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Free network solutions with the DLT method

The photogrammetric inner constraints, which are used for the derivation of free network solutions in close-range applications, are derived for the case where the Direct Linear Transformation model is used in the adjustment of the observations. The derivation is based on the fact that the inner constraint matrix is the transpose of the matrix of the derivatives of the unknown parameters with respect to transformation parameters of the reference frame. This matrix of derivatives is derived directly by studying the variation of the DLT parameters caused by an infinitesimal change of the reference frame. This is achieved from the known corresponding variation of the bundle parameters and the known direct and inverse relations between DLT and bundle parameters.

1. Introduction

The Direct Linear Transformation (DLT) has been proposed as an alternative to the projective equations used in the bundle method, because of the simplicity of its mathematical form, especially for close-range applications (Abdel-Aziz and Karara, 1971, 1974; Karara and Abdel-Aziz, 1974; Karara, 1979). A disadvantage of the DLT method is perhaps the limitation caused by the implicit assumption of different values for each photograph, of certain of the interior orientation parameters, which can be overcome only with the introduction of additional constraints on the parameters (Bopp and Krauss, 1978). The DLT method is typically applied for the solution of the resection problem for each photograph followed by intersections for individual point determinations. However, it can be also applied for "bundle-type" solutions for the simultaneous determination of point coordinates and the DLT parameters of each photograph. An advantage of the bundle method in many close-range applications is the possibility to use the so called "inner constraints" for free-network solutions (Papo, 1987; Dermanis, 1994). In such a solution an optimal introduction of the reference frame is realized from the solution of the "Zero Order Design" optimization problem (Grafarend, 1974).

Here the complete inner constraints for the DLT method are derived, being complete in the sense that they involve not only object coordinates, but also the DLT parameters and thus (implicitly) the elements of exterior orientation of each photograph.

2. Variation of DLT elements due to a small change of the reference frame

The basic model equations of the so called "Direct Linear Transformation" (DLT) are:

$$x = \frac{a_1X + a_2Y + a_3Z + \alpha}{c_1X + c_2Y + c_3Z + 1} = \frac{\boldsymbol{a}^T\boldsymbol{x} + \alpha}{\boldsymbol{c}^T\boldsymbol{x} + 1} \tag{1}$$

$$y = \frac{b_1X + b_2Y + b_3Z + \beta}{c_1X + c_2Y + c_3Z + 1} = \frac{\boldsymbol{b}^T\boldsymbol{x} + \beta}{\boldsymbol{c}^T\boldsymbol{x} + 1} \tag{2}$$

where x, y are the observable photo-coordinates, $\boldsymbol{x} = [X \; Y \; Z]^T$ are the coordinates of the object point and $\boldsymbol{a} = [a_1 \; a_2 \; a_3]^T$, α, $\boldsymbol{b} = [b_1 \; b_2 \; b_3]^T$, β, $\boldsymbol{c} = [c_1 \; c_2 \; c_3]^T$ are the eleven DLT parameters of the particular photograph.

It is known (Shih and Faig, 1987) that the DLT parameters can be related to the six elements of exterior orientation of the photograph (projective centre coordinates X_o, Y_o, Z_o, and orientation angles κ,

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φ, ω) and five elements of interior orientation (principal point coordinates x_o, y_o, focal length f, relative y-scale λ and shear d). As shown in Appendix A (see also Dermanis, 1990) the equations relating the DLT parameters with those of the extended projective equations are:

$$\boldsymbol{a} = q\,\mathbf{R}^{\mathrm{T}}(f\,\boldsymbol{e}_1 - x_o\,\boldsymbol{e}_3) \tag{3}$$

$$\boldsymbol{b} = q\,\mathbf{R}^{\mathrm{T}}(df\,\boldsymbol{e}_1 + \lambda f\,\boldsymbol{e}_2 - y_o\,\boldsymbol{e}_3) \tag{4}$$

$$\boldsymbol{c} = -q\,\mathbf{R}^{\mathrm{T}}\boldsymbol{e}_3 \tag{5}$$

$$\alpha = x_o - qf(\boldsymbol{e}_1^{\mathrm{T}}\mathbf{R}\boldsymbol{y}) \tag{6}$$

$$\beta = y_o - qfd(\boldsymbol{e}_1^{\mathrm{T}}\mathbf{R}\boldsymbol{y}) - qf\lambda(\boldsymbol{e}_2^{\mathrm{T}}\mathbf{R}\boldsymbol{y}) \tag{7}$$

$$q = \frac{1}{\boldsymbol{e}_3^{\mathrm{T}}\mathbf{R}\boldsymbol{y}} \tag{8}$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are the columns of the 3×3 identity matrix $\mathbf{I}$,

$$\boldsymbol{y} = [X_o \quad Y_o \quad Z_o]^{\mathrm{T}} \tag{9}$$

and $\mathbf{R}$ is the orthogonal matrix of rotation from the object frame to the frame of the photograph.

A change of coordinates due to a corresponding change of the reference frame (translation + rotation + scale change) has the form of the similarity transformation:

$$\tilde{\boldsymbol{y}} = (1+\varepsilon)\mathbf{Q}\boldsymbol{y} + \boldsymbol{t} \tag{10}$$

where $(1+\varepsilon)$ is the scale factor, $\mathbf{Q}$ an orthogonal rotation matrix and $\boldsymbol{t}$ the displacement vector. Considering only a small change of reference frame the small orthogonal matrix can be sufficiently approximated by:

$$\mathbf{Q} \approx \begin{bmatrix} 1 & -\vartheta_3 & \vartheta_2 \\ \vartheta_3 & 1 & -\vartheta_1 \\ -\vartheta_2 & \vartheta_1 & 1 \end{bmatrix} = \mathbf{I} + \begin{bmatrix} 0 & -\vartheta_3 & \vartheta_2 \\ \vartheta_3 & 0 & -\vartheta_1 \\ -\vartheta_2 & \vartheta_1 & 0 \end{bmatrix} = \mathbf{I} + [\boldsymbol{\theta}\times] \tag{11}$$

where $[\boldsymbol{\theta}\times]$ is the skew-symmetric "exterior product matrix" of the vector $\boldsymbol{\theta} = [\vartheta_1 \quad \vartheta_2 \quad \vartheta_3]^{\mathrm{T}}$ of three small rotation angles. (If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are any three vectors with matrix representations $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, and such that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c}$, then $[\boldsymbol{a}\times]$ is defined as the 3×3 matrix satisfying the relation $[\boldsymbol{a}\times]\boldsymbol{b} = \boldsymbol{c}$). Inserting $\mathbf{Q}$ from Eq. (11) into Eq. (10) and retaining only first-order terms it follows that:

$$\tilde{\boldsymbol{y}} = (1+\varepsilon)\boldsymbol{y} - [\boldsymbol{y}\times]\boldsymbol{\theta} + \boldsymbol{t} \tag{12}$$

where the property $[\boldsymbol{\theta}\times]\boldsymbol{y} = -[\boldsymbol{y}\times]\boldsymbol{\theta}$, has been used. The orientation matrix $\mathbf{R}$ transforms according to:

$$\tilde{\mathbf{R}} = \mathbf{R}\mathbf{Q}^{\mathrm{T}} = \mathbf{R}(\mathbf{I} - [\boldsymbol{\theta}\times]) \tag{13}$$

since $[\boldsymbol{\theta}\times]^{\mathrm{T}} = -[\boldsymbol{\theta}\times]$ (antisymmetric property). Combination of Eqs. (12) and (13) gives:

$$\tilde{\mathbf{R}}\tilde{\boldsymbol{y}} = \mathbf{R}(\mathbf{I} - [\boldsymbol{\theta}\times])\,[(1+\varepsilon)\boldsymbol{y} - [\boldsymbol{y}\times]\boldsymbol{\theta} + \boldsymbol{t}] \approx (1+\varepsilon)\mathbf{R}\boldsymbol{y} + \mathbf{R}\boldsymbol{t} \tag{14}$$

where second-order terms in the small quantities ε, $\boldsymbol{\theta}$, $\boldsymbol{t}$, have been neglected. The auxiliary parameter q transforms according to:

$$\tilde{q}^{-1} = \boldsymbol{e}_3^{\mathrm{T}}\tilde{\mathbf{R}}\tilde{\boldsymbol{y}} = (1+\varepsilon)\,\boldsymbol{e}_3^{\mathrm{T}}\mathbf{R}\boldsymbol{y} + \boldsymbol{e}_3^{\mathrm{T}}\mathbf{R}\boldsymbol{t} = (1+\varepsilon)\frac{1}{q} + \boldsymbol{e}_3^{\mathrm{T}}\mathbf{R}\boldsymbol{t} \tag{15}$$

$$\tilde{q} = \frac{q}{1+\varepsilon+q\,\boldsymbol{e}_3^{\mathrm{T}}\mathbf{R}\boldsymbol{t}} \approx q(1-\varepsilon-q\,\boldsymbol{e}_3^{\mathrm{T}}\mathbf{R}\boldsymbol{t}) \tag{16}$$

Note that $\tilde{f} = f$, $\tilde{x}_o = x_o$, $\tilde{y}_o = y_o$, $\tilde{d} = d$, $\tilde{\lambda} = \lambda$, since they do not depend on the used reference frame. The DLT parameters transform as follows:

$$\begin{aligned}\tilde{\boldsymbol{a}} &= \tilde{q}\,\tilde{\mathbf{R}}^{\mathrm{T}}(f\,\mathbf{e}_1 - x_o\,\mathbf{e}_3) = q(1 - \varepsilon - q\,\boldsymbol{e}_3^{\mathrm{T}}\mathbf{R}\boldsymbol{t})\{\mathbf{I} + [\boldsymbol{\theta}\times]\}\mathbf{R}^{\mathrm{T}}(f\,\mathbf{e}_1 - x_o\,\mathbf{e}_3) \\ &= (1 - \varepsilon - q\,\boldsymbol{e}_3^{\mathrm{T}}\mathbf{R}\boldsymbol{t})\{\mathbf{I} + [\boldsymbol{\theta}\times]\}\boldsymbol{a} = \boldsymbol{a} - \boldsymbol{a}\,\varepsilon - q\,\boldsymbol{a}\,\boldsymbol{e}_3^{\mathrm{T}}(\mathbf{R}\boldsymbol{t}) - [\boldsymbol{a}\times]\boldsymbol{\theta}\end{aligned} \tag{17}$$

In a similar way it can be shown that:

$$\tilde{\boldsymbol{b}} = \boldsymbol{b} - \boldsymbol{b}\,\varepsilon - q\,\boldsymbol{b}\,\boldsymbol{e}_3^{\mathrm{T}}(\mathbf{R}\boldsymbol{t}) - [\boldsymbol{b}\times]\boldsymbol{\theta} \tag{18}$$

$$\tilde{\boldsymbol{c}} = \boldsymbol{c} - \boldsymbol{c}\,\varepsilon - q\,\boldsymbol{c}\,\boldsymbol{e}_3^{\mathrm{T}}(\mathbf{R}\boldsymbol{t}) - [\boldsymbol{c}\times]\boldsymbol{\theta} \tag{19}$$

while:

$$\begin{aligned}\tilde{\alpha} &= x_o - \tilde{q} f(\boldsymbol{e}_1^{\mathrm{T}}\tilde{\mathbf{R}}\tilde{\boldsymbol{y}}) = x_o - qf(1 - \varepsilon - q\,\boldsymbol{e}_3^{\mathrm{T}}\mathbf{R}\boldsymbol{t})[(1+\varepsilon)\boldsymbol{e}_1^{\mathrm{T}}\mathbf{R}\boldsymbol{y} + \boldsymbol{e}_1^{\mathrm{T}}\mathbf{R}\boldsymbol{t}] \\ &= x_o - (1 - \varepsilon - q\,\boldsymbol{e}_3^{\mathrm{T}}\mathbf{R}\boldsymbol{t})[(1+\varepsilon)(x_o - \alpha) + qf\,\boldsymbol{e}_1^{\mathrm{T}}\mathbf{R}\boldsymbol{t}] \\ &= \alpha - q[f\,\boldsymbol{e}_1 + (x_o + \alpha)\boldsymbol{e}_3]^{\mathrm{T}}\mathbf{R}\boldsymbol{t} \approx \alpha - q[f\,\boldsymbol{e}_1 + \alpha\,\boldsymbol{e}_3]^{\mathrm{T}}\mathbf{R}\boldsymbol{t}\end{aligned} \tag{20}$$

since x_o is a small quantity. Using:

$$d(\boldsymbol{e}_1^{\mathrm{T}}\tilde{\mathbf{R}}\tilde{\boldsymbol{y}}) + \lambda(\boldsymbol{e}_2^{\mathrm{T}}\tilde{\mathbf{R}}\tilde{\boldsymbol{y}}) = [d\,\boldsymbol{e}_1^{\mathrm{T}} + \lambda\boldsymbol{e}_2^{\mathrm{T}}][(1+\varepsilon)\mathbf{R}\boldsymbol{y} + \mathbf{R}\boldsymbol{t}] \tag{21}$$

$$\begin{aligned}\tilde{\beta} &= y_o - qf[d(\boldsymbol{e}_1^{\mathrm{T}}\tilde{\mathbf{R}}\tilde{\boldsymbol{y}}) + \lambda(\boldsymbol{e}_2^{\mathrm{T}}\tilde{\mathbf{R}}\tilde{\boldsymbol{y}})] \\ &= y_o - qf(1 - [\varepsilon q]\boldsymbol{e}_3^{\mathrm{T}}\mathbf{R}\boldsymbol{t})[d\boldsymbol{e}_1 + \lambda\,\boldsymbol{e}_2]^{\mathrm{T}}[(1+\varepsilon)\mathbf{R}\boldsymbol{y} + \mathbf{R}\boldsymbol{t}] \\ &= \beta - q[fd\,\boldsymbol{e}_1 + f\lambda\,\boldsymbol{e}_2 - (y_o - \beta)\boldsymbol{e}_3]^{\mathrm{T}}(\mathbf{R}\boldsymbol{t})\end{aligned} \tag{22}$$

Since d, y_o and $(\lambda - 1)$ are small quantities which will be multiplied with the small vector $\mathbf{R}\boldsymbol{t}$, it holds with sufficient approximation that:

$$\tilde{\beta} \approx \beta - q(f\,\boldsymbol{e}_2 + \beta\,\boldsymbol{e}_3)^{\mathrm{T}}(\mathbf{R}\boldsymbol{t}) \tag{23}$$

Combining the above equations, the complete transformation of the DLT elements in linear approximation is:

$$\begin{bmatrix}\tilde{\boldsymbol{a}} \\ \tilde{\alpha} \\ \tilde{\boldsymbol{b}} \\ \tilde{\beta} \\ \tilde{\boldsymbol{c}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{a} \\ \alpha \\ \boldsymbol{b} \\ \beta \\ \boldsymbol{c}\end{bmatrix} + \begin{bmatrix} q\,\boldsymbol{a}\,\boldsymbol{e}_3^{\mathrm{T}} & [\boldsymbol{a}\times] & \boldsymbol{a} \\ q(f\,\boldsymbol{e}_1 + \alpha\,\boldsymbol{e}_3)^{\mathrm{T}} & 0 & 0 \\ q\,\boldsymbol{b}\,\boldsymbol{e}_3^{\mathrm{T}} & [\boldsymbol{b}\times] & \boldsymbol{b} \\ q(f\,\boldsymbol{e}_2 + \beta\,\boldsymbol{e}_3)^{\mathrm{T}} & 0 & 0 \\ q\,\boldsymbol{c}\,\boldsymbol{e}_3^{\mathrm{T}} & [\boldsymbol{c}\times] & \boldsymbol{c}\end{bmatrix}\begin{bmatrix}-\mathbf{R}\boldsymbol{t} \\ -\boldsymbol{\theta} \\ -\varepsilon\end{bmatrix} \tag{24}$$

Here instead of the original transformation parameters ε, $\boldsymbol{\theta}$, $\boldsymbol{t}$, the equally valid parameters $-\mathbf{R}\boldsymbol{t}$, $-\boldsymbol{\theta}$, $-\varepsilon$, appear in the last vector which can be considered as new transformation parameters $\boldsymbol{t}' = -\mathbf{R}\boldsymbol{t}$, $\boldsymbol{\theta}' = -\boldsymbol{\theta}$, $\varepsilon' = -\varepsilon$. Therefore the last matrix in Eq. (24) is a vector of valid transformation parameters.

The above simplified equations have the advantage that they depend only on DLT parameters except for the auxiliary parameter q and f. For a solution within the DLT framework the q and f must be expressed as functions of DLT parameters. When the more precise equations for α and β are used the terms $(f\,\boldsymbol{e}_1 + \alpha\,\boldsymbol{e}_3)^{\mathrm{T}}$ and $(f\,\boldsymbol{e}_2 + \beta\,\boldsymbol{e}_3)^{\mathrm{T}}$ in Eq. (24) must be replaced by $[f\,\boldsymbol{e}_1 - (x_o - \alpha)\boldsymbol{e}_3]^{\mathrm{T}}$ and $[fd\,\boldsymbol{e}_1 + f\lambda\,\boldsymbol{e}_2 - (y_o - \beta)\boldsymbol{e}_3]^{\mathrm{T}}$, respectively, in which case x_o, y_o, d and λ, expressed as functions of DLT parameters, are also needed. The necessary relations are derived in Appendix A, where q is given by Eq. (A19), f by Eq. (A21), x_o and y_o by Eq. (A20), d by Eq. (A22), and λ by Eq. (A23).

3. Derivation of the inner constraints

The DLT method can be used in analytical close-range photogrammetry, with two or more photographs, for the determination of the coordinates of a number of selected object points. It is also possible to have a combination of photogrammetric with geodetic observations in a mixed type of solution. This type of application is similar to the bundle method and differs from the usual application of the DLT method in two steps: resections which determine the DLT elements of each photograph from known object points, followed by intersections from two or more photographs for the determination of new points. In this two-stage approach it is necessary to have known points for the intersections, and the reference frame is introduced through the known coordinates. There is therefore no room for the concept of inner constraints and free network solutions.

The combined approach without any known points differs from the bundle approach in the following respect: the DLT elements are different for each photograph and this corresponds to a bundle solution where not only the exterior orientation elements ($X_o, Y_o, Z_o, \kappa, \varphi, \omega$), but also some additional interior orientation elements (x_o, y_o, f, d, λ) vary from photograph to photograph. This leads to a certain overparametrization in comparison to the bundle method with additional parameters, where a set of interior orientation elements common to all photographs are included in the unknowns. The advantage of the DLT method lies, of course, in the simplicity of the basic mathematical model.

As shown in Appendix B, the determination of the inner constraints can be based on the equations of transformation of the unknowns due to a corresponding transformation of the reference frame. The unknowns include the corrections to the approximate values of the DLT elements:

$$z_j = \begin{bmatrix} \delta a \\ \delta \alpha \\ \delta b \\ \delta \beta \\ \delta c \end{bmatrix}_j, \qquad j = 1, 2, \ldots, n \tag{25}$$

for each photograph j, as well as the corrections to the approximate values of the coordinates:

$$\boldsymbol{x}_i = \begin{bmatrix} \delta X \\ \delta Y \\ \delta Z \end{bmatrix}_i, \qquad i = 1, 2, \ldots, m \tag{26}$$

for each object point i.

For each pair of observations x_{ji}, y_{ji} of the ith object point on the jth photograph, the linearized observation equations:

$$\boldsymbol{b}_{ji} = \mathbf{A}_{zji} z_j + \mathbf{A}_{xji} \boldsymbol{x}_i + \boldsymbol{v}_{ji} \tag{27}$$

are formulated, which are combined into the total observation equations:

$$\boldsymbol{b} = \mathbf{A}\boldsymbol{x} + \boldsymbol{v} \tag{28}$$

where

$$\boldsymbol{x} = [z_1^T z_2^T \ldots z_n^T \boldsymbol{x}_1^T \boldsymbol{x}_2^T \ldots \boldsymbol{x}_m^T]^T \tag{29}$$

The free network solution is achieved with the use of the inner constraints:

$$\mathbf{E}\boldsymbol{x} = \mathbf{0} \tag{30}$$

which are combined with the normal equations:

$$\mathbf{N}\boldsymbol{x} = \boldsymbol{u} \qquad (\mathbf{N} = \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}, \quad \boldsymbol{u} = \mathbf{A}^{\mathrm{T}}\mathbf{P}\boldsymbol{b}) \tag{31}$$

corresponding to the observation equation (28), in order to obtain the least-squares solution with minimum norm and minimum trace of its covariance matrix. The solution algorithm follows one of various techniques which can be classified into two categories. In the first, the coefficient matrix of the normal equations is modified in some way (e.g. by augmentation with $\mathbf{E}$), while in the second the solution is explicitly or implicitly obtained from the S-transformation of another solution.

The explicit form of matrix $\mathbf{E}$ of the inner constraints is necessary in all the above solutions. It can be obtained either from the property $\mathbf{A}\mathbf{E}^{\mathrm{T}} = \mathbf{0}$, as done, e.g. in Dermanis (1994), or as shown in Appendix B, from $\mathbf{E} = \mathbf{G}^{\mathrm{T}}$, where $\mathbf{G}$ is the coefficient matrix in the equation:

$$\tilde{\boldsymbol{x}} = \boldsymbol{x} + \mathbf{G}\boldsymbol{p} \tag{32}$$

for the transformation of the vector $\boldsymbol{x}$ of the unknowns caused by a differential change of the reference frame described by a set of transformation parameters in $\boldsymbol{p}$ (translations, rotation angles, scale factor).

As shown in the previous section, the DLT parameters transform according to:

$$\tilde{z}_j = z_j + \mathbf{G}_{zj}\boldsymbol{p} \tag{33}$$

where $\mathbf{G}_{zj}$ is the matrix in Eq. (24), now referring to the DLT elements of a specific photograph j. With a redefinition of the frame transformation parameters in Eq. (24), the vector $\boldsymbol{p}$ becomes:

$$\boldsymbol{p} = \begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{\theta} \\ \varepsilon \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \\ t_z \\ \vartheta_x \\ \vartheta_y \\ \vartheta_z \\ \varepsilon \end{bmatrix} \tag{34}$$

According to Eq. (24) the analytic form of the matrix $\mathbf{G}_{zj}$ is:

$$\mathbf{G}_{zj} = \left[\begin{array}{ccc|ccc|c} 0 & 0 & qa_1 & 0 & -a_3 & a_2 & a_1 \\ 0 & 0 & qa_2 & a_3 & 0 & -a_1 & a_2 \\ 0 & 0 & qa_3 & -a_2 & a_1 & 0 & a_3 \\ \hline qf & 0 & q\alpha & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & qb_1 & 0 & -b_3 & b_2 & b_1 \\ 0 & 0 & qb_2 & b_3 & 0 & -b_1 & b_2 \\ 0 & 0 & qb_3 & -b_2 & b_1 & 0 & b_3 \\ \hline 0 & qf & q\beta & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & qc_1 & 0 & -c_3 & c_2 & c_1 \\ 0 & 0 & qc_2 & c_3 & 0 & -c_1 & c_2 \\ 0 & 0 & qc_3 & -c_2 & c_1 & 0 & c_3 \end{array}\right] \tag{35}$$

where all DLT parameters are those of the jth photograph and

$$q = \sqrt{c_1^2 + c_2^2 + c_3^2} \tag{36}$$

The coordinate corrections transform according to:

$$\tilde{\boldsymbol{x}}_i = \boldsymbol{x}_i + \mathbf{G}_{xi}\,\boldsymbol{p} \tag{37}$$

where the form of the matrix $\mathbf{G}_{xi}$ is well known from the geodetic literature (Meissl, 1965, 1969; van Mierlo, 1980):

$$\mathbf{G}_{xi}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -Z & Y \\ Z & 0 & -X \\ -Y & X & 0 \\ X & Y & Z \end{bmatrix}_i \tag{38}$$

With $\mathbf{G}_{zj}$ from Eq. (35) and $\mathbf{G}_{xi}$ from Eq. (38) the matrix $\mathbf{G}$ becomes:

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_{z1} \\ \vdots \\ \mathbf{G}_{zn} \\ \mathbf{G}_{x1} \\ \vdots \\ \mathbf{G}_{xm} \end{bmatrix} \tag{39}$$

and the corresponding matrix of inner constraint $\mathbf{E} = \mathbf{G}^{\mathrm{T}}$ becomes:

$$\mathbf{E} = [\mathbf{E}_{z1} \ \ldots \ \mathbf{E}_{zn}\, \mathbf{E}_{x1} \ \ldots \ \mathbf{E}_{xm}] \tag{40}$$

where $\mathbf{E}_{zj} = \mathbf{G}_{zj}^{\mathrm{T}}$ and $\mathbf{E}_{xi} = \mathbf{G}_{xi}^{\mathrm{T}}$.

Appendix A — Transformations between projective equations and DLT parameters

The DLT Eqs. (1) and (2) can be directly derived from the projective equations:

$$x = x_{\mathrm{o}} - f\frac{u}{w}, \qquad y = y_{\mathrm{o}} - f\frac{v}{w} \tag{A1}$$

where:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{R}\begin{bmatrix} X - X_{\mathrm{o}} \\ Y - Y_{\mathrm{o}} \\ Z - Z_{\mathrm{o}} \end{bmatrix} = \mathbf{R}(\boldsymbol{x} - \boldsymbol{y}) \tag{A2}$$

However, in this case the eleven DLT parameters are not independent since they depend on only nine parameters $(\boldsymbol{x}, \boldsymbol{y}, f, x_o, y_o)$. As a consequence, the easily derived transformation from projective to DLT parameters is an overdetermined system of eleven equations in nine unknowns and cannot be inverted. For this reason, extended projective equations must be used which contain eleven parameters, so that an one-to-one correspondence of DLT to projective parameters is established. This extension is possible by considering two more parameters which reflect primary characteristics of film deformation. Considering an affine transformation of the original coordinates x, y in the projective equations, the extended projective equations become:

$$\begin{aligned}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} x_o \\ y_o \end{bmatrix} + \begin{bmatrix} \alpha_{xx} & \alpha_{xy} \\ \alpha_{yx} & \alpha_{yy} \end{bmatrix} \begin{bmatrix} -f\dfrac{u}{w} \\ -f\dfrac{v}{w} \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix} \\ &= \begin{bmatrix} x_o + b_x \\ y_o + b_y \end{bmatrix} + \begin{bmatrix} 1 & \dfrac{\alpha_{xy}}{\alpha_{xx}} \\ \dfrac{\alpha_{yx}}{\alpha_{xx}} & \dfrac{\alpha_{yy}}{\alpha_{xx}} \end{bmatrix} \begin{bmatrix} -(f\alpha_{xx})\dfrac{u}{w} \\ -(f\alpha_{xx})\dfrac{v}{w} \end{bmatrix} = \begin{bmatrix} \overline{x}_o \\ \overline{y}_o \end{bmatrix} + \begin{bmatrix} 1 & \omega \\ \delta & \lambda \end{bmatrix} \begin{bmatrix} -\overline{f}\dfrac{u}{w} \\ -\overline{f}\dfrac{v}{w} \end{bmatrix}\end{aligned} \tag{A3}$$

where $\overline{x}_o = x_o + b_x$, $\overline{y}_o = y_o + b_y$, $\overline{f} = f\alpha_{xx}$ are modified parameters resulting from the fact that it is impossible to separate f from a_{xx} and x_o, y_o from b_x, b_y, respectively. The original parameters α_{yy}, α_{xy}, α_{yx} have also been replaced by λ, ω, δ, respectively, i.e., with their ratios with α_{xx}. Since α_{xy}, α_{yx} are small quantities while α_{xx}, α_{yy} differ slightly from 1, it follows that ω, δ are small quantities while λ is close to 1. The modified transformation matrix can be analyzed into:

$$\begin{bmatrix} 1 & \omega \\ \delta & \lambda \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} + \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \omega+\delta & 0 \end{bmatrix} \tag{A4}$$

The first term corresponds to the relative scaling λ in the y-direction with respect to that in the x-direction, the common scaling being absorbed into f. The second antisymmetric term corresponds to a small rotation which is equivalent and cannot be distinguished from an opposite rotation of the reference frame in the photo plane. Since it is not related to deformation it should not be included. The last term corresponds to a shear in the plane. Including only the first and third term, setting $d = \omega+\delta$, and dropping the overbars from $\overline{x}_o$, $\overline{y}_o$, $\overline{f}$, Eq. (A3) becomes:

$$x = x_o - f\frac{u}{w} \tag{A5}$$

$$y = y_o - df\frac{u}{w} - \lambda f\frac{v}{w} \tag{A6}$$

which are the equivalent to the DLT "extended projective equations". Since from Eq. (A2):

$$u = \boldsymbol{e}_1^{\mathrm{T}}\mathbf{R}(\boldsymbol{x}-\boldsymbol{y}), \qquad v = \boldsymbol{e}_2^{\mathrm{T}}\mathbf{R}(\boldsymbol{x}-\boldsymbol{y}), \qquad w = \boldsymbol{e}_3^{\mathrm{T}}\mathbf{R}(\boldsymbol{x}-\boldsymbol{y}) \tag{A7}$$

it follows immediately that:

$$x = \frac{x_o w - fu}{w} = \frac{(x_o\boldsymbol{e}_3 - f\boldsymbol{e}_1)^{\mathrm{T}}\mathbf{R}\boldsymbol{x} - x_o\boldsymbol{e}_3^{\mathrm{T}}\mathbf{R}\boldsymbol{y} + f\boldsymbol{e}_1^{\mathrm{T}}\mathbf{R}\boldsymbol{y}}{\boldsymbol{e}_3^{\mathrm{T}}\mathbf{R}\boldsymbol{x} - \boldsymbol{e}_3^{\mathrm{T}}\mathbf{R}\boldsymbol{y}} \tag{A8}$$

Introducing:

$$q = \frac{1}{\boldsymbol{e}_3^{\mathrm{T}}\mathbf{R}\boldsymbol{y}} \tag{A9}$$

and multiplying nominator and denominator of Eq. (A8) with $-q$ gives:

$$x = \frac{-q(x_o e_3 - f e_1)^T \mathbf{R}\boldsymbol{x} + x_o - qf e_1^T \mathbf{R}\boldsymbol{y}}{-q e_3^T \mathbf{R}\boldsymbol{x} + 1} \tag{A10}$$

In a similar way:

$$y = \frac{y_o w - dfu - \lambda f v}{w} = \frac{q(df e_1 + \lambda f e_2 - y_o e_3)^T \mathbf{R}\boldsymbol{x} + y_o - qf(de_1 + \lambda e_2)\mathbf{R}\boldsymbol{y}}{-q e_3^T \mathbf{R}\boldsymbol{x} + 1} \tag{A11}$$

The equations for the DLT parameters follow directly by comparison of Eqs. (A10) and (A11) with Eqs. (1) and (2), respectively:

$$\boldsymbol{a} = q\,\mathbf{R}^T(f e_1 - x_o e_3) \tag{A12}$$

$$\alpha = x_o - qf(e_1^T \mathbf{R}\boldsymbol{y}) \tag{A13}$$

$$\boldsymbol{b} = q\,\mathbf{R}^T(df e_1 + \lambda f e_2 - y_o e_3) \tag{A14}$$

$$\beta = y_o - qf(d e_1 + \lambda e_2)^T \mathbf{R}\boldsymbol{y} \tag{A15}$$

$$\boldsymbol{c} = -q\,\mathbf{R}^T e_3 \tag{A16}$$

These equations must be inverted. For the derivation of x_o, y_o, f, λ and d note that:

$$[\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}] = q\,\mathbf{R}^T \begin{bmatrix} f & df & 0 \\ 0 & \lambda f & 0 \\ -x_o & -y_o & -1 \end{bmatrix} \tag{A17}$$

Multiplication of $[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]$ with its transpose from the left gives:

$$\begin{bmatrix} \boldsymbol{a}^T\boldsymbol{a} & \boldsymbol{a}^T\boldsymbol{b} & \boldsymbol{a}^T\boldsymbol{c} \\ \boldsymbol{b}^T\boldsymbol{a} & \boldsymbol{b}^T\boldsymbol{b} & \boldsymbol{b}^T\boldsymbol{c} \\ \boldsymbol{c}^T\boldsymbol{a} & \boldsymbol{c}^T\boldsymbol{b} & \boldsymbol{c}^T\boldsymbol{c} \end{bmatrix} = q^2 \begin{bmatrix} f^2 + x_o^2 & df^2 + x_o y_o & x_o \\ df^2 + x_o y_o & d^2 f^2 + \lambda^2 f^2 + y_o^2 & y_o \\ x_o & y_o & 1 \end{bmatrix} \tag{A18}$$

and it follows immediately that:

$$q = \sqrt{\boldsymbol{c}^T\boldsymbol{c}} \tag{A19}$$

$$x_o = \frac{\boldsymbol{a}^T\boldsymbol{c}}{\boldsymbol{c}^T\boldsymbol{c}}, \qquad y_o = \frac{\boldsymbol{b}^T\boldsymbol{c}}{\boldsymbol{c}^T\boldsymbol{c}} \tag{A20}$$

$$f^2 = \frac{\boldsymbol{a}^T\boldsymbol{a}}{\boldsymbol{c}^T\boldsymbol{c}} - x_o^2 = \frac{\boldsymbol{a}^T\boldsymbol{a}}{\boldsymbol{c}^T\boldsymbol{c}} - \left(\frac{\boldsymbol{a}^T\boldsymbol{c}}{\boldsymbol{c}^T\boldsymbol{c}}\right)^2 \tag{A21}$$

$$d = \frac{1}{f^2}\left[\frac{\boldsymbol{a}^T\boldsymbol{b}}{\boldsymbol{c}^T\boldsymbol{c}} - x_o y_o\right] = \frac{(\boldsymbol{a}^T\boldsymbol{b})(\boldsymbol{c}^T\boldsymbol{c}) - (\boldsymbol{a}^T\boldsymbol{c})(\boldsymbol{b}^T\boldsymbol{c})}{(\boldsymbol{a}^T\boldsymbol{a})(\boldsymbol{c}^T\boldsymbol{c}) - (\boldsymbol{a}^T\boldsymbol{c})^2} \tag{A22}$$

$$\lambda^2 = \frac{1}{f^2}\left[\frac{\boldsymbol{b}^T\boldsymbol{b}}{\boldsymbol{c}^T\boldsymbol{c}} - d^2 f^2 - y_o^2\right] = \frac{(\boldsymbol{b}^T\boldsymbol{b})(\boldsymbol{c}^T\boldsymbol{c}) - (\boldsymbol{b}^T\boldsymbol{c})^2}{(\boldsymbol{a}^T\boldsymbol{a})(\boldsymbol{c}^T\boldsymbol{c}) - (\boldsymbol{a}^T\boldsymbol{c})^2} - \left[\frac{(\boldsymbol{a}^T\boldsymbol{b})(\boldsymbol{c}^T\boldsymbol{c}) - (\boldsymbol{a}^T\boldsymbol{c})(\boldsymbol{b}^T\boldsymbol{c})}{(\boldsymbol{a}^T\boldsymbol{a})(\boldsymbol{c}^T\boldsymbol{c}) - (\boldsymbol{a}^T\boldsymbol{c})^2}\right]^2 \tag{A23}$$

Eq. (A17) can be solved for $\mathbf{R}$:

$$\mathbf{R} = q^{-1} \begin{bmatrix} f & 0 & -x_o \\ df & \lambda f & -y_o \\ 0 & 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \end{bmatrix}^T = \frac{1}{q\lambda f} \begin{bmatrix} \lambda & 0 & -\lambda x_o \\ -d & 1 & x_o d - y_o \\ 0 & 0 & -\lambda f \end{bmatrix} \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \end{bmatrix}^T \tag{A24}$$

where $\mathbf{R}$ is expressed as a function of the DLT elements with x_o, y_o, f, d and λ from Eqs. (A20), (A20), (A21), (A22), and (A23), respectively.

From Eqs. (A13), (A15), (A9) follow the equations:

$$\mathbf{R}y = \begin{bmatrix} e_1^T \mathbf{R}y \\ e_2^T \mathbf{R}y \\ e_3^T \mathbf{R}y \end{bmatrix} = \frac{1}{\lambda q f} \begin{bmatrix} \lambda(x_o - \alpha) \\ y_o - \beta - d(x_o - \alpha) \\ \lambda f \end{bmatrix} \tag{A25}$$

Multiplying the transpose of Eq. (A17) with y and taking Eq. (A25) into account leads to:

$$\begin{bmatrix} a & b & c \end{bmatrix}^T y = q \begin{bmatrix} f & 0 & -x_o \\ df & \lambda f & -y_o \\ 0 & 0 & -1 \end{bmatrix} \mathbf{R}y = \begin{bmatrix} -\alpha \\ -\beta \\ -1 \end{bmatrix} \tag{A26}$$

and finally:

$$y = \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix}^{-T} \begin{bmatrix} -\alpha \\ -\beta \\ -1 \end{bmatrix} \tag{A27}$$

The above relations are equivalent to similar relations given by Bopp and Krauss (1978), Hådem (1981), Shih and Faig (1987). They are useful not only for the derivation of inner constraints but for the modelling within the DLT frame of additional constraints and observations, such as angular and distance observations in combined geodetic–photogrammetric applications. These equations are initially easily expressed in terms of bundle parameters, and they can be transformed to expressions with DLT parameters with the help of the relations derived above. In this way the DLT method becomes competitive to the bundle method, which has the obvious advantage of using physically meaningful parameters.

Appendix B — Inner constraints in relation to transformations of the reference frame

The basic equation for the least squares adjustment with the method of the (linearized) observation equations is:

$$b = \mathbf{A}x + v \tag{B1}$$

where x are the corrections to the approximate values of the unknowns, b are the reduced observations (observed values minus computed from approximate values of parameters), $\mathbf{A}$ is the design matrix and v are the residuals. Of interest here is the case where the parameters x depend on the particular reference frame used, while the observations do not contain any relevant information, because they are invariant under reference frame transformations (Grafarend and Schaffrin, 1976). As a consequence, the matrix $\mathbf{A}$ has a rank deficiency equal to the number of parameters needed to define the reference frame. Furthermore, there is an infinite number of parameters x which satisfy the least-squares principle $v^T \mathbf{P} v = \min$.

If x is any least-squares solution of Eq. (B1), another least-squares solution $x - x'$ can be obtained by subtracting from x any vector x', such that $\mathbf{A}x' = \mathbf{0}$, because in this case $v = b - \mathbf{A}x = b - \mathbf{A}(x - x')$. We say in this case that x' belongs to the nullspace of $\mathbf{A}$, denoted by $x' \in N(\mathbf{A})$. The problem is to choose x' from $N(\mathbf{A})$ in such a way that $x_E = x - x'$ satisfies the minimum norm principle $\|x_E\|^2 = x_E^T x_E = \min$. Since $\|x - x'\|$ is the distance of the given solution x from an element $x' \in N(\mathbf{A})$, this distance is minimized when x' is chosen so that $x_E = x - x' \perp N(\mathbf{A})$. This condition is satisfied when $x' = x - x_E$ is the projection of the given x on the subspace $N(\mathbf{A})$. In order to get this projection a basis is needed for the subspace $N(\mathbf{A})$.

Let $\mathbf{E}$ be a matrix whose rows, form a basis for $N(\mathbf{A})$, that is, all their linear combinations are elements of $N(\mathbf{A})$ and vice-versa. The set of all linear combinations of the rows of $\mathbf{E}$ (= columns of $\mathbf{E}^T$) is called the range of $\mathbf{E}^T$ and is denoted by $R(\mathbf{E}^T)$. In other words, $N(\mathbf{A}) = R(\mathbf{E}^T)$. The projection of x on $R(\mathbf{E}^T)$ is given by:

$$x' = \mathbf{E}^T(\mathbf{E}\ \mathbf{E}^T)^{-1}\mathbf{E}x \tag{B2}$$

and therefore $x_E = x - x'$ is given by:

$$x_E = [\mathbf{I} - \mathbf{E}^T(\mathbf{E}\ \mathbf{E}^T)^{-1}\mathbf{E}]x \tag{B3}$$

It can be easily verified from Eq. (B3) that the minimum norm least-squares solution is perpendicular to $N(\mathbf{A}) = R(\mathbf{E}^T)$ and satisfies $\mathbf{E}x = 0$, the set of the so called inner constraints.

A different approach to determine the minimum norm least-squares solution is to start from any least-squares solution x and to formulate the transformation of x into another solution $\tilde{x}$ caused by a small change in the reference frame. This transformation has the form:

$$\tilde{x} = x + \mathbf{G}p \tag{B4}$$

derived from the linearization of the known equations $\tilde{x}^a = \tilde{x}^a(x^a, p)$, where x^a refers to the original parameters and not the corrections x to approximate values which appear in the linearized model (B1). In order to find the set of transformation parameters p which transform x into the minimum norm solution x_E, the minimization problem:

$$x = -\mathbf{G}p + \tilde{x}, \qquad \tilde{x}^T\tilde{x} = \min \tag{B5}$$

is formulated which has the well known solution:

$$p = -(\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T x \tag{B6}$$

Setting this value in Eq. (B4) the transformation from x to x_E becomes:

$$x_E = [\mathbf{I} - \mathbf{G}(\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T]x \tag{B7}$$

Comparison of Eq. (B7) with Eq. (B3) shows that the matrix $\mathbf{E}$ of the inner constraints $\mathbf{E}x = \mathit{0}$ is simply $\mathbf{E} = \mathbf{G}^T$. The transformation from any solution x to the minimun norm solution x_E is called the S-transformation (Baarda, 1973; Ebner, 1974; van Mierlo, 1980).

$$x_E = [\mathbf{I} - \mathbf{E}^T(\mathbf{E}\ \mathbf{E}^T)^{-1}\mathbf{E}]x = \mathbf{S}x \tag{B8}$$

References

Abdel-Aziz, Y.I and Karara, H.M., 1971. Direct linear transformation from comparator coordinates into object–space coordinates. ASP Symp. Close-Range Photogrammetry, 1971, Univeristy of Illinois, Urbana, Ill., pp. 1–18.

Abdel-Aziz, Y.I and Karara, H.M., 1974. Photogrammetric Potentials of Non-Metric Cameras. Civil Engineering Studies, Photogrammetry Series No. 36, University of Illinois, Urbana, Ill., 119 pp.

Baarda, W., 1973. S-transformations and Criterion Matrices. Netherlands Geodetic Commission, Publications on Geodesy, New Series, Vol. 5, No. 1.

Bopp, H. and Krauss, H., 1978. An orientation and calibration method for non-topographic applications. Photogramm. Eng. Remote Sensing, 44 (9): 1191–1196.

Dermanis, A., 1990. Analytical Photogrammetry. Editions Ziti, Thessaloniki, 429 pp. (in Greek).

Dermanis, A., 1994. The photogrammetric inner constraints. ISPRS J. Photogramm. Remote Sensing, 49 (1): 25–39.

Ebner, H., 1974. Analysis of covariance matrices. Proc. Symp. Int. Soc. of Photogrammetry, Commission III, Stuttgart, Sept. 1974. Deutsche Geodätische Kommission, Reihe B, Heft Nr. 214, pp. 111–121.

Grafarend, E., 1974. Optimization of geodetic networks. Boll. Geodesia Sci. Affini, 33 (4): 351–406.

Grafarend, E. and Schaffrin, B., 1976. Equivalence of estimable quantities and invariants in geodetic networks. Z. Vermessungswesen, 101 (11): 485–491.

Hådem, I., 1981. Bundle adjustment in industrial photogrammetry. Photogrammetria, 37: 45–46.

Karara, H.M. (Editor), 1979. Handbook of Non-Topographic Photogrammetry. American Society of Photogrammetry, Falls Church, Va.

Karara, H.M. and Abdel-Aziz, Y.I., 1974. Accuracy aspects of non-metric cameras. Photogramm. Eng. Remote Sensing, 1974: 1107–1117.

Meissl, P., 1965. Über die Innere Genauigkeit dreidimensionaler Punkthaufen. Z. Vermessungswesen, 90 (4): 109–118.

Meissl, P., 1969. Zusammenfassung und Ausbau der Inneren Fehlertheorie eines Punkthaufens. Deutsche Geodätische Kommission, Reihe A, Nr. 61, pp. 8–21.

Papo, H., 1987. Bases of null-space in analytical photogrammetry. Photogrammetria, 41: 233–244.

Shih, T.W. and Faig, W., 1987. Physical Interpretation of the Extended DLT Model. ASPRS Tech. Pap., ASPRS-ACSM Fall Convention, Reno, Nevada, Oct. 1987, pp. 385–394.

Van Mierlo, J., 1980. Free network adjustment and S-transformations. Deutsche Geodätische Kommission, Reihe B, Heft Nr. 252, pp. 41–54.

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