$ of the coordinate vectors x (represented by column vectors $\mathbf{x}$) of a three-dimensional network with N points, i.e. as R^m equipped with the "flat" geometry induced by the inner product $(x_\alpha,x_\beta)=\mathbf{x}_\alpha^T\mathbf{x}_\beta$. The related metric induces a Riemannian metric on the curved manifold $\mathcal{M}$ and the corresponding Levi-Civita connection. The manifold $\mathcal{M}$ adopts (at least locally) a coordinate system as follows. Suppose an initial solution to the space-time datum problem is available in the form of a known smooth curve $\mathbf{z}(t)$ in X, obtained e.g. by introducing trivial minimal constraints. This curve belongs also to $\mathcal{M}$ and in particular for fixed t the point $\mathbf{z}(t)\in F_{y(t)}\subset\mathcal{M}$. Any other point on the same fiber can be uniquely expressed by

$$\mathbf{x}_i=\lambda\,\mathbf{R}(\boldsymbol{\theta})\mathbf{z}_i(t)+\mathbf{t}=\mathbf{x}_i(\mathbf{z}_i(t),\mathbf{p}),\qquad \mathbf{p}=\begin{bmatrix}\boldsymbol{\theta}\\ \mathbf{t}\\ \lambda\end{bmatrix}. \tag{72}$$

The transformation parameters $\boldsymbol{\theta}$, $\mathbf{t}$, λ, together with time t uniquely determine any point on the 8-dimensional manifold $\mathcal{M}$. Under this particular coordinate system, a solution to the space time datum problem is provided by a suitable choice of 7 functions of time $\boldsymbol{\theta}(t)$, $\mathbf{t}(t)$, $\lambda(t)$, which defines a curve $\mathbf{x}(t)$ on $\mathcal{M}$ determined by

$$\mathbf{x}_i(t)=\lambda(t)\,\mathbf{R}(\boldsymbol{\theta}(t))\mathbf{z}_i(t)+\mathbf{t}(t)=\mathbf{x}_i(\mathbf{z}_i(t),\mathbf{p}(t)) \tag{73}$$

such that for every epoch t, $f(x(t))=y(t)$, where $y(t)$ is the given point on the model manifold $R(f)$.
It remains is to establish criteria for the optimal choice of the solution curve $x(t)$ on $\mathcal{M}$.
We shall first determine the basic differential geometric characteristics on $\mathcal{M}$, i.e. the metric $\bar{g}$ and the connection $\overline{\nabla}$. The manifold $\mathcal{M}$ is defined as a subspace of X by an equation of the form $\mathbf{x}=\mathbf{x}(\mathbf{p},t)$, where $\mathbf{u}=[\mathbf{p}^T t]^T$ are the curvilinear coordinates on $\mathcal{M}$, which is defined pointwise by $\mathbf{x}_i=\mathbf{x}_i(\mathbf{p},t)=\lambda\,\mathbf{R}(\boldsymbol{\theta})\,\mathbf{z}_i(t)+\mathbf{t}$. Note that now $\mathbf{p}$ (i.e., $\boldsymbol{\theta}$, $\mathbf{t}$ and λ) are independent variables and not function of time, while $\mathbf{x}$ depends on t only through the vector $\mathbf{z}$ of the $\mathbf{z}_i$, $\mathbf{x}=\mathbf{x}(\mathbf{p},t)=\mathbf{x}(\mathbf{p},\mathbf{z}(t))$.
The local tangent space at any point of $\mathcal{M}$ is spanned by the local coordinate frame $\mathbf{e}_m=\frac{\partial\mathbf{x}}{\partial p_m}$, $m=1,\ldots,7$ and $\mathbf{e}_8=\frac{\partial\mathbf{x}}{\partial t}$, which we find more convenient to visualize by their extrinsic counterparts in X. The curvilinear coordinates $\mathbf{u}=[p_1\ldots p_7 t]^T$ of $\mathcal{M}$ will be used in order to express intrinsically the components of the metric $\bar{g}_{ik}(\mathbf{u})$ and the connection coefficients $\overline{\Gamma}^k_{mi}(\mathbf{u})$. Since the metric is inherited from the flat space X we get

$$ds^2=d\mathbf{x}^T d\mathbf{x}=d\mathbf{u}^T\left(\frac{\partial\mathbf{x}}{\partial\mathbf{u}}\right)^T\frac{\partial\mathbf{x}}{\partial\mathbf{u}}d\mathbf{u}=d\mathbf{u}^T\overline{\mathbf{G}}d\mathbf{u}, \tag{74}$$

$$\overline{\mathbf{G}}=\left(\frac{\partial\mathbf{x}}{\partial\mathbf{u}}\right)^T\frac{\partial\mathbf{x}}{\partial\mathbf{u}}=\begin{bmatrix}\frac{\partial\mathbf{x}}{\partial\mathbf{p}}&\frac{\partial\mathbf{x}}{\partial t}\end{bmatrix}^T\begin{bmatrix}\frac{\partial\mathbf{x}}{\partial\mathbf{p}}&\frac{\partial\mathbf{x}}{\partial t}\end{bmatrix}=\begin{bmatrix}\left(\frac{\partial\mathbf{x}}{\partial\mathbf{p}}\right)^T\frac{\partial\mathbf{x}}{\partial\mathbf{p}} & \left(\frac{\partial\mathbf{x}}{\partial\mathbf{p}}\right)^T\frac{\partial\mathbf{x}}{\partial t}\\ \left(\frac{\partial\mathbf{x}}{\partial t}\right)^T\frac{\partial\mathbf{x}}{\partial\mathbf{p}} & \left(\frac{\partial\mathbf{x}}{\partial t}\right)^T\frac{\partial\mathbf{x}}{\partial t}\end{bmatrix}\equiv\begin{bmatrix}\mathbf{G}&\mathbf{g}\\ \mathbf{g}^T&\gamma\end{bmatrix} \tag{75}$$

The connection coefficients will be the elements of matrices $\overline{\boldsymbol{\Gamma}}_m=\overline{\mathbf{G}}^{-1}\overline{\mathbf{K}}_m$ and $\overline{\boldsymbol{\Gamma}}_t=\overline{\mathbf{G}}^{-1}\overline{\mathbf{K}}_t$ where

$$\overline{\mathbf{K}}_m = \overline{\mathbf{K}}_{p_m} = \overline{\mathbf{G}\boldsymbol{\Gamma}}_m = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\right)^T \frac{\partial}{\partial p_m}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\right) = \begin{bmatrix} \left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)^T \frac{\partial}{\partial p_m}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right) & \left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)^T \frac{\partial}{\partial p_m}\left(\frac{\partial \mathbf{x}}{\partial t}\right) \\ \left(\frac{\partial \mathbf{x}}{\partial t}\right)^T \frac{\partial}{\partial p_m}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right) & \left(\frac{\partial \mathbf{x}}{\partial t}\right)^T \frac{\partial}{\partial p_m}\left(\frac{\partial \mathbf{x}}{\partial t}\right) \end{bmatrix} = \begin{bmatrix} \mathbf{K}_m & \mathbf{h}_m \\ \tilde{\mathbf{h}}_m^T & \kappa_m \end{bmatrix} \tag{76}$$

$$\overline{\mathbf{K}}_t = \overline{\mathbf{G}\boldsymbol{\Gamma}}_t = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\right)^T \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\right) = \begin{bmatrix} \left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)^T \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right) & \left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)^T \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}}{\partial t}\right) \\ \left(\frac{\partial \mathbf{x}}{\partial t}\right)^T \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right) & \left(\frac{\partial \mathbf{x}}{\partial t}\right)^T \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}}{\partial t}\right) \end{bmatrix} = \begin{bmatrix} \mathbf{K}_t & \mathbf{h}_t \\ \tilde{\mathbf{h}}_t^T & \kappa_t \end{bmatrix} \tag{77}$$

The result of the computations given in Appendix C is

$$\mathbf{g} = \begin{bmatrix} \lambda^2 \boldsymbol{\Omega}^T \mathbf{R}\mathbf{h} \\ N\lambda \mathbf{R}\dot{\bar{\mathbf{z}}} \\ \lambda \mathbf{z}^T \dot{\mathbf{z}} \end{bmatrix}, \qquad \gamma = \lambda^2 \dot{\mathbf{z}}^T \dot{\mathbf{z}}, \tag{78}$$

$$\mathbf{h}_{\theta_m} = \begin{bmatrix} \lambda^2 \boldsymbol{\Omega}^T \mathbf{R}\mathbf{W}_z^T \mathbf{R}^T \boldsymbol{\omega}_m \\ -N\lambda\, \mathbf{R}[\dot{\bar{\mathbf{z}}}\times]\mathbf{R}^T \boldsymbol{\omega}_m \\ -\lambda\, \mathbf{h}^T \mathbf{R}^T \boldsymbol{\omega}_m \end{bmatrix}, \qquad \mathbf{h}_{t_m} = \mathbf{0}, \qquad \mathbf{h}_\lambda = \begin{bmatrix} \lambda \boldsymbol{\Omega}^T \mathbf{R}\mathbf{h} \\ N\mathbf{R}\dot{\bar{\mathbf{z}}} \\ \mathbf{z}^T \dot{\mathbf{z}} \end{bmatrix} \tag{79}$$

$$\tilde{\mathbf{h}}_{\theta_m} = \begin{bmatrix} -\lambda^2 \boldsymbol{\Omega}^T \mathbf{R}\mathbf{W}_z \mathbf{R}^T \boldsymbol{\omega}_m + \lambda^2 \frac{\partial \boldsymbol{\Omega}^T}{\partial \theta_m} \mathbf{R}\mathbf{h} \\ \mathbf{0} \\ \lambda\, \mathbf{h}^T \mathbf{R}^T \boldsymbol{\omega}_m \end{bmatrix}, \qquad \tilde{\mathbf{h}}_{t_m} = \mathbf{0}, \qquad \tilde{\mathbf{h}}_\lambda = \begin{bmatrix} \lambda \boldsymbol{\Omega}^T \mathbf{R}\mathbf{h} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{80}$$

$$\kappa_{\theta_m} = 0, \qquad \kappa_{t_m} = 0, \qquad \kappa_\lambda = \lambda\, \dot{\mathbf{z}}^T \dot{\mathbf{z}}, \tag{81}$$

$$\mathbf{K}_t = \begin{bmatrix} \lambda^2 \boldsymbol{\Omega}^T \mathbf{R}\mathbf{W}_z^T \mathbf{R}^T \boldsymbol{\Omega} & \mathbf{0} & \lambda\, \boldsymbol{\Omega}^T \mathbf{R}\mathbf{h} \\ -N\lambda\, \mathbf{R}[\dot{\bar{\mathbf{z}}}\times]\mathbf{R}^T \boldsymbol{\Omega} & \mathbf{0} & N\mathbf{R}\dot{\bar{\mathbf{z}}} \\ -\lambda\, \mathbf{h}^T \mathbf{R}^T \boldsymbol{\Omega} & \mathbf{0} & \mathbf{z}^T \dot{\mathbf{z}} \end{bmatrix}, \tag{82}$$

$$\mathbf{h}_t = \begin{bmatrix} \lambda^2 \boldsymbol{\Omega}^T \mathbf{R}\dot{\mathbf{h}} \\ N\lambda\, \mathbf{R}\ddot{\bar{\mathbf{z}}} \\ \lambda\, \mathbf{z}^T \ddot{\mathbf{z}} \end{bmatrix}, \qquad \tilde{\mathbf{h}}_t = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \lambda\, \dot{\mathbf{z}}^T \dot{\mathbf{z}} \end{bmatrix}, \qquad \kappa_t = \lambda^2 \dot{\mathbf{z}}^T \ddot{\mathbf{z}}, \tag{83}$$

where $\bar{\mathbf{z}}$ is the center of mass and $\mathbf{h}$ the angular momentum vector of the network (Goldstein, 1950, ch. 1.2)

$$\bar{\mathbf{z}} \equiv \frac{1}{N} \sum_i z_i, \qquad \mathbf{h} \equiv \sum_i [\mathbf{z}_i \times] \dot{\mathbf{z}}_i, \qquad \dot{\mathbf{h}} \equiv \sum_i [\mathbf{z}_i \times] \ddot{\mathbf{z}}_i, \tag{84}$$

$$\mathbf{C}_z = -\sum_i [\mathbf{z}_i \times][\mathbf{z}_i \times] = (\mathbf{z}^T \mathbf{z})\mathbf{I} - \sum_i \mathbf{z}_i \mathbf{z}_i^T, \tag{85}$$

$$\mathbf{W}_z = -\sum_i [\dot{\mathbf{z}}_i \times][\mathbf{z}_i \times] = (\mathbf{z}^T \dot{\mathbf{z}})\mathbf{I} - \sum_i \mathbf{z}_i \dot{\mathbf{z}}_i^T, \qquad \dot{\mathbf{C}}_z = \mathbf{W}_z + \mathbf{W}_z^T. \tag{86}$$

Note that the rows of $\mathbf{K}_t$ are identical to the vectors $\mathbf{h}_{\theta_1}$, $\mathbf{h}_{\theta_2}$, $\mathbf{h}_{\theta_3}$, $\mathbf{h}_{t_1}$, $\mathbf{h}_{t_2}$, $\mathbf{h}_{t_3}$, $\mathbf{h}_{\lambda}$, as a consequence of the fact that $\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}}{\partial p_m}\right)=\frac{\partial}{\partial p_m}\left(\frac{\partial \mathbf{x}}{\partial t}\right)$, which also leads to the symmetry $\overline{\mathbf{K}}^k_{mi}=\overline{\mathbf{K}}^k_{im}$.
The above approach constitutes an "extension" from the geometry of the single fiber to the manifold $\mathcal{M}$ since the matrices $\mathbf{G}$ and $\mathbf{K}_m$ contained in the above equations are the already derived ones for a single fiber.
The connection coefficients (of the second kind) can be obtained from

$$\overline{\mathbf{\Gamma}}_m=\overline{\mathbf{G}}^{-1}\overline{\mathbf{K}}_m=\begin{bmatrix}\mathbf{G} & \mathbf{g}\\ \mathbf{g}^T & \gamma\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{K}_m & \mathbf{h}_m\\ \tilde{\mathbf{h}}_m^T & \kappa_m\end{bmatrix}=\begin{bmatrix}\mathbf{G}^{-1}+\alpha\,\mathbf{G}^{-1}\mathbf{g}\mathbf{g}^T\mathbf{G}^{-1} & -\alpha\,\mathbf{G}^{-1}\mathbf{g}\\ -\alpha\,\mathbf{g}^T\mathbf{G}^{-1} & \alpha\end{bmatrix}\begin{bmatrix}\mathbf{K}_m & \mathbf{h}_m\\ \tilde{\mathbf{h}}_m^T & \kappa_m\end{bmatrix}=$$

$$=\begin{bmatrix}\mathbf{G}^{-1}\mathbf{K}_m+\alpha\,\mathbf{G}^{-1}\mathbf{g}(\mathbf{g}^T\mathbf{G}^{-1}\mathbf{K}_m-\tilde{\mathbf{h}}_m^T) & \mathbf{G}^{-1}\mathbf{h}_m+\alpha\,(\mathbf{g}^T\mathbf{G}^{-1}\mathbf{h}_m-\kappa_m)\mathbf{G}^{-1}\mathbf{g}\\ -\alpha\,(\mathbf{g}^T\mathbf{G}^{-1}\mathbf{K}_m-\tilde{\mathbf{h}}_m^T) & -\alpha\,(\mathbf{g}^T\mathbf{G}^{-1}\mathbf{h}_m-\kappa_m)\end{bmatrix}=$$

$$=\begin{bmatrix}\mathbf{G}^{-1}\mathbf{K}_m+\alpha\,\mathbf{G}^{-1}\mathbf{g}(\mathbf{g}^T\mathbf{G}^{-1}\mathbf{K}_m-\tilde{\mathbf{h}}_m^T) & \mathbf{G}^{-1}\mathbf{h}_m+\alpha\,(\mathbf{g}^T\mathbf{G}^{-1}\mathbf{h}_m-\kappa_m)\mathbf{G}^{-1}\mathbf{g}\\ -\alpha\,(\mathbf{g}^T\mathbf{G}^{-1}\mathbf{K}_m-\tilde{\mathbf{h}}_m^T) & -\alpha\,(\mathbf{g}^T\mathbf{G}^{-1}\mathbf{h}_m-\kappa_m)\end{bmatrix}=$$

$$=\begin{bmatrix}\mathbf{\Gamma}_m+\alpha\,\mathbf{G}^{-1}\mathbf{g}(\mathbf{g}^T\mathbf{\Gamma}_m-\tilde{\mathbf{h}}_m^T) & \mathbf{G}^{-1}\mathbf{h}_m+\alpha\,(\mathbf{g}^T\mathbf{G}^{-1}\mathbf{h}_m-\kappa_m)\mathbf{G}^{-1}\mathbf{g}\\ -\alpha\,(\mathbf{g}^T\mathbf{\Gamma}_m-\tilde{\mathbf{h}}_m^T) & -\alpha\,(\mathbf{g}^T\mathbf{G}^{-1}\mathbf{h}_m-\kappa_m)\end{bmatrix} \qquad (87)$$

where

$$\alpha=\frac{1}{\gamma-\mathbf{g}^T\mathbf{G}^{-1}\mathbf{g}}\,. \qquad (88)$$

The next step is to derive the differential equations for ***geodesics*** on the manifold $\mathcal{M}$. The well known form for curvilinear coordinates u^i and a general parameter τ (not necessarily the length along the curve s) is given by

$$\frac{d^2u^i}{d\tau^2}-\frac{\ddot{s}}{\dot{s}}\frac{du^i}{d\tau}+\overline{\Gamma}^i_{mk}\frac{du^m}{d\tau}\frac{du^k}{d\tau}=0\,,\qquad i=1,\ldots,8\,,\qquad \dot{s}=\frac{ds}{d\tau}\,,\qquad \ddot{s}=\frac{d^2s}{d\tau^2}\,. \qquad (89)$$

The usual form of the geodesic equation is with the second term missing when the curve parameter is the arc length. The above form for any curve parameter τ follows by replacing

$$\frac{du^i}{ds}=\frac{1}{\dot{s}}\frac{du^i}{d\tau}\,,\qquad \frac{d^2u^i}{ds^2}=\frac{1}{\dot{s}^2}\frac{d^2u^i}{d\tau^2}-\frac{\ddot{s}}{\dot{s}^3}\frac{du^i}{d\tau}\,. \qquad (90)$$

In our particular case one of the coordinates and the curve parameter are identical, i.e. $u^8=t=\tau$. We shall rewrite the above set of equations in the matrix form

$$\frac{d^2\mathbf{u}}{dt^2}-\frac{\ddot{s}}{\dot{s}}\frac{d\mathbf{u}}{dt}+\left[\overline{\mathbf{\Gamma}}_1\frac{d\mathbf{u}}{dt}\;\cdots\;\overline{\mathbf{\Gamma}}_7\frac{d\mathbf{u}}{dt}\;\;\overline{\mathbf{\Gamma}}_8\frac{d\mathbf{u}}{dt}\right]\frac{d\mathbf{u}}{dt}=\mathbf{0}\,. \qquad (91)$$

A solution $\mathbf{u}(t)$ defines intrinsically the geodesic which has the extrinsic representation $\mathbf{x}(t)=\mathbf{x}(\mathbf{u}(t))$. In terms of the euclidean metric in X we have also

$$\dot{s}^2=\left(\frac{d\mathbf{x}}{dt}\right)^T\frac{d\mathbf{x}}{dt}=\dot{\mathbf{x}}^T\dot{\mathbf{x}}\quad\Rightarrow\quad 2\dot{s}\ddot{s}=2\dot{\mathbf{x}}^T\ddot{\mathbf{x}}\quad\Rightarrow\quad \frac{\ddot{s}}{\dot{s}}=\frac{\dot{\mathbf{x}}^T\ddot{\mathbf{x}}}{\dot{\mathbf{x}}^T\dot{\mathbf{x}}}\,. \qquad (92)$$

Multiplying (91) from the left with $\overline{\mathbf{G}}$ we obtain the more convenient matrix form

$$\overline{\mathbf{G}}\frac{d^2\mathbf{u}}{dt^2}-\frac{\ddot{s}}{\dot{s}}\overline{\mathbf{G}}\frac{d\mathbf{u}}{dt}+\left[\overline{\mathbf{K}}_1\frac{d\mathbf{u}}{dt}\quad\cdots\quad\overline{\mathbf{K}}_7\frac{d\mathbf{u}}{dt}\quad\overline{\mathbf{K}}_8\frac{d\mathbf{u}}{dt}\right]\frac{d\mathbf{u}}{dt}=\mathbf{0}\,. \tag{93}$$

To get more detailed expression we use the explicit forms for $\overline{\mathbf{G}}$, $\overline{\mathbf{K}}_{p_m}$ and $\overline{\mathbf{K}}_t$ from (75), (76) and (77), respectively, and we note that since $\mathbf{u}^T=[\mathbf{p}^T t]$

$$\frac{d\mathbf{u}}{dt}=\begin{bmatrix}\dot{\mathbf{p}}\\1\end{bmatrix},\qquad \frac{d^2\mathbf{u}}{dt^2}=\begin{bmatrix}\ddot{\mathbf{p}}\\0\end{bmatrix}, \tag{94}$$

which leads to

$$\begin{bmatrix}\mathbf{G}\,\ddot{\mathbf{p}}\\ \mathbf{g}^T\ddot{\mathbf{p}}\end{bmatrix}-\frac{\ddot{s}}{\dot{s}}\begin{bmatrix}\mathbf{G}\,\dot{\mathbf{p}}+\mathbf{g}\\ \mathbf{g}^T\dot{\mathbf{p}}+\gamma\end{bmatrix}+\sum_{m=1}^{7}\dot{p}_m\begin{bmatrix}\mathbf{K}_m\dot{\mathbf{p}}+\mathbf{h}_m\\ \tilde{\mathbf{h}}_m^T\dot{\mathbf{p}}+\kappa_m\end{bmatrix}+\begin{bmatrix}\mathbf{K}_t\dot{\mathbf{p}}+\mathbf{h}_t\\ \tilde{\mathbf{h}}_t^T\dot{\mathbf{p}}+\kappa_i\end{bmatrix}=\mathbf{0}\,. \tag{95}$$

With the help of the auxiliary matrices

$$\mathbf{H}=[\mathbf{h}_1\cdots\mathbf{h}_7]=\left[\mathbf{h}_{\theta_1}\mathbf{h}_{\theta_2}\mathbf{h}_{\theta_3}\quad \mathbf{h}_{t_1}\mathbf{h}_{t_2}\mathbf{h}_{t_3}\quad \mathbf{h}_\lambda\right]=[\mathbf{H}_\theta\ \mathbf{0}\ \mathbf{h}_\lambda]=\mathbf{K}_t \tag{96}$$

$$\tilde{\mathbf{H}}=[\tilde{\mathbf{h}}_1\cdots\tilde{\mathbf{h}}_7]=\left[\tilde{\mathbf{h}}_{\theta_1}\tilde{\mathbf{h}}_{\theta_2}\tilde{\mathbf{h}}_{\theta_3}\quad \tilde{\mathbf{h}}_{t_1}\tilde{\mathbf{h}}_{t_2}\tilde{\mathbf{h}}_{t_3}\quad \tilde{\mathbf{h}}_\lambda\right]=[\tilde{\mathbf{H}}_\theta\ \mathbf{0}\ \tilde{\mathbf{h}}_\lambda] \tag{97}$$

$$\boldsymbol{\kappa}^T=[\kappa_1\cdots\kappa_7]=\left[\kappa_{\theta_1}\kappa_{\theta_2}\kappa_{\theta_3}\quad \kappa_{t_1}\kappa_{t_2}\kappa_{t_3}\quad \kappa_\lambda\right]=[\mathbf{0}\ \mathbf{0}\ \kappa_\lambda] \tag{98}$$

the equations of a geodesic take the desired working form

$$\mathbf{G}\,\ddot{\mathbf{p}}-\frac{\ddot{s}}{\dot{s}}(\mathbf{G}\,\dot{\mathbf{p}}+\mathbf{g})+\sum_{m=1}^{7}\dot{p}_m\mathbf{K}_m\dot{\mathbf{p}}+\mathbf{H}\,\dot{\mathbf{p}}+\mathbf{K}_t\dot{\mathbf{p}}+\mathbf{h}_t=\mathbf{0}\,, \tag{99}$$

$$\mathbf{g}^T\ddot{\mathbf{p}}-\frac{\ddot{s}}{\dot{s}}(\mathbf{g}^T\dot{\mathbf{p}}+\gamma)+\dot{\mathbf{p}}^T\tilde{\mathbf{H}}\,\dot{\mathbf{p}}+\boldsymbol{\kappa}^T\dot{\mathbf{p}}+\tilde{\mathbf{h}}_t^T\dot{\mathbf{p}}+\kappa_t=\mathbf{0}\,. \tag{100}$$

The above two equations correspond to 8 differential equations, 7 in the first and 1 in the second. These correspond to 8 unknown functions the 7 ones sought $\boldsymbol{\theta}(t)$, $\mathbf{t}(t)$, $\lambda(t)$ and the "nuisance" arc length function $s(t)$. We can eliminate s, e.g. by solving (100) for $\frac{\ddot{s}}{\dot{s}}$ and replacing in (99). We prefer instead to discard equation (99) altogether and to obtain the required s-dependent terms from the quadratic form of the metric

$$\dot{s}^2=\dot{\mathbf{u}}^T\overline{\mathbf{G}}\dot{\mathbf{u}}=[\dot{\mathbf{p}}^T\dot{t}]\begin{bmatrix}\mathbf{G}&\mathbf{g}\\ \mathbf{g}^T&\gamma\end{bmatrix}\begin{bmatrix}\dot{\mathbf{p}}\\ \dot{t}\end{bmatrix}=\dot{\mathbf{p}}^T\mathbf{G}\,\dot{\mathbf{p}}+2\,\dot{t}\,\mathbf{g}^T\dot{\mathbf{p}}+\dot{t}^2\gamma=\dot{\mathbf{p}}^T\mathbf{G}\,\dot{\mathbf{p}}+2\,\mathbf{g}^T\dot{\mathbf{p}}+\gamma \tag{101}$$

and its derivative

$$\frac{d\dot{s}^2}{dt}=2\dot{s}\ddot{s}=2\dot{\mathbf{u}}^T\overline{\mathbf{G}}\,\ddot{\mathbf{u}}+\dot{\mathbf{u}}^T\dot{\overline{\mathbf{G}}}\,\dot{\mathbf{u}}=2\dot{\mathbf{p}}^T\mathbf{G}\,\ddot{\mathbf{p}}+\dot{\mathbf{p}}^T\dot{\mathbf{G}}\,\dot{\mathbf{p}}+2\,\mathbf{g}^T\ddot{\mathbf{p}}+2\,\dot{\mathbf{p}}^T\dot{\mathbf{g}}+\dot{\gamma}\,. \tag{102}$$

To make sure that no information is lost in discarding equation (100) we shall show that it contains the same information as that from the direct computation of $\frac{\ddot{s}}{\dot{s}}$.

Lemma: Equation (100) is equivalent within the relevant system of equations with the expression for $\frac{\ddot{s}}{\dot{s}}$ obtained by direct computation of $\dot{s}^2$ and its differentiation.

Proof: Note first that

$$\dot{\mathbf{G}}=\sum_m \frac{\partial \mathbf{G}}{\partial p_m}\dot{p}_m+\frac{\partial \mathbf{G}}{\partial t}=\sum_m (\mathbf{K}_m+\mathbf{K}_m^T)\dot{p}_m+(\mathbf{K}_t+\mathbf{K}_t^T)$$

$$\dot{\mathbf{g}}=\sum_m \frac{\partial \mathbf{g}}{\partial p_m}\dot{p}_m+\frac{\partial \mathbf{g}}{\partial t}=\sum_m (\mathbf{h}_m+\tilde{\mathbf{h}}_m)\dot{p}_m+(\mathbf{h}_t+\tilde{\mathbf{h}}_t)=(\mathbf{H}+\tilde{\mathbf{H}})\dot{\mathbf{p}}+(\mathbf{h}_t+\tilde{\mathbf{h}}_t)$$

$$\dot{\gamma}=\sum_m \frac{\partial \gamma}{\partial p_m}\dot{p}_m+\frac{\partial \gamma}{\partial t}=\sum_m 2\kappa_m \dot{p}_m+2\kappa_t=2(\boldsymbol{\kappa}^T\dot{\mathbf{p}}+\kappa_t)$$

so that

$$\dot{s}\ddot{s}=\dot{\mathbf{p}}^T\mathbf{G}\,\ddot{\mathbf{p}}+2\mathbf{g}^T\ddot{\mathbf{p}}+\sum_m (\dot{\mathbf{p}}^T\mathbf{K}_m\dot{\mathbf{p}})\dot{p}_m+\dot{\mathbf{p}}^T\mathbf{K}_t\dot{\mathbf{p}}+\dot{\mathbf{p}}^T(\mathbf{H}+\tilde{\mathbf{H}})\dot{\mathbf{p}}+\dot{\mathbf{p}}^T(\mathbf{h}_t+\tilde{\mathbf{h}}_t)+\boldsymbol{\kappa}^T\dot{\mathbf{p}}+\kappa_t$$

On the other hand equation (100) can be replaced without any loss of information by the sum of itself multiplied from the left with $\dot{\mathbf{p}}^T$ and (99), which sum is

$$\dot{\mathbf{p}}^T\mathbf{G}\,\ddot{\mathbf{p}}+\mathbf{g}^T\ddot{\mathbf{p}}+\sum_{m=1}^{7}\dot{p}_m\dot{\mathbf{p}}^T\mathbf{K}_m\dot{\mathbf{p}}+\dot{\mathbf{p}}^T\mathbf{H}\,\dot{\mathbf{p}}+\dot{\mathbf{p}}^T\tilde{\mathbf{H}}\,\dot{\mathbf{p}}+\dot{\mathbf{p}}^T\mathbf{K}_t\dot{\mathbf{p}}+\dot{\mathbf{p}}^T\mathbf{h}_t+\boldsymbol{\kappa}^T\dot{\mathbf{p}}+\tilde{\mathbf{h}}_t^T\dot{\mathbf{p}}+\kappa_t-$$
$$-\frac{\ddot{s}}{\dot{s}}(\dot{\mathbf{p}}^T\mathbf{G}\,\dot{\mathbf{p}}+2\dot{\mathbf{p}}^T\mathbf{g}+\gamma)=0$$

Solving for $\frac{\ddot{s}}{\dot{s}}$ we obtain exactly the same expression as from the direct differentiation of $\dot{s}^2$:

$$\frac{\ddot{s}}{\dot{s}}=\frac{1}{\dot{\mathbf{p}}^T\mathbf{G}\,\dot{\mathbf{p}}+2\dot{\mathbf{p}}^T\mathbf{g}+\gamma}\left(\dot{\mathbf{p}}^T\mathbf{G}\,\ddot{\mathbf{p}}+\mathbf{g}^T\ddot{\mathbf{p}}+\sum_{m=1}^{7}\dot{p}_m\dot{\mathbf{p}}^T\mathbf{K}_m\dot{\mathbf{p}}+\dot{\mathbf{p}}^T\mathbf{H}\,\dot{\mathbf{p}}+\dot{\mathbf{p}}^T\tilde{\mathbf{H}}\,\dot{\mathbf{p}}+\dot{\mathbf{p}}^T\mathbf{K}_t\dot{\mathbf{p}}+\dot{\mathbf{p}}^T\mathbf{h}_t+\boldsymbol{\kappa}^T\dot{\mathbf{p}}+\tilde{\mathbf{h}}_t^T\dot{\mathbf{p}}+\kappa_t\right)\quad\square$$

Using the explicit values of the terms appearing in the above equations (detailed computations are given in Appendix D) we obtain the following detailed 8 second order differential equations, 3 for the rotational parameters $\boldsymbol{\theta}$, 3 for the displacement parameters **t**, one for the scale parameter λ and one for the time parameter *t*. Without any loss of generality, we give the simplified case where $\bar{\mathbf{z}}=\mathbf{0}$ and $\mathbf{z}^T\mathbf{z}=$ constant:

$\boldsymbol{\theta}$ - equation:

$$\left\{2\dot{\lambda}\,\boldsymbol{\Omega}^T+\lambda\,\boldsymbol{\Omega}^T[(\dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}})\times]\right\}\mathbf{R}(\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})+\lambda\,\boldsymbol{\Omega}^T\mathbf{R}\left[\dot{\mathbf{h}}+\dot{\mathbf{C}}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\mathbf{C}_z\mathbf{R}^T(\dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}}+\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}})\right]-$$
$$-\frac{\ddot{s}}{\dot{s}}\lambda\,\boldsymbol{\Omega}^T\mathbf{R}(\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})=0 \qquad (103)$$

t - equation:

$$\ddot{\mathbf{t}}-\frac{\ddot{s}}{\dot{s}}\dot{\mathbf{t}}=\mathbf{0} \qquad (104)$$

λ - equation:

$$\ddot{\lambda}\,\mathbf{z}^T\mathbf{z}-\lambda\,\dot{\mathbf{z}}^T\dot{\mathbf{z}}-\lambda\,\dot{\boldsymbol{\theta}}^T\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}-2\lambda\,\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}-\frac{\ddot{s}}{\dot{s}}\dot{\lambda}\,\mathbf{z}^T\mathbf{z}=0 \qquad (105)$$

t - equation:

$$2\dot{\lambda}\lambda\,\dot{\mathbf{z}}^T\mathbf{z}+\lambda^2\dot{\mathbf{z}}^T\ddot{\mathbf{z}}+2\dot{\lambda}\lambda\,\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\kappa^2\mathbf{h}^T\mathbf{R}^T(\dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}}+\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}})-\frac{1}{2}\lambda^2\dot{\boldsymbol{\theta}}^T\boldsymbol{\Omega}^T\mathbf{R}\dot{\mathbf{C}}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}-$$
$$-\frac{\ddot{s}}{\dot{s}}(\lambda^2\dot{\mathbf{z}}^T\dot{\mathbf{z}}+\lambda^2\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})=0 \qquad (106)$$

The last equation (t-equation) will be discarded as explained by the previous lemma while the factor $\frac{\ddot{s}}{\dot{s}}=\frac{\ddot{s}\dot{s}}{\dot{s}^2}$ is given instead by

$$\dot{s}^2=\dot{\lambda}^2\mathbf{z}^T\mathbf{z}+\lambda^2\dot{\mathbf{z}}^T\dot{\mathbf{z}}+2\lambda^2\mathbf{h}^T\mathbf{R}^T\mathbf{\Omega}\dot{\boldsymbol{\theta}}+\lambda^2\dot{\boldsymbol{\theta}}^T\mathbf{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\mathbf{\Omega}\dot{\boldsymbol{\theta}}+N\,\dot{\mathbf{t}}^T\dot{\mathbf{t}} \tag{107}$$

$\mathbf{\Omega}\dot{\boldsymbol{\theta}}$

$$\ddot{s}\dot{s}=\ddot{\lambda}\dot{\lambda}\,\mathbf{z}^T\mathbf{z}+\dot{\lambda}\lambda\,\dot{\mathbf{z}}^T\dot{\mathbf{z}}+\lambda^2\dot{\mathbf{z}}^T\ddot{\mathbf{z}}+2\dot{\lambda}\lambda\,\mathbf{h}^T\mathbf{R}^T\mathbf{\Omega}\dot{\boldsymbol{\theta}}+\lambda^2\mathbf{h}^T\mathbf{R}^T(\mathbf{\Omega}\ddot{\boldsymbol{\theta}}+\dot{\mathbf{\Omega}}\dot{\boldsymbol{\theta}})+\lambda^2\dot{\mathbf{h}}^T\mathbf{R}^T\mathbf{\Omega}\dot{\boldsymbol{\theta}}+\dot{\lambda}\lambda\,\dot{\boldsymbol{\theta}}^T\mathbf{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T\mathbf{\Omega}\dot{\boldsymbol{\theta}}+$$

$$+\frac{1}{2}\lambda^2\dot{\boldsymbol{\theta}}^T\mathbf{\Omega}^T\mathbf{R}\dot{\mathbf{C}}_z\mathbf{R}^T\mathbf{\Omega}\dot{\boldsymbol{\theta}}+\lambda^2\dot{\boldsymbol{\theta}}^T\mathbf{\Omega}^T\mathbf{R}\dot{\mathbf{C}}_z\mathbf{R}^T(\mathbf{\Omega}\ddot{\boldsymbol{\theta}}+\mathbf{\Omega}\dot{\boldsymbol{\theta}})+N\,\dot{\mathbf{t}}^T\dot{\mathbf{t}} \tag{108}$$

In the usual case when $|\mathbf{\Omega}|\neq 0$, the first equations becomes

$\boldsymbol{\theta}$ - equation:

$$\{2\dot{\lambda}+\lambda[(\mathbf{R}^T\mathbf{\Omega}\dot{\boldsymbol{\theta}})\times]\}(\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\mathbf{\Omega}\dot{\boldsymbol{\theta}})+\lambda\left[\dot{\mathbf{h}}+\dot{\mathbf{C}}_z\mathbf{R}^T\mathbf{\Omega}\dot{\boldsymbol{\theta}}+\mathbf{C}_z\mathbf{R}^T(\dot{\mathbf{\Omega}}\dot{\boldsymbol{\theta}}+\mathbf{\Omega}\ddot{\boldsymbol{\theta}})\right]-\frac{\ddot{s}}{\dot{s}}\lambda(\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\mathbf{\Omega}\dot{\boldsymbol{\theta}})=\mathbf{0} \tag{109}$$

8. Solutions to the non-linear space-time datum problem

We must make a choice among all possible motions $x(t)$ on the manifold $\mathscr{M}$ of the all the fibers $F_{y(t)}$ corresponding to a known solution $y(t)$ to the adjustment problem on the model manifold. For every epoch t, $x(t)$ must belong to a specific manifold $F_{y(t)}$. This is achieved with the use of a reference motion $z(t)$ having the desired property. Such a reference solution can be the intersection of $\mathscr{M}$ with a specified section C of the whole fibering $\mathscr{F}$ which is determined by a set of minimal constraints $h(x)=d$. The solution $x(t)$ is in turn specified by a set of 7 functions $\boldsymbol{\theta}(t)$, $\mathbf{t}(t)$, $\lambda(t)$ of the similarity transformation parameters, which at every epoch transform the given $z(t)$ into $x(t)$ by means of

$$\mathbf{x}(t)=\lambda(t)\left[\mathbf{I}_N\otimes\mathbf{R}(\boldsymbol{\theta}(t))\right]\mathbf{z}(t)+\mathbf{1}_N\otimes\mathbf{t}(t) \tag{110}$$

where $\mathbf{I}_N$ is the $N\times N$ identity matrix (N = number of points in the network) and $\mathbf{1}_N$ the $N\times 1$ vector with all elements equal to 1. The choice of $\boldsymbol{\theta}(t)$, $\mathbf{t}(t)$, $\lambda(t)$ must in general lead to a motion $\mathbf{x}(t)$ which is "small" in some sense, thus avoiding any unnecessarily "large" rotations, displacements and scaling along with time. Another important aspect, from a more practical point of view, is the tractability of the solution, i.e. the possibility to be numerically realized without too extensive computational effort. A more esthetic (rather than practical or theoretical) criterion relates to the possibility to define the choice of motion from an intrinsic point of view within $\mathscr{M}$ without having to resort to any extrinsic principle. In all solutions we must keep in mind that the metric on $\mathscr{M}$, which will ultimately define how small a motion is, comes from the euclidean metric in X which is a specific metric: the distance d between two network positions (configurations and placements) $\mathbf{x}_a$ and $\mathbf{x}_b$ is the mean square of the individual ordinary distances $d_i=[(\mathbf{x}_{ai}-\mathbf{x}_{bi})^T(\mathbf{x}_{ai}-\mathbf{x}_{bi})]^{1/2}$ in E^3 between the positions of the same network point i

$$d=\sqrt{d_1^2+d_2^2+\ldots+d_N^2}\ . \tag{111}$$

This metric is not the best in any sense but it dominates analysis because it leads to solutions which are easier to obtain. Two alternatives could be the "uniform" distance $d=\max_i(d_i)$, (in fact a combination of quadratic and uniform) and $d=d_1+d_2+\ldots+d_N$ which is perhaps the more reasonable choice from the physical point of view. With these remarks in mind we proceed with some proposed solutions.

8.1. Principal motions

We call a motion $\mathbf{x}(t)$ ***centered*** if its origin coincides at every epoch with the "mass center" of the network (visualized as a set of mass points with equal masses, say $m_i = 1$). This happens when for every epoch t

$$\bar{\mathbf{x}}(t) = \frac{1}{N} \sum_i \mathbf{x}_i(t) = \mathbf{0} \,. \tag{112}$$

This imposes three of the 7 conditions required to define $\mathbf{x}(t)$ and fixes the origin of the network. To define the orientation we select the ***principal axes of inertia*** as the axes of the reference frame (Munk & MacDonald, 1960, ch. 3.2, Goldstein, 1950, ch. 5.1). These are the axes for which the inertia matrix

$$\mathbf{C}_x = \int [(\mathbf{x}_i^T \mathbf{x}_i)\mathbf{I} - \mathbf{x}_i \mathbf{x}_i^T]\, dm = \sum_i [(\mathbf{x}_i^T \mathbf{x}_i)\mathbf{I} - \mathbf{x}_i \mathbf{x}_i^T]\, m_i = (\mathbf{x}^T \mathbf{x})\, \mathbf{I} - \sum_i \mathbf{x}_i \mathbf{x}_i^T \tag{113}$$

becomes diagonal (unit mass is assigned to all network points). It remains to fix the scale for which we choose to fix the trace of $\mathbf{C}_x$

$$\text{trace}\mathbf{C}_x = (\mathbf{x}^T \mathbf{x})\, \text{trace}(\mathbf{I}) - \sum_i \text{trace}(\mathbf{x}_i \mathbf{x}_i^T) = 3(\mathbf{x}^T \mathbf{x}) - \sum_i \mathbf{x}_i^T \mathbf{x}_i = 2(\mathbf{x}^T \mathbf{x}) = \text{constant} \,, \tag{114}$$

which is equivalent to fixing the norm $|\mathbf{x}| = \sqrt{\mathbf{x}^T \mathbf{x}}$. We shall call any motion satisfying the above requirements a ***principal motion***. Given a reference motion $\mathbf{z}(t)$ we can obtain $\mathbf{x}(t)$ by imposing the above conditions. Since

$$\bar{\mathbf{x}}(t) = \frac{1}{N} \sum_i \mathbf{x}_i(t) = \frac{1}{N} \lambda(t) \mathbf{R}(t) \sum_i \mathbf{z}_i(t) + \frac{1}{N} \sum_i \mathbf{t} = \lambda(t)\mathbf{R}(t)\bar{\mathbf{z}} + \mathbf{t} = \mathbf{0} \tag{115}$$

it follows that

$$\mathbf{t} = -\lambda(t)\mathbf{R}(t)\bar{\mathbf{z}} \tag{116}$$

and the transformation becomes

$$\mathbf{x}_i(t) = \lambda(t)\mathbf{R}(t)\big(\mathbf{z}_i(t) - \bar{\mathbf{z}}\big). \tag{117}$$

Setting $\delta\mathbf{z}_i = \mathbf{z}_i - \bar{\mathbf{z}}$ we can formulate and diagonalize the inertia matrix

$$\mathbf{C}_{\delta z} = \sum_i [\delta\mathbf{z}_i^T \delta\mathbf{z}_i \mathbf{I} - \delta\mathbf{z}_i \delta\mathbf{z}_i^T] = \mathbf{U}\,\mathbf{M}\,\mathbf{U}^T \,, \tag{118}$$

where $\mathbf{U}$ is the orthogonal matrix having the (orthonormal) eigenvectors of $\mathbf{C}_{\delta z}$ as columns and $\mathbf{M}$ is the diagonal matrix with diagonal elements the corresponding eigenvalues of $\mathbf{C}_{\delta z}$. Since $\mathbf{x}_i = \lambda\, \mathbf{R}\delta\mathbf{z}_i$ it follows that

$$\mathbf{C}_x = \sum_i [(\mathbf{x}_i^T \mathbf{x}_i)\mathbf{I} - \mathbf{x}_i \mathbf{x}_i^T] = \lambda^2 \sum_i [(\delta\mathbf{z}_i^T \delta\mathbf{z}_i)\mathbf{I} - \mathbf{R}\delta\mathbf{z}_i \delta\mathbf{z}_i^T \mathbf{R}^T] = \lambda^2 \mathbf{R}\mathbf{C}_{\delta z}\mathbf{R}^T = \lambda^2 \mathbf{R}\mathbf{U}\mathbf{M}\mathbf{U}^T \mathbf{R}^T \tag{119}$$

which means that $\mathbf{C}_x$ will be diagonal only if we choose

$$\mathbf{R}(t)=\mathbf{U}(t)^T . \tag{120}$$

To fix $\text{trace}\mathbf{C}_x=2(\mathbf{x}^T\mathbf{x})=c^2$, where c^2 is a constant we note that with the above choice of **R**

$$c^2=\text{trace}\mathbf{C}_x=\lambda^2\,\text{trace}(\mathbf{U}\mathbf{M}\mathbf{U}^T)=\lambda^2\,\text{trace}(\mathbf{M}\mathbf{U}^T\mathbf{U})=\lambda^2\,\text{trace}\mathbf{M} , \tag{121}$$

which leads to the choice

$$\lambda(t)=\frac{c}{\sqrt{\text{trace}\mathbf{M}(t)}} . \tag{122}$$

Once **R** and λ have been determined, we can determine **t** from (116)

$$\mathbf{t}(t)=-\frac{c}{\sqrt{\text{trace}\mathbf{M}(t)}}\mathbf{U}(t)^T\,\bar{\mathbf{z}}(t) \tag{123}$$

and the transformation from any reference motion $\mathbf{z}(t)$ to a principal motion $\mathbf{x}(t)$ is given by

$$\mathbf{x}_i(t)=\frac{c}{\sqrt{\text{trace}\mathbf{M}(t)}}\mathbf{U}(t)^T[\mathbf{z}_i(t)-\bar{\mathbf{z}}(t)] . \tag{124}$$

Lemma: The principal motion is unique (up to the assigned constant value of $\sqrt{\mathbf{x}^T\mathbf{x}}$).

Proof: We must show the principal motions $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ obtained by two different reference motions $\mathbf{z}(t)$ and $\mathbf{z}'(t)$, respectively, are identical. Since for any epoch *t* both $\mathbf{z}(t)$ and $\mathbf{z}'(t)$ must belong to the same fiber there will exist $\mu(t)$, $\mathbf{Q}(t)$ (orthogonal) and $\mathbf{d}(t)$ such that

$$\mathbf{z}_i'(t)=\mu(t)\mathbf{Q}(t)\mathbf{z}_i(t)+\mathbf{d}(t) .$$

This implies that $\bar{\mathbf{z}}'=\mu\,\mathbf{Q}\bar{\mathbf{z}}+\mathbf{d}$, so that $\delta\mathbf{z}_i'=\mu\,\mathbf{Q}\delta\mathbf{z}_i$ and furthermore

$$\mathbf{U}'\mathbf{M}'\mathbf{U}'^T=\mathbf{C}_{\delta z'}=\sum_i[(\delta\mathbf{z}_i')^T\delta\mathbf{z}_i'\mathbf{I}-\delta\mathbf{z}_i'(\delta\mathbf{z}_i')^T]=\mu^2\sum_i[\delta\mathbf{z}_i^T\delta\mathbf{z}_i\mathbf{I}-\mathbf{Q}\delta\mathbf{z}_i\delta\mathbf{z}_i^T\mathbf{Q}^T]=\mu^2\mathbf{Q}\mathbf{C}_{\delta z}\mathbf{Q}^T=$$
$$=\mu^2\mathbf{Q}\mathbf{U}\mathbf{M}\mathbf{U}^T\mathbf{Q}^T$$

which implies that $\mathbf{U}'=\mathbf{Q}\mathbf{U}$, $\mathbf{M}'=\mu^2\mathbf{M}$.

Consequently for a specific choice of the value of *c*

$$\mathbf{x}_i'=\frac{c}{\sqrt{\text{trace}\mathbf{M}'}}\mathbf{U}'^T(\mathbf{z}_i'-\bar{\mathbf{z}}')=\frac{c}{\sqrt{\mu^2\,\text{trace}\mathbf{M}}}(\mathbf{Q}\mathbf{U})^T(\mu\,\mathbf{Q}\mathbf{z}_i+\mathbf{d}-\mu\,\mathbf{Q}\bar{\mathbf{z}}-\mathbf{d})=\frac{c}{\sqrt{\text{trace}\mathbf{M}}}\mathbf{U}^T(\mathbf{z}_i-\bar{\mathbf{z}})=\mathbf{x}_i \qquad \square$$

The main advantage of the principal motion is that it is uniquely identified (at least up to the choice of the scalar *c*), independently of the reference motion and it does not require any additional arbitrary choice such as a reference point $\mathbf{x}_0$ or an initial point $\mathbf{x}(0)$, as it will be the case with some other choices. It is also optimal in some sense since by fixing the origin to the center of mass, the orientation to the principal axes of inertia and the mean quadratic magnitude $\frac{1}{N}\mathbf{x}^T\mathbf{x}$ of the network to a constant value (c^2/N) avoids in a way extreme displacements, rotations and scaling, respectively. This optimality however has not been given a concrete mathematical substance, e.g. by minimizing some suitable target function of $\mathbf{x}(t)$.

8.2. Motions nearest to a specific point

This solution is simply an extension to the time domain of the corresponding solution for a single epoch and a single fiber. The solution is provided from equations (A22), (A23) which should now be interpreted as time dependent

$$\frac{1}{N}\sum_i [(\mathbf{x}_{0i}-\bar{\mathbf{x}}_0)\times]\mathbf{R}(t)[\mathbf{z}_i(t)-\bar{\mathbf{z}}(t)]=\mathbf{0} \tag{125}$$

to be solved for the determination of the functions $\boldsymbol{\theta}(t)$ present in $\mathbf{R}(\boldsymbol{\theta}(t))$, which in turn will be used for the determination of the $\mathbf{x}_0$-nearest motion from

$$\mathbf{x}_i(t)=\bar{\mathbf{x}}_0+\frac{\sum_i (\mathbf{x}_{0i}-\bar{\mathbf{x}}_0)^T\mathbf{R}(t)[\mathbf{z}_i(t)-\bar{\mathbf{z}}(t)]}{\sum_i [\mathbf{z}_i(t)-\bar{\mathbf{z}}(t)]^T[\mathbf{z}_i(t)-\bar{\mathbf{z}}(t)]}\mathbf{R}(t)[\mathbf{z}_i(t)-\bar{\mathbf{z}}(t)]. \tag{126}$$

The point $\mathbf{x}_0$ is playing a crucial role in the determination of the motion $\mathbf{x}(t)$. It can be a point outside the manifold or a point on the manifold itself, e.g. some given $\mathbf{x}_0\in F_{y(t_0)}$ for some epoch t_0 in the considered time interval. In this last case $\mathbf{x}(t)$ will necessarily pass through $\mathbf{x}_0$ since it must obviously hold that $\mathbf{x}(t_0)=\mathbf{x}_0$. A particular choice is a known solution $\mathbf{x}(0)=\mathbf{x}_0$ for the initial epoch $t=0$.
For every epoch *t* the vector $\mathbf{x}_0-\mathbf{x}(t)$ is orthogonal to the tangent space of the fiber $F_{y(t)}$ at the point $\mathbf{x}(t)$. If the point $\hat{\mathbf{x}}_0$ of $\mathcal{M}$ closer to $\mathbf{x}_0$ is an interior point of $\mathcal{M}$, i.e. not a point on the boundary fibers $F_{y(0)}$ and $F_{y(T)}$, will be a point on a specific fiber $F_{y(t')}$ corresponding to a specific epoch t' ($0<t'<T$). The vector $\mathbf{x}_0-\hat{\mathbf{x}}_0$ is normal to the tangent space of $\mathcal{M}$ at $\hat{\mathbf{x}}_0$ and therefore to any of its subspaces such as the tangent space to the fiber through the same point $\hat{\mathbf{x}}_0$.
Another possibility is to use a given motion $\mathbf{x}_0(t)$ not on the manifold $\mathcal{M}$, and to obtain a motion $\mathbf{x}(t)$ by requiring that for each individual epoch t, $\mathbf{x}(t)$ is the closest point to $\mathbf{x}_0(t)$ on the corresponding fiber.
We have presented the $\mathbf{x}_0$-nearest solution to the datum problem for a single epoch (or for a time invariant network) as the non-linear generalization of the linear S-transformation of Baarda. One might thing that the $\mathbf{x}_0$-nearest motion provides a similar generalization for the non-linear space-time datum problem, but this is not the case in our opinion. To see this consider the situation where data are available at discrete time epochs $t_1, t_2, \ldots$ which are processed in the usual way through linearization. The present solution corresponds to the situation where a common Taylor point $\mathbf{x}_0$ (= set of approximate coordinates) is used for the linearization at every epoch and the linearized problem is solved independently at every epoch using either inner constraints or transforming a minimal constraint solution to the inner one. In this case the final solutions $\delta\mathbf{x}(t_i)=\mathbf{x}(t_i)-\mathbf{x}_0$ are all orthogonal to the tangent space of the fiber $F_{y_0}=F_{f(x_0)}$ at the point $\mathbf{x}_0$.The usual practice for deforming networks is to use the coordinates $\mathbf{x}(t_{i-1})$ of the previous epoch t_{i-1} as the Taylor point for the current epoch t_i and then to obtain an inner constraints solution $\delta\mathbf{x}(t_i)=\mathbf{x}(t_i)-\mathbf{x}(t_{i-1})$, in order to minimize the displacement part of the deformation, i.e. to have $\|\mathbf{x}(t_i)-\mathbf{x}(t_{i-1})\|=\min$. In this case the vector $\mathbf{x}(t_i)-\mathbf{x}(t_{i-1})$ is orthogonal to the fiber $F_{y(t_{i-1})}=F_{f(x(t_{i-1}))}$. This procedure extends the minimum norm principle of inner constraints to the (discrete) time domain and its time-continuous counterpart to be presented by the next solution is a more appropriate candidate for a generalization of the Baarda transformation.

8.3. Motions orthogonal to the fibers

Assume that we have at hand a particular solution to the non-linear datum problem for a particular epoch t, i.e. a point $\mathbf{x}(t)\in F_t$ (we set $F_t=F_{y(t)}=F_{f(x(t))}$ for simplicity) and we seek a solution $\mathbf{x}(t+\varepsilon)\in F_{t+\varepsilon}$ for a nearby epoch $t+\varepsilon$, where ε is a small number. If the solutions are to be as close as possible, we can choose to minimize the distance $\|\mathbf{x}(t+\varepsilon)-\mathbf{x}(t)\|$, which leads to a situation where the vector

$\frac{1}{\varepsilon}[\mathbf{x}(t+\varepsilon)-\mathbf{x}(t)]$ is orthogonal to the tangent space $T_{\mathbf{x}(t+\varepsilon)}(F_{t+\varepsilon})$ of the fiber $F_{t+\varepsilon}$ at the point $\mathbf{x}(t+\varepsilon)$. If we want to consider a continuous rather than a discrete situation with respect to time, we can take the limit as $\varepsilon \to 0$ to obtain a velocity vector $\dot{\mathbf{x}}(t) \equiv \frac{d\mathbf{x}}{dt}(t)$ which is orthogonal to the tangent space $T_{\mathbf{x}(t)}(F_t)$ of the fiber F_t at the point $\mathbf{x}(t)$. A motion $\mathbf{x}(t)$ is called ***orthogonal to the fibers*** when for every epoch t its velocity vector $\dot{\mathbf{x}}(t) \in T_{x(t)}(\mathcal{M})$ is orthogonal to the tangent space $T_{\mathbf{x}(t)}(F_t) \subset T_{\mathbf{x}(t)}(\mathcal{M})$.

In order to obtain the differential equations to be satisfied by this motion it is sufficient to impose the condition that $\dot{\mathbf{x}}(t)$ is orthogonal to the local coordinate basis of $T_{\mathbf{x}(t)}(F_t)$ which in its extrinsic representation consists of the columns of the matrix $\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(\mathbf{x}(t))$. Thus the equation to be satisfied by every motion orthogonal to the fibers is

$$\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}\right)^T \dot{\mathbf{x}} = \mathbf{0} \tag{127}$$

where we have dropped the dependence on t for the sake of simplicity.

Lemma: An orthogonal motion is a solution of the differential equations

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\Omega}^{-1} \mathbf{C}_{\delta z}^{-1} (N[\bar{\mathbf{z}}\times]\dot{\bar{\mathbf{z}}} - \mathbf{h}) \tag{128}$$

$$\dot{\mathbf{t}} = \lambda \frac{\mathbf{z}^T \dot{\mathbf{z}} - N\bar{\mathbf{z}}^T \dot{\bar{\mathbf{z}}}}{\mathbf{z}^T \mathbf{z} - N\bar{\mathbf{z}}^T \bar{\mathbf{z}}} \mathbf{R}\bar{\mathbf{z}} - \lambda\, \mathbf{R}\dot{\bar{\mathbf{z}}} + \lambda\, \mathbf{R}[\bar{\mathbf{z}}\times]\mathbf{C}_{\delta z}^{-1} (N[\bar{\mathbf{z}}\times]\dot{\bar{\mathbf{z}}} - \mathbf{h}) \tag{129}$$

$$\dot{\lambda} = \lambda \frac{\mathbf{z}^T \dot{\mathbf{z}} - N\bar{\mathbf{z}}^T \dot{\bar{\mathbf{z}}}}{\mathbf{z}^T \mathbf{z} - N\bar{\mathbf{z}}^T \bar{\mathbf{z}}} \qquad \Rightarrow \qquad \lambda = \lambda_0 + \left[\frac{1}{\sqrt{\mathbf{z}^T \mathbf{z} - N\bar{\mathbf{z}}^T \bar{\mathbf{z}}}} \right]_0^t \tag{130}$$

where

$$\mathbf{C}_{\delta z} = \mathbf{C}_z + N[\bar{\mathbf{z}}\times][\bar{\mathbf{z}}\times] = (\sum_i \delta\mathbf{z}_i^T \delta\mathbf{z}_i)\,\mathbf{I} - \sum_i \delta\mathbf{z}_i \delta\mathbf{z}_i^T , \qquad (\delta\mathbf{z}_i = \mathbf{z}_i - \bar{\mathbf{z}}) \tag{131}$$

Proof: The explicit form of equation (127) is

$$\sum_i \left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right)^T \dot{\mathbf{x}}_i = \sum_i \lambda\, \boldsymbol{\Omega}^T \mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T \left(\dot{\lambda}\, \mathbf{R}\mathbf{z}_i + \lambda\, \mathbf{R}\dot{\mathbf{z}}_i - \lambda\, \mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + \dot{\mathbf{t}}\right) =$$
$$= \dot{\lambda}\lambda\, \boldsymbol{\Omega}^T \mathbf{R} \sum_i [\mathbf{z}_i\times]\mathbf{z}_i + \lambda^2\, \boldsymbol{\Omega}^T \mathbf{R} \sum_i [\mathbf{z}_i\times]\dot{\mathbf{z}}_i - \lambda^2\, \boldsymbol{\Omega}^T \mathbf{R} \{\sum_i [\mathbf{z}_i\times][\mathbf{z}_i\times]\}\, \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + \lambda\, \boldsymbol{\Omega}^T \mathbf{R} \sum_i [\mathbf{z}_i\times]\, \mathbf{R}^T \dot{\mathbf{t}} =$$
$$= \lambda^2 \boldsymbol{\Omega}^T \mathbf{R}\mathbf{h} + \lambda^2 \boldsymbol{\Omega}^T \mathbf{R}\mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + N\lambda\, \boldsymbol{\Omega}^T \mathbf{R}[\bar{\mathbf{z}}\times]\mathbf{R}^T \dot{\mathbf{t}} = \mathbf{0}$$

$$\sum_i \left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right)^T \dot{\mathbf{x}}_i = \sum_i \left(\dot{\lambda}\, \mathbf{R}\mathbf{z}_i + \lambda\, \mathbf{R}\dot{\mathbf{z}}_i - \lambda\, \mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + \dot{\mathbf{t}}\right) = N\dot{\lambda}\, \mathbf{R}\bar{\mathbf{z}} + N\lambda\, \mathbf{R}\dot{\bar{\mathbf{z}}} - N\lambda\, \mathbf{R}[\bar{\mathbf{z}}\times]\mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + N\, \dot{\mathbf{t}} = \mathbf{0}$$

$$\sum_i \left(\frac{\partial \mathbf{x}_i}{\partial \lambda}\right)^T \dot{\mathbf{x}}_i = \sum_i \mathbf{z}_i^T \mathbf{R}^T \left(\dot{\lambda}\, \mathbf{R}\mathbf{z}_i + \lambda\, \mathbf{R}\dot{\mathbf{z}}_i - \lambda\, \mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + \dot{\mathbf{t}}\right) =$$
$$= \dot{\lambda} \sum_i \mathbf{z}_i^T \mathbf{z}_i + \lambda \sum_i \mathbf{z}_i^T \dot{\mathbf{z}}_i - \lambda \sum_i \mathbf{z}_i^T [\mathbf{z}_i\times]\mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} + \sum_i \mathbf{z}_i^T \mathbf{R}^T \dot{\mathbf{t}} =$$
$$= N\dot{\lambda}\, \mathbf{z}^T \mathbf{z} + N\lambda\, \mathbf{z}^T \dot{\mathbf{z}} + N\, \bar{\mathbf{z}}^T \mathbf{R}^T \dot{\mathbf{t}} = 0$$

and the equations of a motion orthogonal to the fibers become

$$\mathbf{C}_z \mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}} = -\mathbf{h} - \frac{N}{\lambda}[\bar{\mathbf{z}}\times]\mathbf{R}^T \dot{\mathbf{t}}$$

$$\dot{\mathbf{t}} = -\dot{\lambda}\,\mathbf{R}\bar{\mathbf{z}} - \lambda\,\mathbf{R}\dot{\bar{\mathbf{z}}} + \lambda\,\mathbf{R}[\bar{\mathbf{z}}\times]\mathbf{R}^T \boldsymbol{\Omega}\dot{\boldsymbol{\theta}}$$

$$\dot{\lambda}\,\mathbf{z}^T \mathbf{z} = -\lambda\,\mathbf{z}^T \dot{\mathbf{z}} - N\,\bar{\mathbf{z}}^T \mathbf{R}^T \dot{\mathbf{t}}\,.$$

The above three equations can be solved for $\dot{\boldsymbol{\theta}}$, $\dot{\mathbf{t}}$ and $\dot{\lambda}$ as follows: Replace $\dot{\mathbf{t}} = \dot{\mathbf{t}}(\dot{\boldsymbol{\theta}}, \dot{\lambda})$ from the second to the first, replace the resulting $\dot{\boldsymbol{\theta}}$ (which does not depend on $\dot{\lambda}$) back into the second, replace the resulting $\dot{\mathbf{t}} = \dot{\mathbf{t}}(\dot{\lambda})$ into the third, solve the third for $\dot{\lambda}$ and finally replace the resulting $\dot{\lambda}$ back into the second equation. The result is the equations stated in the lemma. □

The three differential equations (128). (129), (130) (in fact 3+3+1=7 equations) provide a solution which gives the parameter functions $\boldsymbol{\theta}(t)$, $\mathbf{t}(t)$ and $\lambda(t)$ of a transformation which transforms the reference motion $\mathbf{z}(t)$ into a motion orthogonal to the fibers. This transformation we call the "non-linear space-time Baarda S-transformation". To solve the differential equations we need initial values $\boldsymbol{\theta}_0$, $\mathbf{t}_0$ and λ_0, which specify a particular one out of all possible motions orthogonal to fibers. For example if we want the motion curve on $\mathcal{M}$ to pass from a particular initial point $\mathbf{x}(0)$, the initial values will be the unique ones for which $\mathbf{x}_i(0) = \lambda_0 \mathbf{R}(\boldsymbol{\theta}_0)\,\mathbf{z}_i(0) + \mathbf{t}_0$.
The first set of equations can be solved by direct integration

$$\boldsymbol{\theta}(t) = \boldsymbol{\theta}_0 + \int_0^t \boldsymbol{\Omega}(\boldsymbol{\theta})^{-1}\,\mathbf{R}(\boldsymbol{\theta})\mathbf{C}_z^{-1}\left(N[\bar{\mathbf{z}}\times]\dot{\bar{\mathbf{z}}} - \mathbf{h}\right) d\boldsymbol{\theta} \tag{132}$$

and this solution together with the derived one for λ can be used in the right side of the second in order to integrate for $\mathbf{t}(t)$.
In the particular case where the reference motion has been chosen so that $\bar{\mathbf{z}} = \mathbf{0}$ (centered motion) the differential equation of the ***non-linear space-time Baarda S-transformation*** become very simple:

$$\dot{\boldsymbol{\theta}} = -\boldsymbol{\Omega}^{-1}\mathbf{R}\mathbf{C}_z^{-1}\mathbf{h}\,, \tag{133}$$

$$\dot{\mathbf{t}} = \mathbf{0}\,, \tag{134}$$

$$\dot{\lambda} = -\lambda \frac{\mathbf{z}^T \dot{\mathbf{z}}}{\mathbf{z}^T \mathbf{z}}\,, \tag{135}$$

or in "solution" form

$$\boldsymbol{\theta}(t) = \boldsymbol{\theta}_0 - \int_0^t \boldsymbol{\Omega}(\boldsymbol{\theta})^{-1}\,\mathbf{R}(\boldsymbol{\theta})\mathbf{C}_z^{-1}\mathbf{h}\, d\boldsymbol{\theta} \tag{136}$$

$$\mathbf{t}(t) = \mathbf{t}_0 = \text{constant} \tag{137}$$

$$\lambda(t) = \lambda_0 + \left[\frac{1}{\sqrt{\mathbf{z}^T \mathbf{z}}}\right]_0^t . \tag{138}$$

If further it has been chosen that $\mathbf{z}^T \mathbf{z} = \text{constant}$ (as in the case of a principal motion) the last equation further reduces to

$$\dot{\lambda} = 0 \qquad \Rightarrow \qquad \lambda = \lambda_0 = \text{constant}\,. \tag{139}$$

Remark: The equations (127) are in fact nothing else than the ***inner constraints*** of Meissl, familiar from the linear and single epoch datum problem. To see that this is the case, we may use the inverse transformation $\mathbf{z}_i = \lambda^{-1}\mathbf{R}^T(\mathbf{x}_i - \mathbf{t})$ to obtain

$$\sum_i \left(\frac{\partial \mathbf{x}_i}{\partial \boldsymbol{\theta}}\right)^T \dot{\mathbf{x}}_i = \sum_i \lambda\, \boldsymbol{\Omega}^T \mathbf{R}[\mathbf{z}_i \times]\mathbf{R}^T \dot{\mathbf{x}}_i = -\boldsymbol{\Omega}^T \sum_i [\mathbf{x}_i \times]\dot{\mathbf{x}}_i + \boldsymbol{\Omega}^T[\mathbf{t}\times]\sum_i \dot{\mathbf{x}}_i = \mathbf{0} \tag{140}$$

$$\sum_i \left(\frac{\partial \mathbf{x}_i}{\partial \mathbf{t}}\right)^T \dot{\mathbf{x}}_i = \sum_i \dot{\mathbf{x}}_i = \mathbf{0} \tag{141}$$

$$\sum_i \left(\frac{\partial \mathbf{x}_i}{\partial \lambda}\right)^T \dot{\mathbf{x}}_i = \sum_i \mathbf{z}_i^T \mathbf{R}^T \dot{\mathbf{x}}_i = \lambda^{-1}(\mathbf{x}_i - \mathbf{t})^T \dot{\mathbf{x}}_i = 0\,. \tag{142}$$

In the general case with $|\boldsymbol{\Omega}| \neq 0$ and $\lambda \neq 0$, taking into account the second equation for the other two, we arrive at the familiar form of the inner constraints:

$$\sum_i [\mathbf{x}_i \times]\dot{\mathbf{x}}_i = \mathbf{0}\,, \qquad \sum_i \dot{\mathbf{x}}_i = \mathbf{0}\,, \qquad \sum_i \mathbf{x}_i^T \dot{\mathbf{x}}_i = 0\,. \tag{143}$$

If $\mathbf{x}_i$ is replaced by its approximate value $\mathbf{x}_i^0$, and $\dot{\mathbf{x}}_i$ is approximated by $\Delta\mathbf{x}_i / \Delta t = (\mathbf{x}_i - \mathbf{x}_i^0)/\Delta t$, elimination of the common factor Δt leads to the inner constraints of the linearization approach:

$$\sum_i [\mathbf{x}_i^0 \times]\Delta\mathbf{x}_i = \mathbf{0}\,, \qquad \sum_i \Delta\mathbf{x}_i = \mathbf{0}\,, \qquad \sum_i (\mathbf{x}_i^0)^T \Delta\mathbf{x}_i = 0\,. \tag{144}$$ □

How do different orthogonal motions relate to each other? Obviously they do not intersect, because if two of them have a common point, they will be identical. In a some sense to be specified, orthogonal motions are therefore "parallel" to each other.

Lemma: If two orthogonal motions $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ share a common point then they are identical.

Proof: Let $\boldsymbol{\theta}$, $\mathbf{t}$, λ and $\boldsymbol{\theta}'$, $\mathbf{t}'$, λ' be the transformation parameters which relate the two motions $\mathbf{x}(t)$ and $\mathbf{x}'(t)$, respectively, to the reference principal motion $\mathbf{z}(t)$ and let $\mathbf{x}(t_0) = \mathbf{x}'(t_0)$ be their common point. This implies that $\boldsymbol{\theta}(t_0) = \boldsymbol{\theta}'(t_0)$, $\mathbf{t}(t_0) = \mathbf{t}'(t_0)$ and $\lambda(t_0) = \lambda'(t_0)$. In view of $\dot{\mathbf{t}}' = \dot{\mathbf{t}} = \mathbf{0}$, $\dot{\lambda}' = \dot{\lambda} = 0$, this means that $\mathbf{t}(t) = \mathbf{t}'(t)$ and $\lambda(t) = \lambda'(t)$. On the other hand $\boldsymbol{\theta}(t)$ and $\boldsymbol{\theta}'(t)$ are solutions of the same first order differential equation (133) with identical initial conditions and therefore identical $\boldsymbol{\theta}(t) = \boldsymbol{\theta}'(t)$. From the identity of the transformation parameters it follows that $\mathbf{x}(t) = \mathbf{x}'(t)$ for any t and the motions are indeed identical. □

Lemma: Two orthogonal motions $\mathbf{x}'(t)$ and $\mathbf{x}(t)$ can be transformed into each other by means of a similarity transformation $\mathbf{x}_i'(t) = \mu\mathbf{Q}(\boldsymbol{\psi})\mathbf{x}_i(t) + \mathbf{d}$, where the transformation parameters $\boldsymbol{\psi}$, $\mathbf{d}$ and μ are independent of time. We shall call $\mathbf{Q}(\boldsymbol{\psi})$, $\mathbf{d}$ and μ, the relative rotation, relative displacement and relative scale of the motion $\mathbf{x}'(t)$ with respect to the motion $\mathbf{x}(t)$. In fact, if $\boldsymbol{\theta}'(t)$, $\mathbf{t}'(t)$, $\lambda'(t)$ and $\boldsymbol{\theta}(t)$, $\mathbf{t}(t)$, $\lambda(t)$ are the transformation parameter functions of the motions $\mathbf{x}'(t)$ and $\mathbf{x}(t)$, respectively, then $\lambda'(t) = \mu\lambda(t)$, $\mathbf{R}'(t) = \mathbf{Q}\mathbf{R}(t)$ and $\mathbf{t}'(t) = \mathbf{d} + \mu\mathbf{Q}\mathbf{t}(t)$.

Proof: $\mathbf{x}_i = \lambda\mathbf{R}\mathbf{z}_i + \mathbf{t} \;\Rightarrow\; \mathbf{z}_i = \lambda^{-1}\mathbf{R}^T(\mathbf{x}_i - \mathbf{t}) \;\Rightarrow$

$$\mathbf{x}_i' = \lambda'\mathbf{R}'\mathbf{z}_i + \mathbf{t}' = \frac{\lambda'}{\lambda}\mathbf{R}'\mathbf{R}^T(\mathbf{x}_i - \mathbf{t}) + \mathbf{t}' = \frac{\lambda'}{\lambda}\mathbf{R}'\mathbf{R}^T\mathbf{x}_i - \frac{\lambda'}{\lambda}\mathbf{R}'\mathbf{R}^T\mathbf{t} + \mathbf{t}' \equiv \mu\mathbf{Q}\mathbf{x}_i + \mathbf{d} \;\Rightarrow$$

$\mu=\frac{\lambda'}{\lambda}$, $\mathbf{Q}=\mathbf{R}'\mathbf{R}^T$, $\mathbf{d}=\mathbf{t}'-\frac{\lambda'}{\lambda}\mathbf{R}'\mathbf{R}^T\mathbf{t}=\mathbf{t}'-\mu\mathbf{Q}\mathbf{t}$.

It remains to show that μ, $\mathbf{Q}$ and $\mathbf{d}$ are time independent:

$$\dot{\lambda}'=\dot{\lambda}=0 \quad \Rightarrow \quad \dot{\mu}=\frac{\dot{\lambda}'}{\lambda}-\frac{\lambda'}{\lambda^2}\dot{\lambda}=0 \quad \Rightarrow \quad \mu=\text{constant.}$$

$$\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}=\mathbf{R}^T\boldsymbol{\omega}_t=-\mathbf{C}_z^{-1}\mathbf{h} \Rightarrow \quad \mathbf{R}'^T\boldsymbol{\omega}'_t=\mathbf{R}^T\boldsymbol{\omega}_t \quad \Rightarrow \quad \boldsymbol{\omega}'_t=\mathbf{R}'\mathbf{R}^T\boldsymbol{\omega}_t=\mathbf{Q}\boldsymbol{\omega}_t \quad \Rightarrow \quad [\boldsymbol{\omega}'_t\times]=\mathbf{Q}[\boldsymbol{\omega}_t\times]\mathbf{Q}^T.$$

Therefore

$$\dot{\mathbf{Q}}=\dot{\mathbf{R}}'\mathbf{R}^T+\mathbf{R}'\dot{\mathbf{R}}^T=[\boldsymbol{\omega}'_t\times]\mathbf{R}'\mathbf{R}^T-\mathbf{R}'\mathbf{R}^T[\boldsymbol{\omega}_t\times]=[\boldsymbol{\omega}'_t\times]\mathbf{Q}-\mathbf{Q}[\boldsymbol{\omega}_t\times]=\mathbf{Q}[\boldsymbol{\omega}_t\times]\mathbf{Q}^T\mathbf{Q}-\mathbf{Q}[\boldsymbol{\omega}_t\times]=\mathbf{0} \quad \Rightarrow$$

$\mathbf{Q}$ = constant.

$$\dot{\mathbf{t}}'=\dot{\mathbf{t}}=\mathbf{0} \quad \Rightarrow \quad \dot{\mathbf{d}}=\dot{\mathbf{t}}'-\dot{\mu}\mathbf{Q}\mathbf{t}-\mu\dot{\mathbf{Q}}\mathbf{t}-\mu\mathbf{Q}\dot{\mathbf{t}}=\mathbf{0} \quad \Rightarrow \quad \mathbf{d}=\text{constant.}$$ □

The above lemma motivates the introduction of the concept of "parallelism" of motions, which of course bears no relation to the standard parallelism of vectors in the sense of Levi-Civita.

Definition: A motion $\mathbf{x}'(t)$ is called ***parallel*** to another motion $\mathbf{x}(t)$ if $\mathbf{x}'(t)$ can be produced from $\mathbf{x}(t)$ by a time independent similarity transformation, i.e., if for every epoch t

$$\mathbf{x}'_i(t)=\mu\,\mathbf{Q}(\boldsymbol{\psi})\mathbf{x}_i(t)+\mathbf{d}, \tag{145}$$

where the transformation parameters $\boldsymbol{\psi}$, $\mathbf{d}$ and μ are independent of time. □

The curvilinear coordinates of two parallel motions are related by

$$\mathbf{R}'=\mathbf{Q}\mathbf{R}, \qquad \mathbf{t}'=\mathbf{d}+\mu\,\mathbf{Q}\mathbf{t}, \qquad \lambda'=\mu\lambda, \tag{146}$$

$$\mathbf{Q}=\mathbf{R}'\mathbf{R}^T, \qquad \mathbf{d}=\mathbf{t}'-\mu\,\mathbf{Q}\mathbf{t}, \qquad \mu=\frac{\lambda'}{\lambda}. \tag{147}$$

In view of the above definition the previous lemma can be restated as follows:

Lemma: All orthogonal motions are parallel to each other.

Lemma: The lengths $L_0'^t$ and L_0^t of two orthogonal motions $\mathbf{x}'(t)$ and $\mathbf{x}(t)$, respectively, from any epoch 0 to any epoch t are proportional, their ratio being equal to their time independent relative scale μ, i.e., $L_0'^t=\mu\,L_0^t$.

Proof:
Since for orthogonal motions $\dot{\mathbf{t}}=\mathbf{0}$, $\dot{\lambda}=0$ and $\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}=-\mathbf{C}_z^{-1}\mathbf{h}$, it holds that

$$\dot{\mathbf{x}}_i=\dot{\lambda}\,\mathbf{R}\mathbf{z}_i-\lambda\,\mathbf{R}[\mathbf{z}_i\times]\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\lambda\,\mathbf{R}\dot{\mathbf{z}}_i+\dot{\mathbf{t}}=\lambda\,\mathbf{R}\{[\mathbf{z}_i\times]\mathbf{C}_z^{-1}\mathbf{h}+\dot{\mathbf{z}}_i\}.$$

The metric on the orthogonal motion curve is given by

$$\frac{ds^2}{dt^2}=\frac{d\mathbf{x}^T d\mathbf{x}}{dt^2}=\left(\frac{d\mathbf{x}}{dt}\right)^T\frac{d\mathbf{x}}{dt}=\sum_i\dot{\mathbf{x}}_i^T\dot{\mathbf{x}}_i=$$
$$=\lambda^2\sum_i\left(-\mathbf{h}^T\mathbf{C}_z^{-1}[\mathbf{z}_i\times][\mathbf{z}_i\times]\mathbf{C}_z^{-1}\mathbf{h}+\dot{\mathbf{z}}_i^T[\mathbf{z}_i\times]\mathbf{C}_z^{-1}\mathbf{h}-\mathbf{h}^T\mathbf{C}_z^{-1}[\mathbf{z}_i\times]\dot{\mathbf{z}}_i+\dot{\mathbf{z}}_i^T\dot{\mathbf{z}}_i\right)=\lambda^2\left(\dot{\mathbf{z}}^T\dot{\mathbf{z}}-\mathbf{h}^T\mathbf{C}_z^{-1}\mathbf{h}\right).$$

It follows that

$$L_0'^{\,t} = \int_0^t \lambda' \left(\dot{\mathbf{z}}^T \dot{\mathbf{z}} - \mathbf{h}^T \mathbf{C}_z^{-1} \mathbf{h}\right)^{1/2} dt = \int_0^t \mu\, \lambda \left(\dot{\mathbf{z}}^T \dot{\mathbf{z}} - \mathbf{h}^T \mathbf{C}_z^{-1} \mathbf{h}\right)^{1/2} dt = \mu\, L_0^t \,. \qquad \square$$

8.4. Geodesic motions

Among various choices for the optimal motion, geodesics on the space-time manifold $\mathcal{M}$ are the more appealing from the theoretical point of view since they provide the shortest (and thus motion minimizing) route between any two given points $\mathbf{x}(0)$ and $\mathbf{x}(t)$ at the two extreme epochs of the time interval $(0,t)$. Unfortunately, the equations that have to be solved in order to obtain a geodesic are 7 rather complicated second order non-linear differential equations. We shall rewrite the equations (109), (104), (105) here in a more convenient form by replacing $\boldsymbol{\theta}$ with a new rotational parameter

$$\boldsymbol{\varphi} = \mathbf{R}^T \boldsymbol{\Omega} \dot{\boldsymbol{\theta}} \tag{148}$$

with time derivative

$$\dot{\boldsymbol{\varphi}} = \mathbf{R}^T (\dot{\boldsymbol{\Omega}} \dot{\boldsymbol{\theta}} + \boldsymbol{\Omega} \ddot{\boldsymbol{\theta}}) \tag{149}$$

We discard as before the 8th equation of the geodesic corresponding to the coordinate t and use instead the directly evaluated values of $\ddot{s}\dot{s}$ and $\dot{s}^2$. The new form (assuming also $\lambda \neq 0$) is for the $\boldsymbol{\theta}$ - equation:

$$\{2\frac{\dot{\lambda}}{\lambda} + [\boldsymbol{\varphi}\times]\}\,(\mathbf{h} + \mathbf{C}_z \boldsymbol{\varphi}) + (\dot{\mathbf{h}} + \dot{\mathbf{C}}_z \boldsymbol{\varphi} + \mathbf{C}_z \dot{\boldsymbol{\varphi}}) - \frac{\ddot{s}\dot{s}}{\dot{s}^2}(\mathbf{h} + \mathbf{C}_z \boldsymbol{\varphi}) = 0 \tag{150}$$

for the $\mathbf{t}$ - equation:

$$\ddot{\mathbf{t}} - \frac{\ddot{s}\dot{s}}{\dot{s}^2}\dot{\mathbf{t}} = \mathbf{0} \tag{151}$$

and for the λ - equation:

$$\frac{\ddot{\lambda}}{\lambda}\mathbf{z}^T \mathbf{z} - \dot{\mathbf{z}}^T \dot{\mathbf{z}} - (\mathbf{h} + \mathbf{C}_z \boldsymbol{\varphi})^T \boldsymbol{\varphi} - \mathbf{h}^T \boldsymbol{\varphi} - \frac{\ddot{s}\dot{s}}{\dot{s}^2}\frac{\dot{\lambda}}{\lambda}\mathbf{z}^T \mathbf{z} = 0 \tag{152}$$

where

$$\frac{\dot{s}^2}{\lambda^2} = \frac{\dot{\lambda}^2}{\lambda^2}\mathbf{z}^T \mathbf{z} + \dot{\mathbf{z}}^T \dot{\mathbf{z}} + \mathbf{h}^T \boldsymbol{\varphi} + (\mathbf{h} + \mathbf{C}_z \boldsymbol{\varphi})^T \boldsymbol{\varphi} + N\frac{\dot{\mathbf{t}}^T \dot{\mathbf{t}}}{\kappa^2} \tag{153}$$

$$\frac{\ddot{s}\dot{s}}{\lambda^2} = \frac{\ddot{\lambda}}{\lambda}\frac{\dot{\lambda}}{\lambda}\mathbf{z}^T \mathbf{z} + \frac{\dot{\lambda}}{\lambda}\dot{\mathbf{z}}^T \dot{\mathbf{z}} + \dot{\mathbf{z}}^T \ddot{\mathbf{z}} + \frac{\dot{\lambda}}{\lambda}\mathbf{h}^T \boldsymbol{\varphi} + \frac{1}{2}\dot{\mathbf{h}}^T \boldsymbol{\varphi} + \frac{1}{2}\mathbf{h}^T \dot{\boldsymbol{\varphi}} + (\mathbf{h} + \mathbf{C}_z \boldsymbol{\varphi})^T (\frac{\dot{\lambda}}{\lambda}\boldsymbol{\varphi} + \frac{1}{2}\dot{\boldsymbol{\varphi}}) +$$
$$+ \frac{1}{2}(\dot{\mathbf{h}} + \dot{\mathbf{C}}_z \boldsymbol{\varphi} + \mathbf{C}_z \dot{\boldsymbol{\varphi}})^T \boldsymbol{\varphi} + N\frac{\dot{\mathbf{t}}^T \ddot{\mathbf{t}}}{\lambda^2} \,. \tag{154}$$

Note also that

$$(\dot{\mathbf{h}} + \dot{\mathbf{C}}_z \boldsymbol{\varphi} + \mathbf{C}_z \dot{\boldsymbol{\varphi}}) = \frac{d}{dt}(\mathbf{h} + \mathbf{C}_z \boldsymbol{\varphi}) \,. \tag{155}$$

To define a geodesic we must introduce, in a more or less arbitrary way either the two end points $\mathbf{x}(0)$ and $\mathbf{x}(T)$ or the initial point $\mathbf{x}(0)$ and the initial tangent (velocity) vector $\dot{\mathbf{x}}(0)$. The choice of $\mathbf{x}(0)$ is arbitrary as it will be shortly explained. For a given initial point $\mathbf{x}(0)$ we may consider the following possibilities:

- Geodesics with nearest end points (in X):

  Obtain the element $\mathbf{x}(t)\in F_t$ which the nearest to $\mathbf{x}(0)$ element of the fiber F_t, and use the geodesic through $\mathbf{x}(0)$ and $\mathbf{x}(t)$. This solution is not intrinsic to $\mathcal{M}$, since the minimized distance $\|\mathbf{x}(t)-\mathbf{x}(0)\|$ is measured in X along a straight line lying outside $\mathcal{M}$.

- Geodesics of shortest length:

  Among all geodesics through $\mathbf{x}(0)$ and any $\mathbf{x}(T)\in F_T$ choose the one of shortest length. This is a standard problem of the calculus of variations with variable end-point . The solution is intrinsic to the manifold $\mathcal{M}$. It is also possible to consider both end points as variable and seek the shortest geodesic joining the two fibers (and submanifolds of $\mathcal{M}$) F_0 and F_T. However this problem is not always meaningful as it will be shown in the following.

- Geodesics orthogonal to fiber at the initial point:

  Use the geodesic passing through $\mathbf{x}(0)$ and having initial velocity vector $\dot{\mathbf{x}}(0)$ which is orthogonal to the local fiber F_0. This intrinsic solution combines in a certain sense the optimality of the geodesic with the concept of the orthogonal motion, though only at the initial point.

The above definitions give rise to some questions:
Are orthogonal motions geodesics and in particular geodesics of shortest length?
Is it possible to obtain a global optimum through the choice of $\mathbf{x}(0)$? That is, can we get an optimal solution by choosing $\mathbf{x}(0)$ in such a way that (a) the euclidean distance in X between $\mathbf{x}(0)$ and its nearest $\mathbf{x}(T)$ is minimized. (b) The length of the geodesic of shortest length between $\mathbf{x}(0)$ and F_T is minimized. (c) the length of the geodesic through $\mathbf{x}(0)$ orthogonal to F_0 is minimized.
The answer to the second question is negative as it follows from the following two lemmata.

Lemma: The motion $\mathbf{x}'(t)$ which is parallel to a given geodesic $\mathbf{x}(t)$ is itself a geodesic.

Proof: A look at the geodesic equations (apart from the t-equation) shows that the terms depending on the transformation parameters are only:

$$\frac{\ddot{\lambda}}{\lambda},\ \frac{\dot{\lambda}}{\lambda},\ \frac{\dot{\mathbf{t}}^T\dot{\mathbf{t}}}{\lambda^2},\ \frac{\dot{\mathbf{t}}^T\ddot{\mathbf{t}}}{\lambda^2},\ \boldsymbol{\varphi}=\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}} \text{ and } \boldsymbol{\varphi}=\frac{d\boldsymbol{\varphi}}{dt}=\mathbf{R}^T(\dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}}+\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}}).$$

We shall show that all these terms are invariant under the parallel transformation with parameters μ, $\mathbf{Q}(\boldsymbol{\psi})$ and $\mathbf{d}$. Recall that $\lambda'=\mu\lambda$, $\mathbf{t}'=\mathbf{d}+\mu\mathbf{Q}\mathbf{t}$ and $\mathbf{R}'=\mathbf{Q}\mathbf{R}$.

$$\lambda'=\mu\lambda \quad\Rightarrow\quad \frac{\ddot{\lambda}'}{\lambda'}=\frac{\mu\ddot{\lambda}}{\mu\lambda}=\frac{\ddot{\lambda}}{\lambda} \text{ and } \frac{\dot{\lambda}'}{\lambda'}=\frac{\mu\dot{\lambda}}{\mu\lambda}=\frac{\dot{\lambda}}{\lambda}.$$

We have already shown that $\boldsymbol{\omega}'_t=\mathbf{Q}\boldsymbol{\omega}_t$ so that $\dot{\boldsymbol{\omega}}'_t=\mathbf{Q}\dot{\boldsymbol{\omega}}_t$. Similarly $\boldsymbol{\omega}'_m=\mathbf{Q}\boldsymbol{\omega}_m$ and $\boldsymbol{\Omega}'=\mathbf{Q}\boldsymbol{\Omega}$.

$$\boldsymbol{\varphi}'=\mathbf{R}'^T\boldsymbol{\Omega}'\dot{\boldsymbol{\theta}}'=\mathbf{R}'^T\boldsymbol{\omega}'_t=\mathbf{R}^T\mathbf{Q}^T\mathbf{Q}\boldsymbol{\omega}_t=\mathbf{R}^T\boldsymbol{\omega}_t=\boldsymbol{\varphi}$$

$$\dot{\boldsymbol{\varphi}}'=\mathbf{R}'^T(\dot{\boldsymbol{\Omega}}'\dot{\boldsymbol{\theta}}'+\boldsymbol{\Omega}'\ddot{\boldsymbol{\theta}}')=\mathbf{R}'^T\dot{\boldsymbol{\omega}}'_t=\mathbf{R}^T\mathbf{Q}^T\mathbf{Q}\dot{\boldsymbol{\omega}}_t=\mathbf{R}^T\dot{\boldsymbol{\omega}}_t=\dot{\boldsymbol{\varphi}}$$

$$\mathbf{t}'=\mathbf{d}+\mu\mathbf{Q}\mathbf{t} \quad\Rightarrow\quad \dot{\mathbf{t}}'=\mu\mathbf{Q}\dot{\mathbf{t}} \text{ and } \ddot{\mathbf{t}}'=\mu\mathbf{Q}\ddot{\mathbf{t}} \quad\Rightarrow\quad \dot{\mathbf{t}}'^T\dot{\mathbf{t}}'=\mu^2\dot{\mathbf{t}}^T\dot{\mathbf{t}} \text{ and } \dot{\mathbf{t}}'^T\ddot{\mathbf{t}}'=\mu^2\dot{\mathbf{t}}^T\ddot{\mathbf{t}} \quad\Rightarrow$$

$$\frac{\dot{\mathbf{t}}'^T\dot{\mathbf{t}}'}{\lambda'^2}=\frac{\dot{\mathbf{t}}^T\dot{\mathbf{t}}}{\lambda^2} \text{ and } \frac{\dot{\mathbf{t}}'^T\ddot{\mathbf{t}}'}{\lambda'^2}=\frac{\dot{\mathbf{t}}^T\ddot{\mathbf{t}}}{\lambda^2}.$$

For the t-equation we have

$$\dot{\mathbf{t}}'=\mu\mathbf{Q}\dot{\mathbf{t}} \text{ and } \ddot{\mathbf{t}}'=\mu\mathbf{Q}\ddot{\mathbf{t}} \quad\Rightarrow\quad \ddot{\mathbf{t}}'-\frac{\dot{s}'\ddot{s}'}{\dot{s}'^2}\dot{\mathbf{t}}'=\mu\mathbf{Q}\ddot{\mathbf{t}}-\frac{\dot{s}\ddot{s}}{\dot{s}^2}\mu\mathbf{Q}\dot{\mathbf{t}}=\mu\mathbf{Q}(\ddot{\mathbf{t}}-\frac{\dot{s}\ddot{s}}{\dot{s}^2}\dot{\mathbf{t}})=\mathbf{0}.$$

Replacing the above invariant terms in the equations for $\mathbf{x}(t)$ we obtain the identical equations for $\mathbf{x}'(t)$, which is therefore also a geodesic. □

Lemma: An orthogonal motion is not a geodesic.

Proof: For an orthogonal motion we have $\dot{\lambda}=\ddot{\lambda}=0$, $\ddot{\mathbf{t}}=\dot{\mathbf{t}}=\mathbf{0}$ and $\mathbf{h}+\mathbf{C}_z\boldsymbol{\varphi}=\mathbf{0}$. A look at the geodesic equations (150), (151), (152) shows that they are all obviously satisfied except for the λ - equation (152), which with $\boldsymbol{\varphi}=-\mathbf{C}_z^{-1}\mathbf{h}$ becomes $\dot{\mathbf{z}}^T\dot{\mathbf{z}}-\mathbf{h}^T\mathbf{C}_z^{-1}\mathbf{h}=0$.
This equation does not hold in general but only for a restricted class of problems where the adjusted observations $\mathbf{y}(t)$ are such that the corresponding principal motion satisfies this condition. □

Lemma: The lengths $L_0'^t$ and L_0^t of two parallel motions $\mathbf{x}'(t)$ and $\mathbf{x}(t)$, respectively, from any epoch 0 to any epoch *t*, are proportional and their ratio is equal to their time independent relative scale μ, i.e., $L_0'^t=\mu L_0^t$.

Proof:

$$\frac{ds^2}{dt^2}=\lambda^2\left(\frac{\dot{\lambda}^2}{\lambda^2}\mathbf{z}^T\mathbf{z}+\dot{\mathbf{z}}^T\dot{\mathbf{z}}+2\mathbf{h}^T\boldsymbol{\varphi}+\boldsymbol{\varphi}^T\mathbf{C}_z\boldsymbol{\varphi}+\frac{N}{\lambda^2}\dot{\mathbf{t}}^T\dot{\mathbf{t}}\right)$$

The length of a curve is

$$L_0^t=\int_0^t\lambda\left(\frac{\dot{\lambda}^2}{\lambda^2}\mathbf{z}^T\mathbf{z}+\dot{\mathbf{z}}^T\dot{\mathbf{z}}+2\mathbf{h}^T\boldsymbol{\varphi}+\boldsymbol{\varphi}^T\mathbf{C}_z\boldsymbol{\varphi}+\frac{N}{\lambda^2}\dot{\mathbf{t}}^T\dot{\mathbf{t}}\right)^{1/2}dt\,.$$

For a parallel motion we have $\lambda'=\mu\lambda$, $\boldsymbol{\varphi}'=\boldsymbol{\varphi}$, $\frac{\dot{\lambda}'}{\lambda'}=\frac{\dot{\lambda}}{\lambda}$, as we have already seen. Since $\mathbf{t}'=\mathbf{d}+\mu\,\mathbf{Q}\mathbf{t}$, $\dot{\mathbf{t}}'=\mu\,\mathbf{Q}\dot{\mathbf{t}}$, $\dot{\mathbf{t}}'^T\dot{\mathbf{t}}'=\mu^2\dot{\mathbf{t}}^T\dot{\mathbf{t}}$ and $\frac{\dot{\mathbf{t}}'^T\dot{\mathbf{t}}'}{\lambda'^2}=\frac{\dot{\mathbf{t}}^T\dot{\mathbf{t}}}{\lambda^2}$, the length of the parallel motion is

$$L_0'^t=\int_0^t\lambda'\left(\frac{\dot{\lambda}^2}{\lambda^2}\mathbf{z}^T\mathbf{z}+\dot{\mathbf{z}}^T\dot{\mathbf{z}}+2\mathbf{h}^T\boldsymbol{\varphi}+\boldsymbol{\varphi}^T\mathbf{C}_z\boldsymbol{\varphi}+\frac{N}{\lambda^2}\dot{\mathbf{t}}^T\dot{\mathbf{t}}\right)^{1/2}dt=\mu\,L_0^t\,.$$ □

Corollary: The lengths $L_0'^t$ and L_0^t of two parallel geodesics $\mathbf{x}'(t)$ and $\mathbf{x}(t)$, respectively, from any epoch 0 to any epoch *t*, are proportional and their ratio is equal to their time independent relative scale μ, i.e., $L_0'^t=\mu L_0^t$.

Remark: The fact that parallelism modifies the length of orthogonal and geodesic motions by the relative scale factor leads to the adoption of equivalent classes of motions. All motions which are parallel are equivalent and they can be represented by a single element of the class, for example the one passing through $\mathbf{z}(0)$, where $\mathbf{z}(t)$ is a fixed principal motion (fixed by deciding the constant value of $\mathbf{z}^T\mathbf{z}$).
Any member of the class is as good a candidate for a solution to the space-time datum problem as any other, and in this sense the choice of initial point $\mathbf{x}(0)$ is arbitrary. If this point of view is not adopted, then we can "improve" a solution by making it "shorter" by simply making the scale smaller. A global optimum though cannot be obtained since the "optimal" value $\lambda=0$, leads to the trivial identity $\mathbf{x}(t)=\mathbf{0}$, i.e. to a network shrunk until all its points coincide. This is of course not acceptable and the point **0**, where in fact all fibers converge has to be excluded from $X=E^{3N}$, otherwise the single epoch solution manifolds will not constitute a fibering (fibers are by definition disjoint).
Coming to the network itself parallelism has the following interpretation. For two parallel motions $\mathbf{x}(t)$ and $\mathbf{x}'(t)$, the displacement curves in E^3 $\mathbf{x}_i(t)$ and $\mathbf{x}_i'(t)$ of any point *i* have identical shape since they differ only by a relative scale and orientation determined from the placements at the initial epoch. This follows from the fact that $\mathbf{x}_i'(t)-\mathbf{x}_i'(0)=\mu\,\mathbf{Q}[\mathbf{x}_i(t)-\mathbf{x}_i(0)]$, the parameters μ, **Q** (and **t**) been uniquely determined from the network initial placements, e.g. from 3 network points, solving the redundant equations $\mathbf{x}_1'(0)=\mu\,\mathbf{Q}\mathbf{x}_1(0)+\mathbf{t}$, $\mathbf{x}_2'(0)=\mu\,\mathbf{Q}\mathbf{x}_2(0)+\mathbf{t}$, $\mathbf{x}_3'(0)=\mu\,\mathbf{Q}\mathbf{x}_3(0)+\mathbf{t}$. □

Lemma: A given motion $\mathbf{z}(t)$ [i.e. a mapping $z:(0,t)\rightarrow X:t\rightarrow z(t)$ such that $f(z(t))=y(t)$] is a geodesic if it satisfies the following conditions:

$$(\dot{\mathbf{z}}^T\dot{\mathbf{z}})\dot{\mathbf{h}}=(\dot{\mathbf{z}}^T\ddot{\mathbf{z}})\mathbf{h}\,, \qquad (\dot{\mathbf{z}}^T\dot{\mathbf{z}})\ddot{\bar{\mathbf{z}}}=(\dot{\mathbf{z}}^T\ddot{\mathbf{z}})\dot{\bar{\mathbf{z}}} \tag{156}$$

Proof: The motion z(t) can be used as a reference motion in which case it can be "auto-represented" by the time-independent transformation parameters $\boldsymbol{\theta}(t)=\mathbf{0}$, $\mathbf{t}(t)=\mathbf{0}$, $\lambda(t)=1$. In this case we have $\dot{\mathbf{p}}=\mathbf{0}$, $\ddot{\mathbf{p}}=\mathbf{0}$ and the general geodesic equations (99) and (100) become

$$-\frac{\dot{\mathbf{z}}^T\ddot{\mathbf{z}}}{\dot{\mathbf{z}}^T\dot{\mathbf{z}}}\mathbf{g}+\mathbf{h}_t=\mathbf{0}\,, \qquad -\frac{\dot{\mathbf{z}}^T\ddot{\mathbf{z}}}{\dot{\mathbf{z}}^T\dot{\mathbf{z}}}\gamma+\kappa_t=0\,.$$

With $\mathbf{R}=\mathbf{I}$ and $\boldsymbol{\Omega}=-\mathbf{I}$ we get the values

$$\mathbf{g}=\begin{bmatrix}-\mathbf{h}\\ N\dot{\bar{\mathbf{z}}}\\ \mathbf{z}^T\dot{\mathbf{z}}\end{bmatrix}, \qquad \gamma=\dot{\mathbf{z}}^T\dot{\mathbf{z}}\,, \qquad \mathbf{h}_t=\begin{bmatrix}-\dot{\mathbf{h}}\\ N\ddot{\bar{\mathbf{z}}}\\ \mathbf{z}^T\ddot{\mathbf{z}}\end{bmatrix}, \qquad \kappa_t=\dot{\mathbf{z}}^T\ddot{\mathbf{z}}$$

which turn the geodesic equations into

$$-\dot{\mathbf{h}}+\frac{\dot{\mathbf{z}}^T\ddot{\mathbf{z}}}{\dot{\mathbf{z}}^T\dot{\mathbf{z}}}\mathbf{h}=\mathbf{0}\,, \qquad \ddot{\bar{\mathbf{z}}}-\frac{\dot{\mathbf{z}}^T\ddot{\mathbf{z}}}{\dot{\mathbf{z}}^T\dot{\mathbf{z}}}\dot{\bar{\mathbf{z}}}=\mathbf{0}\,, \qquad \mathbf{z}^T\ddot{\mathbf{z}}-\frac{\dot{\mathbf{z}}^T\ddot{\mathbf{z}}}{\dot{\mathbf{z}}^T\dot{\mathbf{z}}}\mathbf{z}^T\dot{\mathbf{z}}=0\,, \qquad \dot{\mathbf{z}}^T\ddot{\mathbf{z}}-\frac{\dot{\mathbf{z}}^T\ddot{\mathbf{z}}}{\dot{\mathbf{z}}^T\dot{\mathbf{z}}}\dot{\mathbf{z}}^T\dot{\mathbf{z}}=0$$

the last two being identical and trivially satisfied. □

Remark: The conditions of the above lemma are differential equations with solutions

$$\mathbf{h}=\sqrt{\dot{\mathbf{z}}^T\dot{\mathbf{z}}}\;\mathbf{c}_h\,, \qquad \dot{\bar{\mathbf{z}}}=\sqrt{\dot{\mathbf{z}}^T\dot{\mathbf{z}}}\;\mathbf{c}_z \tag{157}$$

where $\mathbf{c}_h$ and $\mathbf{c}_z$ are arbitrary constant vectors. □

8.5. Geodesic motions as energy minimizing motions

In flat space straight lines are the optimal choices for connecting points since they are the geodesics. However when motion is involved straight lines have another interpretation, that of the orbits of particles in the absence of any forces (free motion). According to Lagrangean dynamics the motion is governed by the principle of least action (Hamilton's principle) i.e. it is such that the integral of the Lagrangean function $L=T-U$ is minimized. In the case of free motion the potential function U is vanishing and the minimized function becomes the kinetic energy $T=\frac{1}{2}v^2=\frac{1}{2}\dot{\mathbf{x}}^T\dot{\mathbf{x}}$. In the case of the a flat (euclidean) space, the minimization of the integral (a factor 2 is added for convenience)

$$2\int_0^T T(t)\,dt=\int_0^T \dot{\mathbf{x}}^T(t)\dot{\mathbf{x}}(t)\,dt=\min \tag{158}$$

leads to curves which coincide with geodesics. It is a well known fact that this coincidence of minimum length curves extends to the case of a Riemannian manifold (see e.g., Choquet-Bruhat, DeWitt-Morette, Dillard-Bleick (1977) ch. V.C, Bishop & Goldberg (1968) ch. 5.13, Sternberg (1964) p. 149,.168,) such as our space-time manifold. The relevant proofs in the literature are quite involved and for this reason we present a simplified proof for our particular case:

Lemma: The vanishing terms in the Euler equations for geodesics (minimization of $F=\sqrt{\dot{\mathbf{x}}^T\dot{\mathbf{x}}}$) and the ones for the minimization of the energy $L=\dot{\mathbf{x}}^T\dot{\mathbf{x}}$ are related by the following relation:

$$\left(\frac{\partial F}{\partial \mathbf{u}}\right)^T-\frac{d}{d\tau}\left(\frac{\partial F}{\partial \dot{\mathbf{u}}}\right)^T=\left\{\left(\frac{\partial L}{\partial \mathbf{u}}\right)^T-\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\mathbf{u}}}\right)^T\right\}+\frac{1}{2}L^{-1}\frac{dL}{d\tau}\left(\frac{\partial L}{\partial \dot{\mathbf{u}}}\right)^T \tag{159}$$

Proof:
Minimum energy curves minimize the integral of $L=\dot{\mathbf{x}}^T\dot{\mathbf{x}}$, while geodesics minimize the integral of $F=\sqrt{\dot{\mathbf{x}}^T\dot{\mathbf{x}}}=L^{1/2}$. The Euler-Lagrange equations for geodesics are

$$0=\frac{\partial F}{\partial u_i}-\frac{d}{d\tau}\left(\frac{\partial F}{\partial \dot{u}_i}\right)=\frac{1}{2}L^{-1/2}\frac{\partial L}{\partial u_i}-\frac{d}{d\tau}\left(\frac{1}{2}L^{-1/2}\frac{\partial L}{\partial \dot{u}_i}\right)=\frac{1}{2}L^{-1/2}\frac{\partial L}{\partial u_i}+\frac{1}{4}L^{-3/2}\frac{dL}{d\tau}\frac{\partial L}{\partial \dot{u}_i}-\frac{1}{2}L^{-1/2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{u}_i}\right)=$$

$$=\frac{1}{2}L^{-1/2}\left[\frac{\partial L}{\partial u_i}-\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{u}_i}\right)\right]+\frac{1}{4}L^{-3/2}\frac{dL}{d\tau}\frac{\partial L}{\partial \dot{u}_i}$$

and since $L\neq 0$ the equation of the geodesic becomes

$$\frac{\partial L}{\partial u_i}-\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{u}_i}\right)+\frac{1}{2}L^{-1}\frac{dL}{d\tau}\frac{\partial L}{\partial \dot{u}_i}=0$$

or in matrix form

$$\left(\frac{\partial L}{\partial \mathbf{u}}\right)^T-\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\mathbf{u}}}\right)^T+\frac{1}{2}L^{-1}\frac{dL}{d\tau}\left(\frac{\partial L}{\partial \dot{\mathbf{u}}}\right)^T=\mathbf{0}\,.$$

□

Lemma: A motion on $\mathcal{M}$ is a geodesic motion if and only if its minimizes the energy integral.

Proof: To show that a geodesic is also a minimum energy curve (and vice versa) it suffices to show that $\frac{dL}{d\tau}=0$, in which case equation (159) becomes the Euler-Lagrange equation for the energy minimization problem. Recalling that $L=\dot{\mathbf{x}}^T\dot{\mathbf{x}}=\dot{\mathbf{u}}^T\overline{\mathbf{G}}\dot{\mathbf{u}}=\dot{s}^2$ we get

$$\frac{dL}{d\tau}=2\dot{\mathbf{u}}^T\overline{\mathbf{G}}\ddot{\mathbf{u}}+\dot{\mathbf{u}}^T\dot{\overline{\mathbf{G}}}\dot{\mathbf{u}}\,,\quad \frac{dL}{d\dot{\mathbf{u}}}=2\dot{\mathbf{u}}^T\overline{\mathbf{G}}\qquad\Rightarrow\qquad \frac{dL}{d\tau}\left(\frac{dL}{d\dot{\mathbf{u}}}\right)^T=2\dot{\overline{\mathbf{G}}}\dot{\mathbf{u}}+2\overline{\mathbf{G}}\ddot{\mathbf{u}}\,,$$

$$\frac{\partial L}{\partial u_m}=\dot{\mathbf{u}}^T\frac{\partial\overline{\mathbf{G}}}{\partial u_m}\dot{\mathbf{u}}\qquad\Rightarrow\qquad \frac{\partial L}{\partial \mathbf{u}}\dot{\mathbf{u}}=\sum_m\frac{\partial L}{\partial u_m}\dot{u}_m=\dot{\mathbf{u}}^T\left(\sum_m\frac{\partial\overline{\mathbf{G}}}{\partial u_m}\dot{u}_m\right)\dot{\mathbf{u}}=\dot{\mathbf{u}}^T\dot{\overline{\mathbf{G}}}\dot{\mathbf{u}}$$

Multiplying the Euler equation of the geodesic with $\dot{\mathbf{u}}$ from the right gives

$$\mathbf{0}=\left(\frac{\partial L}{\partial \mathbf{u}}\right)\dot{\mathbf{u}}-\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\mathbf{u}}}\right)\dot{\mathbf{u}}+\frac{1}{2}L^{-1}\frac{dL}{d\tau}\left(\frac{\partial L}{\partial \dot{\mathbf{u}}}\right)\dot{\mathbf{u}}=\dot{\mathbf{u}}^T\dot{\overline{\mathbf{G}}}\dot{\mathbf{u}}-(2\dot{\overline{\mathbf{G}}}\dot{\mathbf{u}}+2\overline{\mathbf{G}}\ddot{\mathbf{u}})^T\dot{\mathbf{u}}+\frac{1}{2}L^{-1}\frac{dL}{d\tau}(2\dot{\overline{\mathbf{G}}}\dot{\mathbf{u}}+2\overline{\mathbf{G}}\ddot{\mathbf{u}})^T\dot{\mathbf{u}}=$$

$$=-\dot{\mathbf{u}}^T\dot{\overline{\mathbf{G}}}\dot{\mathbf{u}}-2\dot{\mathbf{u}}^T\overline{\mathbf{G}}\ddot{\mathbf{u}}+\frac{1}{2}L^{-1}\frac{dL}{d\tau}(2\dot{\mathbf{u}}^T\dot{\overline{\mathbf{G}}}\dot{\mathbf{u}}+2\dot{\mathbf{u}}^T\overline{\mathbf{G}}\ddot{\mathbf{u}})=-\frac{dL}{d\tau}+\frac{dL}{d\tau}\frac{\dot{\mathbf{u}}^T\dot{\overline{\mathbf{G}}}\dot{\mathbf{u}}+\dot{\mathbf{u}}^T\overline{\mathbf{G}}\ddot{\mathbf{u}}}{\dot{\mathbf{u}}^T\overline{\mathbf{G}}\dot{\mathbf{u}}}=\frac{dL}{d\tau}\left[\frac{\dot{\mathbf{u}}^T\dot{\overline{\mathbf{G}}}\dot{\mathbf{u}}+\dot{\mathbf{u}}^T\overline{\mathbf{G}}\ddot{\mathbf{u}}}{\dot{\mathbf{u}}^T\overline{\mathbf{G}}\dot{\mathbf{u}}}-1\right]$$

$$\Rightarrow\qquad\frac{dL}{d\tau}=0$$

We have proved the sufficiency part of the lemma. To prove necessity i.e. to show that an energy minimizing motion is a geodesic we consider the Euler equations for the energy

$$\left(\frac{\partial L}{\partial \mathbf{u}}\right)^T-\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\mathbf{u}}}\right)^T=\mathbf{0}\qquad\Rightarrow$$

$$\mathbf{0}=\left(\frac{\partial L}{\partial \mathbf{u}}\right)\dot{\mathbf{u}}-\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\mathbf{u}}}\right)\dot{\mathbf{u}}=\dot{\mathbf{u}}^T\dot{\overline{\mathbf{G}}}\dot{\mathbf{u}}-(2\dot{\overline{\mathbf{G}}}\dot{\mathbf{u}}+2\overline{\mathbf{G}}\ddot{\mathbf{u}})^T\dot{\mathbf{u}}=\dot{\mathbf{u}}^T\dot{\overline{\mathbf{G}}}\dot{\mathbf{u}}-2\dot{\mathbf{u}}^T\dot{\overline{\mathbf{G}}}\dot{\mathbf{u}}-2\dot{\mathbf{u}}^T\overline{\mathbf{G}}\ddot{\mathbf{u}}=-\dot{\mathbf{u}}^T\dot{\overline{\mathbf{G}}}\dot{\mathbf{u}}-2\dot{\mathbf{u}}^T\overline{\mathbf{G}}\ddot{\mathbf{u}}=-\frac{dL}{d\tau}=0$$

□

The energy minimization principle is used in Appendix E for the derivation of the differential equations of the geodesic motions using the Euler equations, in order to have a check on the direct results of Appendix D, since in both cases the computations involved are quite lengthy.

We finally look into the global minimization problem of choosing the best geodesic motion joining the initial and terminal fibers F_0 and F_T, respectively. As explained in a previous remark this problem has no solution and since parallelism of motions has been decided to be interpreted as motion equivalence we have to turn to the more simple problem where the initial point $\mathbf{x}(0)$ is arbitrarily chosen. Any motion from F_0 to F_T is parallel (and thus equivalent) to a motion through $\mathbf{x}(0)$.

Lemma: The geodesic through a given point $\mathbf{x}(0)$ on the initial fiber F_0, with the shortest length is the one which at the terminal fiber F_T is orthogonal to this fiber, i.e. the vector $\dot{\mathbf{x}}(T)$ tangent to the geodesic is orthogonal to the tangent space of F_T at the point $\mathbf{x}(T)$.

Proof: This is a standard result in the calculus of variations. See, e.g., Loomis & Sternberg, 1980, § 13.9., p. 535-536, eq. 9.10. and Sternberg, 1964, § IV.2, theorem 2.2. □

The analytical form of the orthogonality condition is $\left(\frac{\partial \mathbf{x}}{\partial \mathbf{p}}(T)\right)^T \dot{\mathbf{x}}(T)=\mathbf{0}$, which is a local application of the orthogonality criterion used in the case of orthogonal motions and has therefore the explicit form

$$\dot{\boldsymbol{\theta}}(T)=-\boldsymbol{\Omega}^{-1}(\boldsymbol{\theta}(T))\mathbf{R}(\boldsymbol{\theta}(T))\mathbf{C}_z^{-1}(T)\mathbf{h}(T) \tag{160}$$

$$\dot{\mathbf{t}}(T)=\mathbf{0} \tag{161}$$

$$\dot{\lambda}(T)=-\lambda(T)\frac{\mathbf{z}^T(T)\dot{\mathbf{z}}(T)}{\mathbf{z}^T(T)\mathbf{z}(T)}. \tag{162}$$

8.6. The case of rigid transformations

When distance measurements have been carried out in the geodetic network, then scale is determined from the available observations and the datum problem is restricted to the choice of origin and orientation of the reference frame. As a consequence only rigid transformations on each single epoch fiber are allowed, being the ones that leave the (adjusted) observables invariant. For such transformations generated by

$$\mathbf{x}_i=\mathbf{R}\mathbf{z}_i+\mathbf{t} \tag{163}$$

the derivation of the various differential geometric characteristics follows the same lines as in the case of the similarity transformations. However all relevant results can be obtained also as special cases of the ones for the similarity transformation by simply setting $\lambda=1$, $\dot{\lambda}=\ddot{\lambda}=0$ and discarding equations corresponding to the λ coordinate. We give directly the results of this approach.

The non-linear equations for a ***motion nearest to a specific point*** $\mathbf{x}_0$ in the rigid case are:

$$\frac{1}{N}\sum_i[(\mathbf{x}_{0i}-\bar{\mathbf{x}}_0)\times]\mathbf{R}(t)[\mathbf{z}_i(t)-\bar{\mathbf{z}}(t)]=\mathbf{0} \tag{164}$$

$$\mathbf{x}_i(t)=\bar{\mathbf{x}}_0+\mathbf{R}(t)[\mathbf{z}_i(t)-\bar{\mathbf{z}}(t)] \tag{165}$$

The ***principal motion*** in the rigid case is given by

$$\mathbf{x}_i(t)=\mathbf{U}(t)^T[\mathbf{z}_i(t)-\bar{\mathbf{z}}(t)] \tag{166}$$

where $\mathbf{z}_i(t)$ is any reference motion, $\overline{\mathbf{z}}(t)=\frac{1}{N}\sum_i \mathbf{z}_i(t)$ and **U** is the orthogonal matrix with columns the orthonormal eigenvectors of the matrix

$$\mathbf{C}_{\delta z}=\sum_i[(\mathbf{z}_i-\overline{\mathbf{z}})^T(\mathbf{z}_i-\overline{\mathbf{z}})\,\mathbf{I}-(\mathbf{z}_i-\overline{\mathbf{z}})(\mathbf{z}_i-\overline{\mathbf{z}})^T]\,. \tag{167}$$

The differential equations for ***orthogonal motions*** in the rigid case are:

$$\dot{\boldsymbol{\theta}}=-\boldsymbol{\Omega}^{-1}\mathbf{R}\mathbf{C}_z^{-1}\mathbf{h}\,,\qquad \boldsymbol{\theta}(t)=\boldsymbol{\theta}_0-\int_0^t \boldsymbol{\Omega}(\boldsymbol{\theta})^{-1}\mathbf{R}(\boldsymbol{\theta})\mathbf{C}_z^{-1}\mathbf{h}\,d\boldsymbol{\theta} \tag{168}$$

$$\dot{\mathbf{t}}=\mathbf{0}\,,\qquad \mathbf{t}(t)=\mathbf{t}_0=\text{const.} \tag{169}$$

where a "centered" reference motions ($\overline{\mathbf{z}}=\mathbf{0}$) is assumed. The ***inner constraints*** become

$$\sum_i[\mathbf{x}_i\times]\dot{\mathbf{x}}_i=\mathbf{0}\,,\qquad \sum_i\dot{\mathbf{x}}_i=\mathbf{0}\,. \tag{170}$$

The ***geodesic equations*** for the rigid case are:

$\boldsymbol{\theta}$ – equation:

$$\boldsymbol{\Omega}[(\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})\times]\mathbf{R}(\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})+\boldsymbol{\Omega}^T\mathbf{R}[\dot{\mathbf{h}}+\dot{\mathbf{C}}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\mathbf{C}_z\mathbf{R}^T(\dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}}+\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}})]-\frac{\ddot{s}}{\dot{s}}\boldsymbol{\Omega}^T\mathbf{R}(\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})=\mathbf{0} \tag{171}$$

t – equation:

$$\ddot{\mathbf{t}}-\frac{\ddot{s}}{\dot{s}}\dot{\mathbf{t}}=\mathbf{0} \tag{172}$$

t – equation:

$$\dot{\mathbf{z}}^T\ddot{\mathbf{z}}+\mathbf{h}^T\mathbf{R}^T(\dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}}+\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}})-\frac{1}{2}\dot{\boldsymbol{\theta}}^T\boldsymbol{\Omega}^T\mathbf{R}\dot{\mathbf{C}}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}-\frac{\ddot{s}}{\dot{s}}(\dot{\mathbf{z}}^T\dot{\mathbf{z}}+\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})=0\,. \tag{173}$$

The last equation is not used. Instead $\frac{\ddot{s}}{\dot{s}}$ is computed from the ratio of the following two expressions:

$$\dot{s}^2=\dot{\mathbf{z}}^T\dot{\mathbf{z}}+\mathbf{h}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+(\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+N\,\dot{\mathbf{t}}^T\dot{\mathbf{t}} \tag{174}$$

$$\ddot{s}\dot{s}=\dot{\mathbf{z}}^T\ddot{\mathbf{z}}+\mathbf{h}^T\mathbf{R}^T(\dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}}+\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}})+\dot{\mathbf{h}}^T\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\frac{1}{2}\dot{\boldsymbol{\theta}}^T\boldsymbol{\Omega}^T\mathbf{R}\dot{\mathbf{C}}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\dot{\boldsymbol{\theta}}^T\boldsymbol{\Omega}^T\mathbf{R}\mathbf{C}_z\mathbf{R}^T(\dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}}+\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}})+N\,\dot{\mathbf{t}}^T\ddot{\mathbf{t}} \tag{175}$$

In the general $|\boldsymbol{\Omega}|\neq 0$ and the first equation (171) simplifies to

$\boldsymbol{\theta}$ – equation:

$$[(\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})\times](\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})+[\dot{\mathbf{h}}+\dot{\mathbf{C}}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}+\mathbf{C}_z\mathbf{R}^T(\dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}}+\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}})]-\frac{\ddot{s}}{\dot{s}}(\mathbf{h}+\mathbf{C}_z\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}})=\mathbf{0} \tag{176}$$

In compact form using $\boldsymbol{\varphi}=\mathbf{R}^T\boldsymbol{\Omega}\dot{\boldsymbol{\theta}}$ and $\dot{\boldsymbol{\varphi}}=\mathbf{R}^T(\dot{\boldsymbol{\Omega}}\dot{\boldsymbol{\theta}}+\boldsymbol{\Omega}\ddot{\boldsymbol{\theta}})$, the geodesic equations are

$\boldsymbol{\theta}$ – equation:

$$[\boldsymbol{\varphi}\times](\mathbf{h}+\mathbf{C}_z\boldsymbol{\varphi})+\frac{d}{dt}(\mathbf{h}+\mathbf{C}_z\boldsymbol{\varphi})-\frac{\ddot{s}\dot{s}}{\dot{s}^2}(\mathbf{h}+\mathbf{C}_z\boldsymbol{\varphi})=\mathbf{0} \tag{177}$$

t – equation:

$$\ddot{\mathbf{t}}-\frac{\ddot{s}}{\dot{s}}\dot{\mathbf{t}}=\mathbf{0} \tag{178}$$

where

$$\dot{s}^2=\dot{\mathbf{z}}^T\dot{\mathbf{z}}+\mathbf{h}^T\boldsymbol{\varphi}+(\mathbf{h}+\mathbf{C}_z\boldsymbol{\varphi})^T\boldsymbol{\varphi}+N\,\dot{\mathbf{t}}^T\dot{\mathbf{t}} \tag{179}$$

$$\ddot{s}\dot{s}=\dot{\mathbf{z}}^T\ddot{\mathbf{z}}+\frac{1}{2}\dot{\mathbf{h}}^T\boldsymbol{\varphi}+\frac{1}{2}\mathbf{h}^T\dot{\boldsymbol{\varphi}}+\frac{1}{2}(\mathbf{h}+\mathbf{C}_z\boldsymbol{\varphi})^T\dot{\boldsymbol{\varphi}}+\frac{1}{2}\left[\frac{d}{dt}(\mathbf{h}+\mathbf{C}_z\boldsymbol{\varphi})\right]^T\boldsymbol{\varphi}+N\,\dot{\mathbf{t}}^T\ddot{\mathbf{t}}\,. \tag{180}$$

We can now reproduce some of the results of the similarity case, specialized to the rigid transformation. The proofs are simplified versions of the corresponding proofs of the similarity case and are therefore omitted.

Lemma: The principal motion is unique.

Lemma: If two orthogonal motions $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ share a common point then they are identical.

Lemma : All orthogonal motions are parallel to each other.

Lemma: The motion $\mathbf{x}'(t)$ which is parallel to a given geodesic $\mathbf{x}(t)$ is itself a geodesic. Here parallel means that $\mathbf{x}'(t)$ is produced from $\mathbf{x}(t)$ by a time independent rigid transformation, i.e. for every epoch *t*

$$\mathbf{x}'_i(t)=\mathbf{Q}(\boldsymbol{\psi})\mathbf{x}_i(t)+\mathbf{d}\,, \tag{181}$$

where the transformation parameters $\boldsymbol{\psi}$ and **d** are independent of time.

Lemma: An orthogonal motion is a geodesic.

Proof: For an orthogonal motion $\dot{\mathbf{t}}=\mathbf{0}$ and $\mathbf{h}+\mathbf{C}_z\boldsymbol{\varphi}=\mathbf{0}$ so that both geodesic equations (177) and (178) are obviously satisfied. □

Lemma: The lengths $L_0'^{\,t}$ and L_0^t of two parallel motions $\mathbf{x}'(t)$ and $\mathbf{x}(t)$, respectively, from any epoch 0 to any epoch t, are equal, $L_0'^{\,t}=L_0^t$.

Corollary: The lengths $L_0'^{\,t}$ and L_0^t of two parallel orthogonal motions $\mathbf{x}'(t)$ and $\mathbf{x}(t)$, respectively, from any epoch 0 to any epoch *t* are equal, $L_0'^{\,t}=L_0^t$.

Corollary: The lengths $L_0'^{\,t}$ and L_0^t of two parallel geodesics $\mathbf{x}'(t)$ and $\mathbf{x}(t)$, respectively, from any epoch 0 to any epoch t, are equal, $L_0'^{\,t}=L_0^t$.

Remark: The fact that parallelism preserves the length of orthogonal and geodesic motions leads to the adoption of equivalent classes of motions. All motions which are parallel are equivalent and they can be represented by a single element of the class, for example the one passing through $\mathbf{z}(0)$, where $\mathbf{z}(t)$ is the principal motion. Any member of the class is as good a candidate for a solution to the space-time datum problem as any other, and in this sense the choice of initial point $\mathbf{x}(0)$ is arbitrary.

Coming to the network itself parallelism has the following interpretation. For two parallel motions $\mathbf{x}(t)$ and $\mathbf{x}'(t)$, the displacement curves in E^3 $\mathbf{x}_i(t)$ and $\mathbf{x}'_i(t)$ of any point *i* have identical shape and size since they differ only by a relative orientation determined from the placements at the initial epoch. This follows from the fact that $\mathbf{x}'_i(t)-\mathbf{x}'_i(0)=\mathbf{Q}[\mathbf{x}_i(t)-\mathbf{x}_i(0)]$, the parameters $\mathbf{Q}$ (and $\mathbf{t}$) been uniquely determined from the network initial placements, e.g. from 3 network points, solving the redundant equations $\mathbf{x}'_1(0)=\mathbf{Q}\,\mathbf{x}_1(0)+\mathbf{t}$, $\mathbf{x}'_{21}(0)=\mathbf{Q}\,\mathbf{x}_2(0)+\mathbf{t}$, $\mathbf{x}'_3(0)=\mathbf{Q}\,\mathbf{x}_3(0)+\mathbf{t}$. □

Lemma: A given motion $\mathbf{z}(t)$ [i.e. a mapping $z:(0,t)\to X:t\to z(t)$ such that $f(z(t))=y(t)$] is a geodesic if it satisfies the following differential equations:

$$(\dot{\mathbf{z}}^T\dot{\mathbf{z}})\dot{\mathbf{h}}=(\dot{\mathbf{z}}^T\ddot{\mathbf{z}})\mathbf{h}\,, \qquad (\dot{\mathbf{z}}^T\dot{\mathbf{z}})\ddot{\bar{\mathbf{z}}}=(\dot{\mathbf{z}}^T\ddot{\mathbf{z}})\dot{\bar{\mathbf{z}}} \tag{182}$$

with obvious solutions

$$\mathbf{h}=\sqrt{\dot{\mathbf{z}}^T\dot{\mathbf{z}}}\;\mathbf{c}_h\,, \qquad \dot{\bar{\mathbf{z}}}=\sqrt{\dot{\mathbf{z}}^T\dot{\mathbf{z}}}\;\mathbf{c}_z \tag{183}$$

where $\mathbf{c}_h$ and $\mathbf{c}_z$ are arbitrary constant vectors. □

Unlike the similarity case, the global minimization problem for the length of motions is meaningful in the rigid case. However we cannot expect to have a unique solution since parallel motions have the same length.

Lemma: Any orthogonal motion has minimum length among all possible motions.

Proof: The solution to the problem of finding the shortest curve joining two submanifolds of a given Riemannian manifold is well known to be the geodesic which is orthogonal to the submanifolds at its two endpoints (see again Loomis & Sternberg, 1980, §13.9., p. 535-536, eq. 9.10. and Sternberg, 1964, § IV.2, theorem 2.2.). An orthogonal motion is a geodesic which satisfies an orthogonality condition for all fibers and therefore at the initial and terminal fibers, which are the above mentioned submanifolds in our case. □

This final lemma justifies the introduction of orthogonal motions as generalizations to the space-time domain of the well known Baarda transformation (*S*-transformation or rather *R*-transformation for the rigid case) originally introduced for the linear (or linearized) case and only the spatial domain. In contrast to this positive result, in case of the similarity transformation the additional scale parameter "destroys" in some sense this global optimality property of orthogonal motions. In a way that is not quite natural, we have imposed a point of view where parallel motions are equivalent although they have different lengths. A way out of this problem is perhaps the redefinition of the metric on the space-time manifold $\mathcal{M}$, instead of the one inherited from the Euclidean space *X*, in a way that its important aspects are preserved but the lengths of parallel motions turn out to be equal.

8.7. Two-dimensional networks

The previous results can be obtained also for plane networks, in which case we need as auxiliary quantities the antisymmetric matrix

$$\mathbf{W}\equiv\frac{\partial\mathbf{R}}{\partial\theta}\mathbf{R}^T \tag{184}$$

with properties $\mathbf{W}^T=-\mathbf{W}$, $\mathbf{W}^T\mathbf{W}=-\mathbf{W}^2=\mathbf{I}$ and $\mathbf{R}^T\mathbf{W}\mathbf{R}=\mathbf{R}$ and a function analogous to the angular momentum vector

$$h\equiv\sum_i \mathbf{z}_i^T\mathbf{W}\dot{\mathbf{z}}_i\,. \tag{185}$$

The derivations follow exactly the same lines as in the case of three-dimensional networks. We give here only the results.

The similarity transformation case

The differential equations of the ***orthogonal motions*** are

$$(\mathbf{z}^T\mathbf{z})\dot{\theta}-h=0 \tag{186}$$

$$\dot{\mathbf{t}}=\mathbf{0} \tag{187}$$

$$\dot{\lambda}=0\,. \tag{188}$$

The differential equations of the ***geodesic motions*** are

$$\lambda^2(\mathbf{z}^T\mathbf{z})\ddot{\theta}+2\lambda(\mathbf{z}^T\mathbf{z})\dot{\theta}\dot{\lambda}-2\lambda h\dot{\lambda}-\lambda^2\dot{h}-\frac{\ddot{s}}{\dot{s}}\lambda^2[(\mathbf{z}^T\mathbf{z})\dot{\theta}-h]=0 \tag{189}$$

$$\ddot{\mathbf{t}}-\frac{\ddot{s}}{\dot{s}}\dot{\mathbf{t}}=\mathbf{0} \tag{190}$$

$$(\mathbf{z}^T\mathbf{z})\ddot{\lambda}-\lambda(\mathbf{z}^T\mathbf{z})\dot{\theta}^2+2\lambda h\dot{\theta}-\lambda(\dot{\mathbf{z}}^T\dot{\mathbf{z}})-\frac{\ddot{s}}{\dot{s}}(\mathbf{z}^T\mathbf{z})\dot{\lambda}=0 \tag{191}$$

The equation corresponding to t

$$-\lambda^2h\ddot{\theta}-2\lambda h\dot{\lambda}\dot{\theta}+2\lambda\,\dot{\mathbf{z}}^T\dot{\mathbf{z}}\dot{\lambda}+\lambda^2\,\dot{\mathbf{z}}^T\ddot{\mathbf{z}}-\frac{\ddot{s}}{\dot{s}}\lambda^2(\dot{\mathbf{z}}^T\dot{\mathbf{z}}-h\dot{\theta})=0 \tag{192}$$

is not used. Instead the arc length is eliminated using $\frac{\ddot{s}}{\dot{s}}=\frac{\ddot{s}\dot{s}}{\dot{s}^2}$ with

$$\dot{s}^2=\lambda^2(\mathbf{z}^T\mathbf{z})\dot{\theta}^2-2\lambda^2h\dot{\theta}+(\mathbf{z}^T\mathbf{z})\dot{\lambda}^2+\lambda^2\dot{\mathbf{z}}^T\dot{\mathbf{z}}+N\,\dot{\mathbf{t}}^T\dot{\mathbf{t}} \tag{193}$$

$$\ddot{s}\dot{s}=\lambda^2(\mathbf{z}^T\mathbf{z})\dot{\theta}\ddot{\theta}+\lambda(\mathbf{z}^T\mathbf{z})\dot{\lambda}\dot{\theta}^2-\lambda^2\dot{h}\dot{\theta}-2\lambda h\dot{\lambda}\dot{\theta}-\lambda^2h\ddot{\theta}+(\mathbf{z}^T\mathbf{z})\dot{\lambda}\ddot{\lambda}+\lambda(\dot{\mathbf{z}}^T\dot{\mathbf{z}})\dot{\lambda}+\lambda^2\dot{\mathbf{z}}^T\ddot{\mathbf{z}}+N\,\dot{\mathbf{t}}^T\ddot{\mathbf{t}} \tag{194}$$

The orthogonal motions are not geodesic motions. The λ-equation (191) is not satisfied because if we replace $\dot{\lambda}=0$ and $\dot{\theta}=\frac{h}{\mathbf{z}^T\mathbf{z}}$ we obtain

$$(\mathbf{z}^T\mathbf{z})\ddot{\lambda}-\lambda(\mathbf{z}^T\mathbf{z})\dot{\theta}^2+2\lambda h\dot{\theta}-\lambda\dot{\mathbf{z}}^T\dot{\mathbf{z}}-\frac{\ddot{s}}{\dot{s}}(\mathbf{z}^T\mathbf{z})\dot{\lambda}=-\lambda(\mathbf{z}^T\mathbf{z})\dot{\theta}^2+2\lambda h\dot{\theta}-\lambda\dot{\mathbf{z}}^T\dot{\mathbf{z}}=$$

$$=-\lambda(\mathbf{z}^T\mathbf{z})\left(\frac{h}{\mathbf{z}^T\mathbf{z}}\right)^2+2\lambda h\frac{h}{\mathbf{z}^T\mathbf{z}}-\lambda\dot{\mathbf{z}}^T\dot{\mathbf{z}}=\lambda\frac{h^2}{\mathbf{z}^T\mathbf{z}}-\lambda\dot{\mathbf{z}}^T\dot{\mathbf{z}}\neq 0 \tag{195}$$

except for the very special case of problems where $h^2=(\mathbf{z}^T\mathbf{z})(\dot{\mathbf{z}}^T\dot{\mathbf{z}})$.

The rigid transformation case

The differential equations of the ***orthogonal motions*** are

$$(\mathbf{z}^T\mathbf{z})\dot{\theta}-h=0 \tag{196}$$

$$\dot{\mathbf{t}}=\mathbf{0} \tag{197}$$

The differential equations of the ***geodesic motions*** are

$$(\mathbf{z}^T\mathbf{z})\ddot{\theta}-\dot{h}-\frac{\ddot{s}}{\dot{s}}[(\mathbf{z}^T\mathbf{z})\dot{\theta}-h]=0 \tag{198}$$

$$\ddot{\mathbf{t}}-\frac{\ddot{s}}{\dot{s}}\dot{\mathbf{t}}=\mathbf{0} \tag{199}$$

The equation corresponding to t

$$-h\ddot{\theta}+\dot{\mathbf{z}}^T\ddot{\mathbf{z}}-\frac{\ddot{s}}{\dot{s}}(\dot{\mathbf{z}}^T\dot{\mathbf{z}}-h\dot{\theta})=0 \tag{200}$$

is not used. Instead the arc length is eliminated using $\frac{\ddot{s}}{\dot{s}}=\frac{\ddot{s}\dot{s}}{\dot{s}^2}$ with

$$\dot{s}^2=(\mathbf{z}^T\mathbf{z})\dot{\theta}^2-2h\dot{\theta}+\dot{\mathbf{z}}^T\dot{\mathbf{z}}+N\,\dot{\mathbf{t}}^T\dot{\mathbf{t}} \tag{201}$$

$$\ddot{s}\dot{s}=(\mathbf{z}^T\mathbf{z})\dot{\theta}\ddot{\theta}-\dot{h}\dot{\theta}-h\ddot{\theta}+\dot{\mathbf{z}}^T\ddot{\mathbf{z}}+N\,\dot{\mathbf{t}}^T\ddot{\mathbf{t}}\,. \tag{202}$$

The orthogonal motions are in this case geodesic motions and in fact the ones with minimum length among all geodesics joining the extreme fibers $F_{y(0)}$ and $F_{y(T)}$.

References

Baarda, W. (1973): *S-Transformations and Criterion Matrices*. Netherlands Geodetic Commission, Publications on Geodesy, New Series, Vol. 5, No. 1.

Bishop, R.L. and S.I. Goldberg (1968): *Tensor Analysis on Manifolds*. Dover, New York.

Bjerhammar, A. (1951): Rectangular reciprocal matrices with special references to geodetic calculations. *Bulletin Géodesique*, 52, 188-220.

Blaha, G. (1971): *Inner Adjustment Constraints with Emphasis on Range Observations*. Department of Geodetic Science Report No. 148, The Ohio State University.

Cannon, W.H (1979): The Measurement of Earth Rotation on a Deformable Earth by Long Baseline Interferometry. Paper presented at the "Symposium on Radio Interferometry: Techniques for Geodesy", MIT, June 19-21, 1979.

Choquet-Bruhat, Y., C. DeWitt-Morette, M. Dillard-Bleick (1977): *Analysis, Manifolds and Physics.* North Holland, Amsterdam.

Dermanis, A. (1980): VLBI: Principles and Geodynamic Prospectives. *Quaterniones Geodaesiae*, vol. 3, 213-230.

Dermanis, A. (1991): A Unified Approach to Linear Estimation and Prediction. Presented at the IUGG XXth General Assembly, International Association of Geodesy, Vienna, August 11-24, 1991.

Dermanis, A., and E.W. Grafarend (1992): The Finite Element Approach to the Geodetic Computation of Two- and Three-dimensional Deformation Parameters: A Study of Frame Invariance and Parameter Estimability. Proceedings International Conference "Cartography-Geodesy", 5th Centenary of the Americas: 1492-1992, Maracaibo, Venezuela.

Dermanis, A. and F. Sanso (1995): Nonlinear Estimation Problems for Nonlinear Models. *Manuscripta Geodaetica*, 20, 110-122.

do Carmo, M.P. (1992): *Riemannian Geometry*. Birkhäuser, Basel.

Goldstein, H. (1950): *Classical Mechanics*. Addison-Wesley, Reading Massachusetts.

Grafarend, E. (1974). Optimization of Geodetic Networks. *Bollettino di Geodesia e Scienze Affini*, 33, 4, 351-406.

Grafarend, E. and B. Schaffrin (1974): Unbiased Free Net Adjustment. *Survey Review*, vol. XXII, No. 171, 200-218.

Grafarend, E. and B. Schaffrin (1976): Equivalence of Estimable Quantities and Invariants in Geodetic Networks. *Zeitschrift fürVermessungswesen*, 101, 11, 485-491.

Hicks, N. J. (1965): *Notes on Differential Geometry*. Van Nostrand Reinhold, New York.

Koch K.-R. (1982): S-transformations and Projections for Obtaining Estimable Parameters. In: "*40 Years of Thought*", Anniversary volume for Prof. Baarda's 65th Birthday, Delft. v. 1, 136-144.

Loomis, L.H. and S. Sternberg (1980): *Advanced Calculus*. Addison-Wesley, Reading Massachusetts.

Meissl, P. (1965): Über die Innere Genauigkeit dreidimensionaler Punkthaufen. *Zeitschrift für Vermessungswesen*, 90, 4, 109-118.

Meissl, P. (1969): Zusammengfassung und Ausbau der inneren Fehlertheorie eines Punkthaufens. *Deutsche Geodätische Kommission*, Reihe A, Nr. 61, 8-21.

Munk, W.H. and G.J.F. MacDonald (1960): *The Rotation of the Earth*. Cambridge University Press.

Sanso, F. and G. Sona (1995): The Theory of Optimal Linear Estimation for Continuous Fields of Measurements. *Manuscripta Geodaetica*, 20, 204-230.

Sternberg, L. (1964): *Lectures on Differential Geometry*. Prentice-Hall, Englewood Cliffs, N.J.

Takos, I. (1976): Generalized Inverse of Matrices and Its Applications (in Greek). *Bulletin of the Hellenic Army Geographical Service*, 1976, 77-133.

Teunissen, P.J.G. (1985): *The Geometry of Geodetic Inverse Linear Mapping and Non-Linear Adjustment*. Netherlands Geodetic Commission, Publications on Geodesy, New Series, v. 8, No. 1.

Teunissen, P.J.G. (1990): Non-Linear Least Squares. *Manuscripta Geodaetica*, 15, 137-150.

Truesdell, C. (1977): *A First Course in Rational Continuum Mechanics*, vol.1. Academic, New York.

van Mierlo, J. (1980): Free Network Adjustment and S-transformations. *Deutsche Geodätische Kommission*, Reihe B, Heft Nr. 252, 41-54.

Xu, Peiliang (1995): Testing the hypotheses of non-estimable functions in free net adjustment models. *Manuscripta Geodaetica*, 20, 73-81.