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The photogrammetric inner constraints

A derivation of the complete inner constraints, which are required for obtaining "free network" solutions in close- range photogrammetry, is presented. The inner constraints are derived analytically for the bundle method, by exploiting the fact that the rows of their coefficient matrix form a basis for the null subspace of the design matrix used in the linearized observation equations. The derivation is independent of any particular choice of rotational parameters and examples are given for three types of rotation angles used in photogrammetry, as well as for the Rodriguez elements. A convenient algorithm based on the use of the S-transformation is presented, for the computation of free solutions with either inner or partial inner constraints. This approach is finally compared with alternative approaches to free network solutions.

1. Introduction

It is well known that photogrammetric observations can determine the shape of photographed objects but not their scale, position and orientation. The determination of these additional characteristics is called the "datum problem" since it is related to the definition of a reference frame to which object coordinates should refer.

In aerotriangulation, the datum problem is solved with the use of control points with known coordinates. In close-range photogrammetry such control points are mostly not available and a more-or-less arbitrary reference frame must be chosen. In the theory of optimization of geodetic networks the choice of the optimum frame is called the Zero Order Design problem (Grafarend, 1974; Fraser, 1984). The solution to this problem is the use of the so-called inner constraints (Meissl, 1965, 1969; Blaha, 1971).

The use of inner constraints in photogrammetry is first mentioned in Meissl (1965), who elaborates their use with an example for the relative orientation of two photographs. Their use is later considered in Ebner (1974), Dermanis (1975), Grün (1976), Granshaw (1980), Papo and Perelmuter (1980, 1982), Fraser (1980, 1982, 1983, 1984, 1988), Brown (1982), Cooper (1984), Hinsken (1985), Zindorf (1985), Papo (1985, 1987), Fraser and Gründig (1985). Papo and Perelmuter (1982) give an extensive account for the implementation of inner constraints in photogrammetry, along the lines of their previous work for geodetic networks (Perelmuter, 1979; Papo and Perelmuter, 1981). Their approach leads to the modification of the original singular normal equations so that the inner constraints are taken into account. It is also implicitly related to a solution with trivial minimal constraints, where seven coordinates are held fixed.

In most of these works the constraints used are partial inner constraints which do not involve all the photogrammetric parameters. Only coordinates of object points and projective centres are involved, ignoring the orientation angles of the photographs. This allows the direct use of the inner constraints whose analytical form has been taken from that in three-dimensional geodetic networks, without any new derivation for the photogrammetric case. The popularity of the use of partial inner constraints on the object points is due to the fact that the exterior orientation parameters are in a sense "nuisance parameters". When a pseudoinverse solution of the normal equations is computed, the result is equivalent to the use of the complete inner constraints. A derivation of the complete inner constraints, has been given by Granshaw (1980) who includes the orientation angles for the particular case where their approximate values can be set equal to zero. A complete derivation for the most general case has been finally presented

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by Papo (1987). Papo's elegant derivation is essentially based on the same concept as in the original work of Meissl. When the parameters of coordinate transformations which leave the observables invariant are known (Grafarend and Schaffrin, 1976), the coefficient matrix $\mathbf{E}$ of the inner constraints is the matrix of the partial derivatives of the unknown parameters with respect to these transformation parameters.

In the present work the complete inner constraints for all photogrammetric parameters will be analytically derived in a completely different way, which is based on the property $\mathbf{A}\,\mathbf{E}^{\mathrm{T}} = \mathbf{0}$, where $\mathbf{A}$ is the design matrix, following from the linearization of the collinearity equations. This property has been used by Papo (1987) as an analytical test of the derived inner constraints, where "a great deal of patient algebra is involved". The derivation presented here is based on the determination of seven independent solutions to the homogeneous system $\mathbf{A}\boldsymbol{e} = \mathbf{0}$, where $\boldsymbol{e}$ is any of the seven columns of $\mathbf{E}^{\mathrm{T}}$. The algebraic difficulties are greatly diminished, with the use of a very compact derivation in matrix notation of the elements of $\mathbf{A}$ from Dermanis (1990), which is briefly summarized in Appendix A. The derivation is independent of any particular choice of orientation parameters, and is specialized to some usual choices of orientation angles (Appendix B), as well as for the Rodriguez elements (Appendix C).

From the complete inner constraints follows directly any type of partial inner constraints involving any desired subset of the parameters. Furthermore, a detailed algorithm based on S-transformations will be given, for the implementation of complete or partial inner constraints. This algorithm has the advantage that it does not disturb the nice form of the original photogrammetric normal equations, as is the case with the use of modified normal equations.

A derivation of the complete inner constraints for the parameters of the Direct Linear Transformation (DLT) model is given in Dermanis (1994).

2. Derivation of the complete photogrammetric inner constraints

From all the photogrammetric parameters (object point and projection centre coordinates, orientation angles, additional parameters), inner constraints involve only those which are variant under changes of the reference frame. The invariant additional parameters will not therefore be present in the inner constraints. The inner constraints are seven constraints (three for position, three for orientation, one for scale) of the form:

$$\mathbf{E}\boldsymbol{x} = \mathbf{0} \tag{1}$$

which in combination with the linearized observation equations:

$$\mathbf{b} = \mathbf{A}\boldsymbol{x} + \boldsymbol{v} \tag{2}$$

lead to a least squares solution for the unknown parameters $\boldsymbol{x}$, having the properties:

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = \min \qquad \mathrm{trace}(\mathbf{C}_x) = \min \tag{3}$$

$\mathbf{C}_x$ being the covariance matrix of $\boldsymbol{x}$. It is well known (Meissl, 1965, 1969; Blaha, 1971; Koch, 1987) that the matrix $\mathbf{E}$ is a full-row-rank matrix satisfying:

$$\mathbf{A}\,\mathbf{E}^{\mathrm{T}} = \mathbf{0} \tag{4}$$

If $\boldsymbol{e}$ is any row of $\mathbf{E}$, the inner constraints will be determined if 7 linearly independent solutions are found for the system:

$$\mathbf{A}\boldsymbol{e} = \mathbf{0} \tag{5}$$

For any pair of observations of the image coordinates of the object point i on photograph j, the corresponding linearized observation equations have the form:

$$\boldsymbol{b}_{ji} = \dot{\mathbf{A}}_{ji} \begin{bmatrix} \delta\boldsymbol{y}_j \\ \delta\boldsymbol{\theta}_j \end{bmatrix} + \ddot{\mathbf{A}}_{ji}\delta\boldsymbol{x}_i + \boldsymbol{v}_{ji} \tag{6}$$

where $\delta \boldsymbol{y}_j$ are the corrections to the approximate values of the coordinates of the projective centres $\boldsymbol{y}_j = [X_j \; Y_j \; Z_j]^{\mathrm{T}}$, $\delta\boldsymbol{\theta}_j$ are the corrections to the approximate values of the photo-frame orientation parameters $\boldsymbol{\theta}_j$, $\delta \boldsymbol{x}_i$ are the corrections to the approximate values of the object point coordinates $\boldsymbol{x}_i = [X_i \; Y_i \; Z_i]^{\mathrm{T}}$, $\boldsymbol{b}_{ji}$ are the reduced observations (observed minus computed), $\boldsymbol{v}_{ji}$ are the observational errors and $\dot{\mathbf{A}}_{ji}$, $\ddot{\mathbf{A}}_{ji}$ are design matrices of the partial derivatives determined in Appendix A.

Equations (6) are two of the total observation eqs. (2). The corresponding two from eqs. (5) will have the form:

$$\dot{\mathbf{A}}_{ji} \begin{bmatrix} \boldsymbol{h}_j \\ \boldsymbol{k}_j \end{bmatrix} + \ddot{\mathbf{A}}_{ji} \boldsymbol{g}_i = \mathbf{0} \tag{7}$$

where $\boldsymbol{h}_j$, $\boldsymbol{k}_j$ and $\boldsymbol{g}_i$ are the subvectors of $\boldsymbol{e}$ corresponding to the parameters $\delta\boldsymbol{y}_j$, $\delta\boldsymbol{\theta}_j$ and $\delta\boldsymbol{x}_i$, respectively. The submatrix $\dot{\mathbf{A}}_{ji}$ has the form:

$$\dot{\mathbf{A}}_{ji} = [-\ddot{\mathbf{A}}_{ji} \mathbf{C}_{ji}] \tag{8}$$

and eqs. (7) take the form:

$$\ddot{\mathbf{A}}_{ji}(\boldsymbol{g}_i - \boldsymbol{h}_j) + \mathbf{C}_{ji}\boldsymbol{k}_j = \mathbf{0} \tag{9}$$

for all i and j.
The submatrices $\ddot{\mathbf{A}}_{ji}$ and $\mathbf{C}_{ji}$ have, according to Appendix A, the form:

$$\ddot{\mathbf{A}}_{ji} = \begin{bmatrix} \dfrac{\partial x_{ji}}{\partial X_i} & \dfrac{\partial x_{ji}}{\partial Y_i} & \dfrac{\partial x_{ji}}{\partial Z_i} \\ \dfrac{\partial y_{ji}}{\partial X_i} & \dfrac{\partial y_{ji}}{\partial Y_i} & \dfrac{\partial y_{ji}}{\partial Z_i} \end{bmatrix} = \frac{f}{w_{ji}^2} \begin{bmatrix} \boldsymbol{d}_{ij}^{\mathrm{T}}[(\mathbf{R}_j^{\mathrm{T}}\boldsymbol{e}_2)\times] \\ -\boldsymbol{d}_{ij}^{\mathrm{T}}[(\mathbf{R}_j^{\mathrm{T}}\boldsymbol{e}_1)\times] \end{bmatrix} \tag{10}$$

$$\mathbf{C}_{ji} = \begin{bmatrix} \dfrac{\partial x_{ji}}{\partial \alpha_j} & \dfrac{\partial x_{ji}}{\partial \beta_j} & \dfrac{\partial x_{ji}}{\partial \gamma_j} \\ \dfrac{\partial y_{ji}}{\partial \alpha_j} & \dfrac{\partial y_{ji}}{\partial \beta_j} & \dfrac{\partial y_{ji}}{\partial \gamma_j} \end{bmatrix} = \frac{f}{w_{ji}^2} \begin{bmatrix} \boldsymbol{d}_{ji}^{\mathrm{T}}\mathbf{F}_{\alpha_j}\boldsymbol{d}_{ji} & \boldsymbol{d}_{ji}^{\mathrm{T}}\mathbf{F}_{\beta_j}\boldsymbol{d}_{ji} & \boldsymbol{d}_{ji}^{\mathrm{T}}\mathbf{F}_{\gamma_j}\boldsymbol{d}_{ji} \\ -\boldsymbol{d}_{ji}^{\mathrm{T}}\mathbf{E}_{\alpha_j}\boldsymbol{d}_{ji} & -\boldsymbol{d}_{ji}^{\mathrm{T}}\mathbf{E}_{\beta_j}\boldsymbol{d}_{ji} & -\boldsymbol{d}_{ji}^{\mathrm{T}}\mathbf{E}_{\gamma_j}\boldsymbol{d}_{ji} \end{bmatrix} \tag{11}$$

where $\boldsymbol{\theta}_j = [\alpha_j \; \beta_j \; \gamma_j]^{\mathrm{T}}$ are the rotation parameters used for the definition of the rotation matrix $\mathbf{R}_j$, $\boldsymbol{d}_{ji} = \boldsymbol{x}_i - \boldsymbol{y}_j$, while the parameters w_{ji}, $\mathbf{F}_{\alpha_j}$, $\mathbf{F}_{\beta_j}$, $\mathbf{F}_{\gamma_j}$, $\mathbf{E}_{\alpha_j}$, $\mathbf{E}_{\beta_j}$ and $\mathbf{E}_{\gamma_j}$, are defined in Appendix A (eqs. A2, A16 and A17). The vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, are the three columns of the 3×3 identity matrix $\mathbf{I}$. Setting:

$$\boldsymbol{k}_j = [k_{\alpha_j} \; k_{\beta_j} \; k_{\gamma_j}]^{\mathrm{T}} \tag{12}$$

and taking (10) and (11) into account, eq. (9) becomes:

$$\boldsymbol{d}_{ji}^{\mathrm{T}}[(\mathbf{R}_j^{\mathrm{T}}\boldsymbol{e}_2)\times](\boldsymbol{g}_i - \boldsymbol{h}_j) + \boldsymbol{d}_{ji}^{\mathrm{T}}(k_{\alpha_j}\mathbf{F}_{\alpha_j} + k_{\beta_j}\mathbf{F}_{\beta_j} + k_{\gamma_j}\mathbf{F}_{\gamma_j})\boldsymbol{d}_{ji} = \mathbf{0} \tag{13}$$

$$\boldsymbol{d}_{ji}^{\mathrm{T}}[(\mathbf{R}_j^{\mathrm{T}}\boldsymbol{e}_1)\times](\boldsymbol{g}_i - \boldsymbol{h}_j) + \boldsymbol{d}_{ji}^{\mathrm{T}}(k_{\alpha_j}\mathbf{E}_{\alpha_j} + k_{\beta_j}\mathbf{E}_{\beta_j} + k_{\gamma_j}\mathbf{E}_{\gamma_j})\boldsymbol{d}_{ji} = \mathbf{0} \tag{14}$$

Taking into account eqs. (A16), (A17), it follows that:

$$\begin{aligned} k_{\alpha_j}\mathbf{F}_{\alpha_j} + k_{\beta_j}\mathbf{F}_{\beta_j} + k_{\gamma_j}\mathbf{F}_{\gamma_j} &= [(\mathbf{R}_j^{\mathrm{T}}\boldsymbol{e}_2)\times]\{k_{\alpha_j}[\boldsymbol{q}_{\alpha_j}\times] + k_{\beta_j}[\boldsymbol{q}_{\beta_j}\times] + k_{\gamma_j}[\boldsymbol{q}_{\gamma_j}\times]\} \\ &= [(\mathbf{R}_j^{\mathrm{T}}\boldsymbol{e}_2)\times][(k_{\alpha_j}\boldsymbol{q}_{\alpha_j} + k_{\beta_j}\boldsymbol{q}_{\beta_j} + k_{\gamma_j}\boldsymbol{q}_{\gamma_j})\times] \\ &= [(\mathbf{R}_j^{\mathrm{T}}\boldsymbol{e}_2)\times][(\mathbf{Q}_j\boldsymbol{k}_j)\times] \end{aligned} \tag{15}$$

where

$$\mathbf{Q}_j \equiv [\boldsymbol{q}_{\alpha_j}\ \boldsymbol{q}_{\beta_j}\ \boldsymbol{q}_{\gamma_j}] \tag{16}$$

and the columns $\boldsymbol{q}_{\alpha_j}, \boldsymbol{q}_{\beta_j}, \boldsymbol{q}_{\gamma_j}$, are defined by (Appendix A, eq. A10):

$$[\boldsymbol{q}_{\alpha_j}\times] = \mathbf{R}_j^{\mathrm{T}}\partial_{\alpha_j}\mathbf{R}_j, \qquad [\boldsymbol{q}_{\beta_j}\times] = \mathbf{R}_j^{\mathrm{T}}\partial_{\beta_j}\mathbf{R}_j, \qquad [\boldsymbol{q}_{\gamma_j}\times] = \mathbf{R}_j^{\mathrm{T}}\partial_{\gamma_j}\mathbf{R}_j \tag{17}$$

In a similar way:

$$k_{\alpha_j}\mathbf{E}_{\alpha_j} + k_{\beta_j}\mathbf{E}_{\beta_j} + k_{\gamma_j}\mathbf{E}_{\gamma_j} = [(\mathbf{R}_j^{\mathrm{T}}\boldsymbol{e}_1)\times][(\mathbf{Q}_j\boldsymbol{k}_j)\times] \tag{18}$$

and setting:

$$z_j = \mathbf{Q}_j\boldsymbol{k}_j \tag{19}$$

eqs. (13) and (14) finally become:

$$\boldsymbol{d}_{ji}^{\mathrm{T}}[(\mathbf{R}_j^{\mathrm{T}}\boldsymbol{e}_2)\times]\{\boldsymbol{g}_i - \boldsymbol{h}_j + [z_j\times]\boldsymbol{d}_{ji}\} = \mathbf{0} \tag{20a}$$

$$\boldsymbol{d}_{ji}^{\mathrm{T}}[(\mathbf{R}_j^{\mathrm{T}}\boldsymbol{e}_1)\times]\{\boldsymbol{g}_i - \boldsymbol{h}_j + [z_j\times]\boldsymbol{d}_{ji}\} = \mathbf{0} \tag{20b}$$

for any i and j.

Seven independent solutions of the above equations must be found. Two obvious classes of solutions result by setting the quantity in brackets equal to either zero or $\boldsymbol{d}_{ji}$. The last follows from the property:

$$\boldsymbol{d}^{\mathrm{T}}[\boldsymbol{t}\times]\boldsymbol{d} = \mathbf{0} \tag{21}$$

for any $\boldsymbol{d}$ and $\boldsymbol{t}$.
In the first class:

$$\boldsymbol{g}_i - \boldsymbol{h}_j + [z_j\times]\boldsymbol{d}_{ji} = \mathbf{0} \tag{22}$$

an obvious choice is:

$$\boldsymbol{g}_i - \boldsymbol{h}_j = \mathbf{0}, \qquad z_j = \mathbf{0} \tag{23}$$

leading to three independent solutions:

$$\boldsymbol{g}_i^{(1)} = \boldsymbol{h}_j^{(1)} = \boldsymbol{e}_1, \qquad z_j^{(1)} = \mathbf{0} \quad \Longrightarrow \quad \boldsymbol{k}_j^{(1)} = \mathbf{0} \tag{24a}$$

$$\boldsymbol{g}_i^{(2)} = \boldsymbol{h}_j^{(2)} = \boldsymbol{e}_2, \qquad z_j^{(2)} = \mathbf{0} \quad \Longrightarrow \quad \boldsymbol{k}_j^{(2)} = \mathbf{0} \tag{24b}$$

$$\boldsymbol{g}_i^{(3)} = \boldsymbol{h}_j^{(3)} = \boldsymbol{e}_3, \qquad z_j^{(3)} = \mathbf{0} \quad \Longrightarrow \quad \boldsymbol{k}_j^{(3)} = \mathbf{0} \tag{24c}$$

A not so obvious choice follows if (22) is rewritten, using $\boldsymbol{d}_{ji} = \boldsymbol{x}_i - \boldsymbol{y}_j$, in the form:

$$\boldsymbol{g}_i + [z_j\times]\boldsymbol{x}_i = \boldsymbol{h}_j + [z_j\times]\boldsymbol{y}_j \tag{25}$$

for any i and j.
Since i and j refer to independent parameters, (25) can be satisfied only if $z_j = z =$ constant and both sides vanish. Three independent solutions follow from $z = -\boldsymbol{e}_1, z = -\boldsymbol{e}_2, z = -\boldsymbol{e}_3$:

$$\boldsymbol{g}_i^{(4)} = -[\boldsymbol{x}_i\times]\boldsymbol{e}_1, \qquad \boldsymbol{h}_j^{(4)} = -[\boldsymbol{y}_j\times]\boldsymbol{e}_1, \qquad \boldsymbol{k}_j^{(4)} = -\mathbf{Q}_j^{-1}\boldsymbol{e}_1 \tag{26a}$$

$$\boldsymbol{g}_i^{(5)} = -[\boldsymbol{x}_i\times]\boldsymbol{e}_2, \qquad \boldsymbol{h}_j^{(5)} = -[\boldsymbol{y}_j\times]\boldsymbol{e}_2, \qquad \boldsymbol{k}_j^{(5)} = -\mathbf{Q}_j^{-1}\boldsymbol{e}_2 \tag{26b}$$

$$\boldsymbol{g}_i^{(6)} = -[\boldsymbol{x}_i\times]\boldsymbol{e}_3, \qquad \boldsymbol{h}_j^{(6)} = -[\boldsymbol{y}_j\times]\boldsymbol{e}_3, \qquad \boldsymbol{k}_j^{(6)} = -\mathbf{Q}_j^{-1}\boldsymbol{e}_3 \tag{26c}$$

In the second class:

$$g_i - h_j + [z_j\times](x_i - y_j) = x_i - y_j \tag{27}$$

an obvious solution is obtained by choosing $z_j = \mathbf{0}$, in which case:

$$g_i^{(7)} = x_i, \qquad h_j^{(7)} = y_j, \qquad k_j^{(7)} = \mathbf{0} \tag{28}$$

Equations (24), (26) and (28) give the required seven linearly independent solutions from which the seven rows of the matrix $\mathbf{E}$ of the inner constraints are defined. Taking into account that:

$$[g_i^{(1)} \quad g_i^{(2)} \quad g_i^{(3)}] = [e_1 \quad e_2 \quad e_3] = \mathbf{I} \tag{29}$$

$$[h_j^{(1)} \quad h_j^{(2)} \quad h_j^{(3)}] = [e_1 \quad e_2 \quad e_3] = \mathbf{I} \tag{30}$$

$$[k_j^{(1)} \quad k_j^{(2)} \quad k_j^{(3)}] = [\mathbf{0} \quad \mathbf{0} \quad \mathbf{0}] = \mathbf{0} \tag{31}$$

$$[g_i^{(4)} \quad g_i^{(5)} \quad g_i^{(6)}] = -\big[[x_i\times]e_1 \quad [x_i\times]e_2 \quad [x_i\times]e_3\big] = -[x_i\times][e_1 \quad e_2 \quad e_3] = -[x_i\times] \tag{32}$$

$$[h_j^{(4)} \quad h_j^{(5)} \quad h_j^{(6)}] = -\big[[y_j\times]e_1 \quad [y_j\times]e_2 \quad [y_j\times]e_3\big] = -[y_j\times][e_1 \quad e_2 \quad e_3] = -[y_j\times] \tag{33}$$

$$[k_j^{(4)} \quad k_j^{(5)} \quad k_j^{(6)}] = -\big[\mathbf{Q}_j^{-1}e_1 \quad \mathbf{Q}_j^{-1}e_2 \quad \mathbf{Q}_j^{-1}e_3\big] = -\mathbf{Q}_j^{-1}[e_1 \quad e_2 \quad e_3] = -\mathbf{Q}_j^{-1} \tag{34}$$

the matrix $\mathbf{E}$ of the inner constraints becomes:

$$\mathbf{E}^{\mathrm{T}} = \begin{bmatrix} & & & \vdots & & & \\ h_j^{(1)} & h_j^{(2)} & h_j^{(3)} & h_j^{(4)} & h_j^{(5)} & h_j^{(6)} & h_j^{(7)} \\ k_j^{(1)} & k_j^{(2)} & k_j^{(3)} & k_j^{(4)} & k_j^{(5)} & k_j^{(6)} & k_j^{(7)} \\ & & & \vdots & & & \\ g_i^{(1)} & g_i^{(2)} & g_i^{(3)} & g_i^{(4)} & g_i^{(5)} & g_i^{(6)} & g_i^{(7)} \\ & & & \vdots & & & \end{bmatrix} = \begin{bmatrix} & \vdots & \\ \mathbf{I} & -[y_j\times] & y_j \\ \mathbf{0} & -\mathbf{Q}_j^{-1} & \mathbf{0} \\ & \vdots & \\ \mathbf{I} & -[x_i\times] & x_i \\ & \vdots & \end{bmatrix} \tag{35}$$

$$\mathbf{E} = \begin{bmatrix} \cdots & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{I} & \cdots \\ \cdots & [y_j\times] & -\mathbf{Q}_j^{-\mathrm{T}} & \cdots & [x_i\times] & \cdots \\ \cdots & y_j^{\mathrm{T}} & \mathbf{0} & \cdots & x_i^{\mathrm{T}} & \cdots \end{bmatrix} \tag{36}$$

In the above equation x_i and y_j are the vectors of the approximate values of the coordinates of object point i and of the projective centre of photograph j, respectively. The matrix $\mathbf{Q}_j$ is a function of the approximate values of the rotation parameters of photograph j. Its specific value depends on the type of rotation parameters. $\mathbf{Q}_j^{-\mathrm{T}}$ is computed in Appendix B, for various types of orientation angles, and in Appendix C for the Rodriguez elements. An interpretation of the physical meaning of the matrix $\mathbf{Q}$ is given in Appendix D. For the usual choice of κ, φ, ω angles, the inner constraint matrix $\mathbf{E}$ takes the form:

$$\mathbf{E} = \big[\cdots \quad \mathbf{G}_j\mathbf{K}_j \quad \cdots \quad \mathbf{G}_i \quad \cdots\big] \tag{37}$$

where $\mathbf{G}_i$ and $\mathbf{G}_j$ are given by:

$$\mathbf{G}_k = \begin{bmatrix} \mathbf{I} \\ [\mathbf{y}_k \times] \\ \mathbf{y}_k^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -Z_{ok} & Y_{ok} \\ Z_{ok} & 0 & -X_{ok} \\ -Y_{ok} & X_{ok} & 0 \\ X_{ok} & Y_{ok} & Z_{ok} \end{bmatrix} \qquad k = i \text{ or } j \tag{38}$$

and

$$\mathbf{K}_j = \begin{bmatrix} \mathbf{0} \\ -\mathbf{Q}_j^{-\mathrm{T}} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ -\dfrac{\sin\omega_j}{\cos\varphi_j} & \cos\omega_j & \sin\omega_j \tan\varphi_j \\ \dfrac{\cos\omega_j}{\cos\varphi_j} & \sin\omega_j & -\cos\omega_j \tan\varphi_j \\ 0 & 0 & 0 \end{bmatrix} \tag{39}$$

When the angles κ, φ, ω are very small their approximate values can be taken equal to zero (at least for the first step in an iterating adjustment solution), and the matrix $\mathbf{K}_j$ takes the same form, with reverse ordering of the angles, as in Granshaw (1980).

The seven constraints can be separated into the three translational constraints:

$$\sum_j \delta \mathbf{y}_j + \sum_i \delta \mathbf{x}_i = \mathbf{0} \tag{40}$$

the three rotational constraints:

$$\sum_j [\mathbf{y}_j \times] \delta \mathbf{y}_j - \sum_j \mathbf{Q}_j^{-\mathrm{T}} \delta \boldsymbol{\theta}_j + \sum_i [\mathbf{x}_i \times] \delta \mathbf{x}_i = \mathbf{0} \tag{41}$$

and one scale constraint:

$$\sum_j \mathbf{y}_j^{\mathrm{T}} \delta \mathbf{y}_j + \sum_i \mathbf{x}_i^{\mathrm{T}} \delta \mathbf{x}_i = \mathbf{0} \tag{42}$$

where $\delta \mathbf{y}_j$, $\delta \boldsymbol{\theta}_j$, and $\delta \mathbf{x}_i$ are the unknown corrections to perspective centre coordinates, rotation parameters and object point coordinates, respectively. Note that rotation parameter corrections appear only in the rotational constraints. Of these seven constraints only those should be used which correspond to the defect of the network with respect to translation, rotation or scale. For purely photogrammetric observations all seven are needed. In combinations of photogrammetric with geodetic observations, which include distance measurements, only the six first (translational and rotational) constraints should be used.

3. Computation of the inner constraints solution

Apart from the direct use of the pseudoinverse leading to the inner constraints solution, four different approaches can be used for the implementation of minimal constraints:

$$\mathbf{C}\boldsymbol{x} = \mathbf{0} \tag{43}$$

which can be either the inner constraints $\mathbf{E}\boldsymbol{x} = \mathbf{0}$, or partial inner constraints where only the sum of squares of the corrections to only some of the parameters is minimized. The coefficient matrix $\mathbf{C}$ of the partial inner constraints results from the matrix $\mathbf{E}$, by replacing with zeros all the columns corresponding to parameters not included in the minimization.

The *first* approach (Fraser, 1980, 1982, 1983, 1984, 1988; Fraser and Gründig, 1985) is the "bordering" of the matrix $\mathbf{N}$ of the original singular normal equations $\mathbf{N}\boldsymbol{x} = \boldsymbol{u}$. If:

$$\begin{bmatrix} \mathbf{N} & \mathbf{C}^{\mathrm{T}} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{Q} & \mathbf{L}^{\mathrm{T}} \\ \mathbf{L} & \mathbf{0} \end{bmatrix} \tag{44}$$

the solution is:

$$\boldsymbol{x} = \mathbf{Q}\boldsymbol{u}, \qquad \mathbf{C}_x = \sigma^2\mathbf{Q} \tag{45}$$

σ^2 being the a posteriori variance of unit weight and $\mathbf{C}_x$ the covariance matrix of $\boldsymbol{x}$. The advantageous structure of the matrix $\mathbf{N}$ is not destroyed and the symmetric indefinite "bordered" matrix in (44) can be inverted with appropriate algorithms (Fraser, 1982). However, the usual inversion algorithms, e.g. Cholesky's method, which are routinely used in photogrammetric applications cannot be used in this case because they are restricted to the case of positive definite matrices.

The *second* approach is based on the relations:

$$\boldsymbol{x} = (\mathbf{N} + \mathbf{C}^{\mathrm{T}}\mathbf{C})^{-1}\boldsymbol{u}, \qquad \mathbf{C}_x = \sigma^2(\mathbf{N} + \mathbf{C}^{\mathrm{T}}\mathbf{C})^{-1}\mathbf{N}(\mathbf{N} + \mathbf{C}^{\mathrm{T}}\mathbf{C})^{-1} \tag{46}$$

For the inner constraints in particular:

$$\mathbf{C}_x = \sigma^2[(\mathbf{N} + \mathbf{E}^{\mathrm{T}}\mathbf{E})^{-1} - \mathbf{E}^{\mathrm{T}}(\mathbf{E}\mathbf{E}^{\mathrm{T}})^{-2}\mathbf{E}] \tag{47}$$

In this approach the advantageous sparse structure of the matrix $\mathbf{N}$ is destroyed by the addition of the complete matrix $\mathbf{C}^{\mathrm{T}}\mathbf{C}$ or $\mathbf{E}^{\mathrm{T}}\mathbf{E}$.

The *third* approach (Papo and Perelmuter, 1980, 1982; Papo, 1985) is based on the elimination of seven parameters $\boldsymbol{x}_2$, which correspond to a set of *trivial* minimal constraints $\boldsymbol{x}_2 = \mathbf{0}$, from the observation equations and the formulation of the corresponding (modified) normal equations, for the remaining parameters $\boldsymbol{x}_1$. The elimination is based on:

$$\mathbf{C}\boldsymbol{x} = \mathbf{C}_1\boldsymbol{x}_1 + \mathbf{C}_2\boldsymbol{x}_2 = \mathbf{0} \quad \Longrightarrow \quad \boldsymbol{x}_2 = -\mathbf{C}_2^{-1}\mathbf{C}_1\boldsymbol{x}_1 = \mathbf{G}\boldsymbol{x}_1 \tag{48}$$

with $\mathbf{G} = -\mathbf{C}_2^{-1}\mathbf{C}_1$, and the solution is:

$$\mathbf{x} = \begin{bmatrix} \mathbf{I} \\ \mathbf{G} \end{bmatrix} \tilde{\mathbf{N}}^{-1}\tilde{\boldsymbol{u}} \qquad \mathbf{C}_x = \sigma^2 \begin{bmatrix} \mathbf{I} \\ \mathbf{G} \end{bmatrix} \tilde{\mathbf{N}}^{-1}[\mathbf{I}\,\mathbf{G}^{\mathrm{T}}] \tag{49}$$

where

$$\tilde{\mathbf{N}} = [\mathbf{I}\,\mathbf{G}^{\mathrm{T}}]\mathbf{N} \begin{bmatrix} \mathbf{I} \\ \mathbf{G} \end{bmatrix} \qquad \tilde{\boldsymbol{u}} = [\mathbf{I}\,\mathbf{G}^{\mathrm{T}}]\boldsymbol{u} \tag{50}$$

This approach is essentially equivalent to a special case of the S-transformation approach to be presented next.

The *fourth* approach is based on the application of the *S-transformation* (Baarda, 1973; Ebner, 1974; van Mierlo, 1980; Strang van Hess, 1982; Koch, 1982):

$$\boldsymbol{x}^{\mathrm{E}} = \mathbf{S}\boldsymbol{x}, \qquad \mathbf{C}_x^{\mathrm{E}} = \mathbf{S}\mathbf{C}_x\mathbf{S}^{\mathrm{T}}, \qquad \mathbf{S} = \mathbf{I} - \mathbf{E}^{\mathrm{T}}(\mathbf{C}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{C} \tag{51}$$

on an original solution $\boldsymbol{x}$, $\mathbf{C}_x$, which results from the use of any set of minimal constraints. Among possible choices of minimal constraints, the easiest to implement are *trivial constraints*, where as many coordinate corrections as the rank defect are set equal to zero. The addition of the term $\mathbf{C}^{\mathrm{T}}\mathbf{C}$ in eq. (46) corresponds to the addition of "ones" to those diagonal elements of $\mathbf{N}$ which correspond to the fixed coordinates, a procedure which does not interfere with the original advantageous sparse structure of the matrix $\mathbf{N}$. This in fact means that any standard computer program for the bundle adjustment can be used in the first and most time-consuming part of the computations.

Equations (51) are very simple indeed, but their direct programming results in many multiplications and additions with zeros, which unnecessarily delay the computations and accumulate computational errors. For this reason a more appropriate version of the algorithm will be presented.

The inner constraints matrix has the form (37), so that any set of partial inner constraints has the corresponding form:

$$\mathbf{C} = [\ldots\ (\lambda_j \mathbf{G}_j)(\mu_j \mathbf{K}_j)\ \ldots\ (\lambda_i \mathbf{G}_i)\ \ldots] \tag{52}$$

The factors λ_j, μ_j, λ_i, have the value 1 or 0, depending on whether the corresponding parameters (projective centre coordinate corrections $\delta\boldsymbol{y}_j$, rotational parameter corrections $\delta\boldsymbol{\theta}_j$, object point coordinate corrections $\delta\boldsymbol{x}_i$, respectively) participate in the partial inner constraints or not. The equations for the S-transformation, follow directly, if the matrices $\mathbf{E}$ and $\mathbf{C}$ from eqs. (37) and (52), respectively, are replaced in eq. (51).

For the sake of simplification, no distinction needs to be made between projective centres and object points. They are all points of the "network" and $\boldsymbol{y}_j$, $\boldsymbol{x}_i$ can be replaced by a common symbol $\boldsymbol{x}_k$. The submatrices $\mathbf{G}_j$, $\mathbf{G}_i$ will be similarly replaced by $\mathbf{G}_k$, and the factors λ_j, λ_i by λ_k. If K is the set of the indices of all points $\boldsymbol{x}_k$ participating in the partial inner constraints, and J the same set for the participating rotational parameters $\boldsymbol{\theta}_j$, the S-transformation is performed with the following algorithm:

$$\mathbf{R} \equiv \mathbf{C}\mathbf{E}^{\mathrm{T}} = \sum_{k\in K} \mathbf{G}_k \mathbf{G}_k^{\mathrm{T}} + \sum_{j\in J} \mathbf{K}_j \mathbf{K}_j^{\mathrm{T}} \tag{53}$$

$$\boldsymbol{d} \equiv \mathbf{C}\boldsymbol{x} = \sum_{k\in K} \mathbf{G}_k \delta\boldsymbol{x}_k + \sum_{j\in J} \mathbf{K}_j \delta\boldsymbol{\theta}_j \tag{54}$$

$$\boldsymbol{p} = \mathbf{R}^{-1}\boldsymbol{d} \tag{55}$$

$$\delta\boldsymbol{x}_k^{\mathrm{E}} = \delta\boldsymbol{x}_k - \lambda_k \mathbf{G}_k^{\mathrm{T}}\boldsymbol{p}, \qquad (\lambda_k = 0 \text{ or } 1) \tag{56}$$

$$\delta\boldsymbol{\theta}_j^{\mathrm{E}} = \delta\boldsymbol{\theta}_j - \mu_j \mathbf{K}_j^{\mathrm{T}}\boldsymbol{p}, \qquad (\mu_j = 0 \text{ or } 1) \tag{57}$$

The cross-covariance 3 × 3 submatrices $\mathbf{C}(\boldsymbol{x}_k, \boldsymbol{x}_m)$, $\mathbf{C}(\boldsymbol{x}_k, \boldsymbol{\theta}_j)$, $\mathbf{C}(\boldsymbol{\theta}_i, \boldsymbol{\theta}_j)$, of the covariance matrix $\mathbf{C}_x$, are transformed from the minimal to the inner constraint solution, according to the following equations:

$$\begin{aligned}
\mathbf{C}(\boldsymbol{x}_k^{\mathrm{E}}, \boldsymbol{x}_m^{\mathrm{E}}) = {} & \mathbf{C}(\boldsymbol{x}_k, \boldsymbol{x}_m) - \mathbf{G}_k^{\mathrm{T}}\mathbf{R}^{-1}\left\{\sum_{t\in K}\mathbf{G}_t\mathbf{C}(\boldsymbol{x}_t, \boldsymbol{x}_m) + \sum_{r\in J}\mathbf{K}_r\mathbf{C}(\boldsymbol{\theta}_r, \boldsymbol{x}_m)\right\} \\
& - \left\{\sum_{t\in K}\mathbf{C}(\boldsymbol{x}_k, \boldsymbol{x}_t)\mathbf{G}_t^{\mathrm{T}} + \sum_{r\in J}\mathbf{C}(\boldsymbol{x}_k, \boldsymbol{\theta}_r)\mathbf{K}_r^{\mathrm{T}}\right\}\mathbf{R}^{-1}\mathbf{G}_m + \mathbf{G}_k^{\mathrm{T}}\mathbf{R}^{-1}\left\{\sum_{t\in K}\mathbf{G}_t\left[\sum_{s\in K}\mathbf{C}(\boldsymbol{x}_t, \boldsymbol{x}_s)\mathbf{G}_t^{\mathrm{T}} + \sum_{p\in J}\mathbf{C}(\boldsymbol{x}_k, \boldsymbol{\theta}_p)\mathbf{K}_p^{\mathrm{T}}\right]\right. \\
& \left. + \sum_{r\in J}\mathbf{K}_r\left[\sum_{s\in K}\mathbf{C}(\boldsymbol{\theta}_r, \boldsymbol{x}_s)\mathbf{G}_s^{\mathrm{T}} + \sum_{p\in J}\mathbf{C}(\boldsymbol{\theta}_r, \boldsymbol{\theta}_p)\mathbf{K}_p^{\mathrm{T}}\right]\right\}\mathbf{R}^{-1}\mathbf{G}_m
\end{aligned} \tag{58}$$

$$\mathbf{C}(\boldsymbol{x}_k^{\mathrm{E}}, \boldsymbol{\theta}_j^{\mathrm{E}}) = \mathbf{C}(\boldsymbol{x}_k, \boldsymbol{\theta}_j) - \mathbf{G}_k^{\mathrm{T}}\mathbf{R}^{-1}\left\{\sum_{t\in K}\mathbf{G}_t\mathbf{C}(\boldsymbol{x}_t, \boldsymbol{\theta}_j) + \sum_{r\in J}\mathbf{K}_r\mathbf{C}(\boldsymbol{\theta}_r, \boldsymbol{\theta}_j)\right\}$$

$$
-\left\{\sum_{t\in K} \mathbf{C}(x_k, x_t)\mathbf{G}_t^{\mathrm{T}} + \sum_{r\in J} \mathbf{C}(x_k, \theta_r)\mathbf{K}_r^{\mathrm{T}}\right\}\mathbf{R}^{-1}\mathbf{K}_j + \mathbf{G}_k^{\mathrm{T}}\mathbf{R}^{-1}\left\{\sum_{t\in K} \mathbf{G}_t\left[\sum_{s\in K} \mathbf{C}(x_t, x_s)\mathbf{G}_t^{\mathrm{T}} + \sum_{p\in J} \mathbf{C}(x_t, \theta_p)\mathbf{K}_p^{\mathrm{T}}\right]\right.
$$

$$
\left. + \sum_{r\in J} \mathbf{K}_r\left[\sum_{s\in K} \mathbf{C}(\theta_r, x_s)\mathbf{G}_s^{\mathrm{T}} + \sum_{p\in J} \mathbf{C}(\theta_r, \theta_p)\mathbf{K}_p^{\mathrm{T}}\right]\right\}\mathbf{R}^{-1}\mathbf{K}_j \tag{59}
$$

$$
\mathbf{C}(\theta_i^{\mathrm{E}}, \theta_j^{\mathrm{E}}) = \mathbf{C}(\theta_i, \theta_j) - \mathbf{K}_i^{\mathrm{T}}\mathbf{R}^{-1}\left\{\sum_{t\in K} \mathbf{G}_t\mathbf{C}(x_t, \theta_j) + \sum_{r\in J} \mathbf{K}_r\mathbf{C}(\theta_r, \theta_j)\right\}
$$

$$
-\left\{\sum_{t\in K} \mathbf{C}(\theta_i, x_t)\mathbf{G}_t^{\mathrm{T}} + \sum_{r\in J} \mathbf{C}(\theta_i, \theta_r)\mathbf{K}_r^{\mathrm{T}}\right\}\mathbf{R}^{-1}\mathbf{K}_j + \mathbf{K}_i^{\mathrm{T}}\mathbf{R}^{-1}\left\{\sum_{t\in K} \mathbf{G}_t\left[\sum_{s\in K} \mathbf{C}(x_t, x_s)\mathbf{G}_t^{\mathrm{T}} + \sum_{p\in J} \mathbf{C}(x_t, \theta_p)\mathbf{K}_p^{\mathrm{T}}\right]\right.
$$

$$
\left. + \sum_{r\in J} \mathbf{K}_r\left[\sum_{s\in K} \mathbf{C}(\theta_r, x_s)\mathbf{G}_s^{\mathrm{T}} + \sum_{p\in J} \mathbf{C}(\theta_r, \theta_p)\mathbf{K}_p^{\mathrm{T}}\right]\right\}\mathbf{R}^{-1}\mathbf{K}_j \tag{60}
$$

Despite their complex appearance the above equations are very easy to program on a computer. Usually, only the diagonal submatrices of $\mathbf{C}_x$ are computed using eq. (58) with $k = m$ and eq. (60) with $i = j$. A usual technique in geodetic problems is to transform the original coordinates into "barycentric" ones, in which case the matrix $\mathbf{R}$ becomes diagonal and the computations are further simplified.

As a final remark, let it be pointed out that the covariance matrices so obtained have no meaning of their own because they correspond to non-estimable parameters (coordinates), i.e. parameters that cannot be determined from the photogrammetric observations alone. They are only a sort of depository of information which can be further used for the computation of the variances and covariances of estimable parameters, such as length ratios and angles (or lengths when distances have also been observed).

Appendix A — Partial derivatives of the projective equations

The partial derivatives of the observed photo coordinates with respect to the photogrammetric parameters, present in the projective equation, are well known and can be found in photogrammetry texts. Here they will be derived in a compact form, based on the notion of the "cross product" matrix, which greatly facilitates the derivation of the inner constraints.

Let X_o, Y_o, Z_o be the coordinates of the projective centre in the object reference frame, X, Y, Z the coordinates of an object point in the object frame and u, v, w the coordinates of the same point in the reference frame of the photograph. Setting:

$$
y = [X_o \quad Y_o \quad Z_o]^{\mathrm{T}}, \qquad x = [X \quad Y \quad Z]^{\mathrm{T}}, \qquad p = [u \quad v \quad w]^{\mathrm{T}} \tag{A1}
$$

it holds that:

$$
p = \mathbf{R}(x - y) \tag{A2}
$$

where $\mathbf{R}$ is the orthogonal matrix of rotation from the object reference frame to that of the photograph. If f is the focal length of the camera and x, y the coordinates of the point image in the plane of the photograph, the well known collinearity equations are:

$$
x = -f\frac{u}{w}, \qquad y = -f\frac{v}{w} \tag{A3}
$$

For the derivation of the partial derivatives of x, y with respect to the parameters of exterior orientation and the object point coordinates, use will be made of the "cross product" matrix $[z\times]$ defined for every vector z as the matrix:

$$[z\times] \equiv \begin{bmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{bmatrix} \qquad z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{A4}$$

Two properties of these matrices will be also used:

$$[(\mathbf{Q}z)\times] = \mathbf{Q}[\mathbf{z}\times]\mathbf{Q}^{\mathrm{T}} \tag{A5}$$

where $\mathbf{Q}$ is any orthogonal matrix, and:

$$[\boldsymbol{a}\times][\boldsymbol{b}\times] = \boldsymbol{b}\boldsymbol{a}^{\mathrm{T}} - (\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b})\mathbf{I} \tag{A6}$$

Direct differentiation gives:

$$\frac{\partial x}{\partial \boldsymbol{p}} = \frac{f}{w^2}\boldsymbol{p}^{\mathrm{T}}[\boldsymbol{e}_2\times] = \frac{f}{w^2}(\boldsymbol{x}-\boldsymbol{y})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}[\boldsymbol{e}_2\times] = \frac{f}{w^2}(\boldsymbol{x}-\boldsymbol{y})^{\mathrm{T}}[(\mathbf{R}^{\mathrm{T}}\boldsymbol{e}_2)\times]\mathbf{R}^{\mathrm{T}} \tag{A7}$$

$$\frac{\partial y}{\partial \boldsymbol{p}} = -\frac{f}{w^2}\boldsymbol{p}^{\mathrm{T}}[\boldsymbol{e}_1\times] = -\frac{f}{w^2}(\boldsymbol{x}-\boldsymbol{y})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}[\boldsymbol{e}_1\times] = -\frac{f}{w^2}(\boldsymbol{x}-\boldsymbol{y})^{\mathrm{T}}[(\mathbf{R}^{\mathrm{T}}\boldsymbol{e}_1)\times]\mathbf{R}^{\mathrm{T}} \tag{A8}$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, are the columns of the 3×3 identity matrix,

$$\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{x}} = \mathbf{R}, \qquad \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{y}} = -\mathbf{R}, \qquad \frac{\partial \boldsymbol{p}}{\partial \vartheta} = \partial_\vartheta \mathbf{R}(\boldsymbol{x}-\boldsymbol{y}) \tag{A9}$$

where ϑ is any of the three parameters α, β, γ defining $\boldsymbol{R}$. For any orthogonal matrix $\mathbf{R}$, differentiation of $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$, with respect to a parameter ϑ shows that:

$$\mathbf{R}_\vartheta^{\mathrm{T}}\mathbf{R} + \mathbf{R}^{\mathrm{T}}\mathbf{R}_\vartheta = \mathbf{0} \quad \Longrightarrow \quad \mathbf{R}^{\mathrm{T}}\mathbf{R}_\vartheta = -(\mathbf{R}^{\mathrm{T}}\mathbf{R}_\vartheta)^{\mathrm{T}} \equiv [\boldsymbol{q}_\vartheta\times] \tag{A10}$$

the existence of the vector $\boldsymbol{q}_\vartheta$ being guarantied from the fact that $\mathbf{R}^{\mathrm{T}}\mathbf{R}_\vartheta$ is obviously an antisymmetric matrix. It is therefore easy to compute by direct differentiation three vectors $\boldsymbol{q}_\alpha, \boldsymbol{q}_\beta, \boldsymbol{q}_\gamma$, corresponding to the three parameters describing the rotation matrix $\mathbf{R}$. The derivatives of $\mathbf{R}$ have the form:

$$\partial_\vartheta \mathbf{R} \equiv \mathbf{R}_\vartheta = \mathbf{R}[\boldsymbol{q}_\vartheta\times], \qquad \vartheta = \alpha,\ \beta,\ \gamma \tag{A11}$$

The above derivatives are combined with the use of the chain rule to yield the desired derivatives:

$$\frac{\partial x}{\partial \boldsymbol{x}} = \frac{\partial x}{\partial \boldsymbol{p}}\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{x}} = \frac{f}{w^2}(\boldsymbol{x}-\boldsymbol{y})^{\mathrm{T}}[(\mathbf{R}^{\mathrm{T}}\boldsymbol{e}_2)\times] = -\frac{\partial x}{\partial \boldsymbol{y}} \tag{A12}$$

$$\frac{\partial y}{\partial \boldsymbol{x}} = \frac{\partial y}{\partial \boldsymbol{p}}\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{x}} = -\frac{f}{w^2}(\boldsymbol{x}-\boldsymbol{y})^{\mathrm{T}}[(\mathbf{R}^{\mathrm{T}}\boldsymbol{e}_1)\times] = -\frac{\partial y}{\partial \boldsymbol{y}} \tag{A13}$$

$$\frac{\partial x}{\partial \vartheta} = \frac{\partial x}{\partial \boldsymbol{p}}\frac{\partial \boldsymbol{p}}{\partial \vartheta} = \frac{f}{w^2}(\boldsymbol{x}-\boldsymbol{y})^{\mathrm{T}}\mathbf{F}_\vartheta(\boldsymbol{x}-\boldsymbol{y}) \tag{A14}$$

$$\frac{\partial y}{\partial \vartheta} = \frac{\partial y}{\partial \boldsymbol{p}}\frac{\partial \boldsymbol{p}}{\partial \vartheta} = -\frac{f}{w^2}(\boldsymbol{x}-\boldsymbol{y})^{\mathrm{T}}\mathbf{E}_\vartheta(\boldsymbol{x}-\boldsymbol{y}) \tag{A15}$$

where, taking property (A6) into account,

$$\mathbf{F}_\vartheta = [(\mathbf{R}^{\mathrm{T}}\boldsymbol{e}_2)\times][\boldsymbol{q}_\vartheta\times] = \boldsymbol{q}_\vartheta\boldsymbol{e}_2^{\mathrm{T}}\mathbf{R} - (\boldsymbol{e}_2^{\mathrm{T}}\mathbf{R}\boldsymbol{q}_\vartheta)\mathbf{I} \tag{A16}$$

$$\mathbf{E}_\vartheta = [(\mathbf{R}^{\mathrm{T}}\boldsymbol{e}_1)\times][\boldsymbol{q}_\vartheta\times] = \boldsymbol{q}_\vartheta\boldsymbol{e}_1^{\mathrm{T}}\mathbf{R} - (\boldsymbol{e}_1^{\mathrm{T}}\mathbf{R}\boldsymbol{q}_\vartheta)\mathbf{I} \tag{A17}$$

Appendix B — Inner constraints for various choices of orientation angles

In this appendix the matrices (one for each photograph j):

$$\mathbf{Q}^{-T} = [\boldsymbol{q}_\alpha\ \boldsymbol{q}_\beta\ \boldsymbol{q}_\gamma]^{-T} \tag{B1}$$

appearing in the inner constraints will be evaluated for three different choices where orientation angles are used as rotation parameters α, β, γ. The columns $\boldsymbol{q}_\alpha$, $\boldsymbol{q}_\beta$, $\boldsymbol{q}_\gamma$, are defined according to Appendix A, by:

$$[\boldsymbol{q}_\alpha \times] = \mathbf{R}^T \partial_\alpha \mathbf{R}, \qquad [\boldsymbol{q}_\beta \times] = \mathbf{R}^T \partial_\beta \mathbf{R}, \qquad [\boldsymbol{q}_\gamma \times] = \mathbf{R}^T \partial_\gamma \mathbf{R} \tag{B2}$$

B1. Orientation angles κ, φ, ω

For the usual choice of orientation angles:

$$\mathbf{R} = \mathbf{R}_3(\kappa)\mathbf{R}_2(\varphi)\mathbf{R}_1(\omega) \tag{B3}$$

$$\mathbf{R}_\kappa \equiv \partial_\kappa \mathbf{R} = -\mathbf{R}\,\mathbf{R}_1(-\omega)\mathbf{R}_2(-\varphi)[\boldsymbol{e}_3 \times]\mathbf{R}_2(\varphi)\mathbf{R}_1(\omega) \tag{B4}$$

$$\mathbf{R}_\varphi \equiv \partial_\varphi \mathbf{R} = -\mathbf{R}\,\mathbf{R}_1(-\omega)[\boldsymbol{e}_2 \times]\mathbf{R}_1(\omega) \tag{B5}$$

$$\mathbf{R}_\omega \equiv \partial_\omega \mathbf{R} = -\mathbf{R}[\boldsymbol{e}_1 \times] \tag{B6}$$

where the property $\partial_\vartheta \mathbf{R}_i(\vartheta) = -[\boldsymbol{e}_i \times]\mathbf{R}_i(\vartheta) = -\mathbf{R}_i(\vartheta)[\boldsymbol{e}_i \times]$ has been used.
According to (B2):

$$[\boldsymbol{q}_\kappa \times] = \mathbf{R}^T\mathbf{R}_\kappa = -\mathbf{R}_1(-\omega)\mathbf{R}_2(-\varphi)[\boldsymbol{e}_3 \times]\mathbf{R}_2(\varphi)\mathbf{R}_1(\omega) \tag{B7}$$

$$[\boldsymbol{q}_\varphi \times] = \mathbf{R}^T\mathbf{R}_\varphi = -\mathbf{R}_1(-\omega)[\boldsymbol{e}_2 \times]\mathbf{R}_1(\omega) \tag{B8}$$

$$[\boldsymbol{q}_\omega \times] = \mathbf{R}^T\mathbf{R}_\varphi = -[\boldsymbol{e}_1 \times] \tag{B9}$$

With the use of property (A5) it follows that:

$$\boldsymbol{q}_\kappa = -\mathbf{R}_1(-\omega)\mathbf{R}_2(-\varphi)\,\boldsymbol{e}_3 \tag{B10}$$

$$\boldsymbol{q}_\varphi = -\mathbf{R}_1(-\omega)\,\boldsymbol{e}_2 \tag{B11}$$

$$\boldsymbol{q}_\omega = -\boldsymbol{e}_1 \tag{B12}$$

and

$$\mathbf{Q} = [\boldsymbol{q}_\kappa\ \boldsymbol{q}_\varphi\ \boldsymbol{q}_\omega] = \begin{bmatrix} -\sin\varphi & 0 & -1 \\ \sin\omega\cos\varphi & -\cos\omega & 0 \\ -\cos\omega\cos\varphi & -\sin\omega & 0 \end{bmatrix} \tag{B13}$$

$$-\mathbf{Q}^{-T} = \begin{bmatrix} 0 & 0 & 1 \\ -\dfrac{\sin\omega}{\cos\varphi} & \cos\omega & \sin\omega\ \tan\varphi \\ \dfrac{\cos\omega}{\cos\varphi} & \sin\omega & -\cos\omega\ \tan\varphi \end{bmatrix} \tag{B14}$$

B2. Azimuth, swing and tilt (α, S, t)

The rotation angles, azimuth α, swing S and tilt t, are defined by:

$$\mathbf{R} = \mathbf{R}_3(180° - S)\mathbf{R}_1(t)\mathbf{R}_3(-\alpha) \tag{B15}$$

Following the same approach as above we obtain:

$$q_\alpha = e_3, \qquad q_t = -\mathbf{R}_3(\alpha)\, e_1, \qquad q_S = \mathbf{R}_3(\alpha)\mathbf{R}_1(-t)\, e_3 \tag{B16}$$

$$-\mathbf{Q}^{-T} = -\begin{bmatrix} 0 & -\cos\alpha & -\sin\alpha\sin t \\ 0 & \sin\alpha & -\cos\alpha\sin t \\ 1 & 0 & \cos t \end{bmatrix}^{-\mathrm{T}} = \begin{bmatrix} -\sin\alpha\cot t & \cos\alpha & \dfrac{\sin\alpha}{\sin t} \\ -\cos\alpha\cot t & -\sin\alpha & \dfrac{\cos\alpha}{\sin t} \\ -1 & 0 & 0 \end{bmatrix} \tag{B17}$$

B3. Azimuth, vertical angle and swing (α, ω, κ)

In close-range photogrammetry, especially when photogrammetric observations are to be adjusted in combination with geodetic observations, an appropriate choice of rotation parameters is:

$$\mathbf{R} = \mathbf{R}_3(\kappa)\mathbf{R}_1(90° + \omega)\mathbf{R}_3(-\alpha) \tag{B18}$$

where the azimuth α and the vertical angle ω are the same as those measured by theodolites. In a similar way as before we obtain:

$$q_\alpha = e_3 \quad q_\omega = -\mathbf{R}_3(\alpha)\, e_1 \quad q_\kappa = -\mathbf{R}_3(\alpha)\mathbf{R}_1(-90° - \omega)\, e_3 \tag{B19}$$

$$-\mathbf{Q}^{-\mathrm{T}} = -\begin{bmatrix} 0 & -\cos\alpha & \sin\alpha\cos\omega \\ 0 & \sin\alpha & \cos\alpha\cos\omega \\ 1 & 0 & \sin\omega \end{bmatrix}^{-\mathrm{T}} = \begin{bmatrix} \sin\alpha\tan\omega & \cos\alpha & -\dfrac{\sin\alpha}{\cos\omega} \\ \cos\alpha\tan\omega & -\sin\alpha & -\dfrac{\cos\alpha}{\cos\omega} \\ -1 & 0 & 0 \end{bmatrix} \tag{B20}$$

Appendix C — Inner constraints with Rodriguez elements

One choice of rotation parameters are the Rodriguez elements based on the representation of an orthogonal matrix in the form:

$$\mathbf{R} = (\mathbf{I} - [z\times])(\mathbf{I} + [z\times])^{-1} \tag{C1}$$

where z is a vector in the direction of the rotation axis (eigenvector of $\mathbf{R}$) having length $|z| = \tan(\psi/2)$, where ψ is the angle of rotation around z. A usual choice for z is:

$$z = \frac{1}{d}[abc]^{\mathrm{T}}, \qquad (a^2 + b^2 + c^2 + d^2 = 1) \tag{C2}$$

leading to the rotation matrix:

$$\mathbf{R} = 2\begin{bmatrix} \frac{1}{2} - b^2 - c^2 & ab + cd & ac - bd \\ ab - cd & \frac{1}{2} - a^2 - c^2 & bc + ad \\ ac + bd & bc - ad & \frac{1}{2} - a^2 - b^2 \end{bmatrix} \tag{C3}$$

The procedure for the determination of the coefficients of the inner constraints corresponding to the Rodriguez elements a, b, c, is the one described in section 2, for the various choices of rotation angles. The derivatives $\mathbf{R}_a$, $\mathbf{R}_b$, $\mathbf{R}_c$ of $\mathbf{R}$ with respect to the Rodriguez elements a, b, c, respectively, are:

$$\mathbf{R}_a = \frac{2}{d}\begin{bmatrix} 0 & bd-ac & cd+ab \\ bd+ac & -2ad & d^2-a^2 \\ cd-ab & a^2-d^2 & -2ad \end{bmatrix} \tag{C4}$$

$$\mathbf{R}_b = \frac{2}{d}\begin{bmatrix} -2bd & ad-bc & b^2-d^2 \\ ad+bc & 0 & cd-ab \\ d^2-b^2 & cd+ab & -2bd \end{bmatrix} \tag{C5}$$

$$\mathbf{R}_c = \frac{2}{d}\begin{bmatrix} -2cd & d^2-c^2 & ad+bc \\ c^2-d^2 & -2cd & bd-ac \\ ad-bc & bd+ac & 0 \end{bmatrix} \tag{C6}$$

Multiplying with $\mathbf{R}^{\mathrm{T}}$ from the left it follows that:

$$[\boldsymbol{q}_a\times] = \mathbf{R}^{\mathrm{T}}\mathbf{R}_a = \frac{2}{d}\begin{bmatrix} 0 & ac-bd & -ab-cd \\ bd-ac & 0 & a^2+d^2 \\ ab+cd & -(a^2+d^2) & 0 \end{bmatrix} \tag{C7}$$

$$[\boldsymbol{q}_b\times] = \mathbf{R}^{\mathrm{T}}\mathbf{R}_b = \frac{2}{d}\begin{bmatrix} 0 & ad+bc & -(b^2+d^2) \\ -ad-bc & 0 & ab-cd \\ b^2+d^2 & cd-ab & 0 \end{bmatrix} \tag{C8}$$

$$[\boldsymbol{q}_c\times] = \mathbf{R}^{\mathrm{T}}\mathbf{R}_c = \frac{2}{d}\begin{bmatrix} 0 & c^2+d^2 & ad-bc \\ -(c^2+d^2) & 0 & bd+ac \\ bc-ad & -bd-ac & 0 \end{bmatrix} \tag{C9}$$

and finally:

$$-\mathbf{Q}^{-\mathrm{T}} = \frac{d}{2}\begin{bmatrix} a^2+d^2 & ab-cd & ac+bd \\ ab+cd & b^2+d^2 & bc-ad \\ ac-bd & bc+ad & c^2+d^2 \end{bmatrix}^{-\mathrm{T}} = \frac{1}{2}\begin{bmatrix} d & c & -b \\ -c & d & a \\ b & -a & d \end{bmatrix} \tag{C10}$$

The rotation of each photograph j is described by a corresponding set of Rodriguez elements a_j, b_j, c_j, whose approximate values are used for the formulation of the $-\mathbf{Q}_j^{-\mathrm{T}}$ submatrices according to (C10). These submatrices are used for the formulation of the matrix of inner constraints $\mathbf{E}$, according to eq. (36).

Appendix D — Interpretation of the matrix Q

An interpretation of the matrix $\mathbf{Q}$ is possible along the lines of Papo (1987). Let $\boldsymbol{x}$ be the representation of any vector in object space, which the orthogonal matrix $\mathbf{R} = \mathbf{R}(\boldsymbol{\theta}) = \mathbf{R}(\alpha, \beta, \gamma)$ transforms into $\mathbf{R}\boldsymbol{x}$, i.e., its representation in the photograph frame. A small rotation of the object reference frame transforms x into:

$$x + dx = \mathbf{R}_3(dm_3)\mathbf{R}_2(dm_2)\mathbf{R}_1(dm_1)x = \begin{bmatrix} 1 & dm_3 & -dm_2 \\ -dm_3 & 1 & dm_1 \\ dm_2 & -dm_1 & 1 \end{bmatrix} x = (\mathbf{I} - [dm\times])x$$

$$= x - [dm\times]x \tag{D1}$$

where dm_1, dm_2, dm_3 are the three differentially small rotation angles. Since $\mathbf{R}x$ is invariant under this object frame transformation, its differential will vanish so that:

$$d(\mathbf{R}x) = d\mathbf{R}x + \mathbf{R}\,dx = d\mathbf{R}x - \mathbf{R}[dm\times]x = \mathbf{0} \tag{D2}$$

Since this relation holds for any x, it follows that:

$$[dm\times] = \mathbf{R}^T d\mathbf{R} = \mathbf{R}^T(\partial_\alpha \mathbf{R}\,d\alpha + \partial_\beta \mathbf{R}\,d\beta + \partial_\gamma \mathbf{R}\,d\gamma)$$

$$= [q_\alpha\times]d\alpha + [q_\beta\times]d\beta + [q_\gamma\times]d\gamma = [(q_\alpha\,d\alpha + q_\beta\,d\beta + q_\gamma\,d\gamma)\times] \tag{D3}$$

and

$$dm = q_\alpha\,d\alpha + q_\beta\,d\beta + q_\gamma\,d\gamma = [q_\alpha \quad q_\beta \quad q_\gamma]\begin{bmatrix} d\alpha \\ d\beta \\ d\gamma \end{bmatrix} = \mathbf{Q}\,d\theta \tag{D4}$$

where the definitions (16) and (17) have been used. From (D4) it is immediately seen that:

$$\mathbf{Q} = \frac{\partial m}{\partial \theta}, \quad \mathbf{Q}^{-1} = \frac{\partial \theta}{\partial m} \tag{D5}$$

Therefore each submatrix $\mathbf{Q}_j^{-1}$ of the matrix $\mathbf{E}^T$, is the matrix of the partial derivatives of the orientation parameters $\theta_j = [\alpha_j \quad \beta_j \quad \gamma_j]^T$ of photograph j with respect to three angles m_1, m_2, m_3 of a transformation of the object reference frame. The above approach is in fact identical to that of Papo (1987), for the derivation of $\partial\theta/\partial m$ as a submatrix of the inner constraint matrix $\mathbf{E}^T$.

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