Dilatation, Shear, Rotation and Energy Analysis of Map Projections

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Summary — The analysis of the strain tensor structure is carried out concerning map projections with particular interest on some scalar strain parameters, e.g., dilatation, maximum shear strain, rotation and strain energy. In this way a detailed classification of map projections could be done even within families of maps with the same traditional properties, e.g., conformality, equivalence, etc.

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1. — INTRODUCTION

The first application of the Theory of Elasticity in Cartography is due to Tissot (1881), who analyzed the deformations implied by the mapping of the spherical grid of meridians and parallels onto a plane through an one-to-one correspondence (see also Fiorini, 1881). Tissot’s approach is a standard technique in Cartography, where the deformation analysis is done through the well known Tissot ellipse (or indicatrix), see, e.g., Biernacki, 1965, Swonarew, 1953. In the mid-fifties, Bernasconi, 1953, 1956 utilizing the principle of the minimization of elastic energy, studied some isotropic projections, as well as the deformation implied by mapping the spheroid onto a sphere.

In this paper we present a complete strain analysis for map projections focusing our interest on some scalar strain parameters, i.e., dilatation, maximum shear strain, rotational deformation and strain energy dissipated in the mapping. These parameters being quantities of obvious geometrical and physical meaning, allow a deeper understanding of map distortion and provide additional tools for comparing maps of even the same traditional properties, e.g., conformality, equivalence, aphyllacticity. This could also be of interest when dealing with modern Thematic Cartography. Applications are given in Dermanis et al., 1982.

2. — THE STRAIN TENSOR

An one-to-one correspondence of a plane space with another one is locally represented by the differential mapping

\[
G = \frac{\partial (\beta_1, \beta_2)}{\partial (\alpha_1, \alpha_2)} = \begin{bmatrix}
\frac{\partial \beta_1}{\partial \alpha_1} & \frac{\partial \beta_1}{\partial \alpha_2} \\
\frac{\partial \beta_2}{\partial \alpha_1} & \frac{\partial \beta_2}{\partial \alpha_2}
\end{bmatrix}
\]
where $\alpha = (\alpha_1, \alpha_2)^T$ and $\beta = (\beta_1, \beta_2)^T$ the respective sets of orthogonal cartesian coordinates, the relevant metrics being

$$ds^2_\alpha = d\alpha^T d\alpha = d\beta^T (G^{-1})^T G^{-1} d\beta $$

$$ds^2_\beta = d\beta^T d\beta = d\alpha^T G^T G d\alpha.$$  \hspace{1cm} (2)

It is hereon assumed that $\text{det} (G) \neq 0$ at the points of interest.

In the Lagrangian approach the strain tensor $E_L$ is a function of the objective surface, $E_L(\alpha)$, defined by the relation

$$ds^2_\beta - ds^2_\alpha \overset{\text{def}}{=} 2 d\alpha^T E_L d\alpha$$  \hspace{1cm} (3)

and consequently

$$E_L = \frac{1}{2} (G^T G - I)$$  \hspace{1cm} (4)

where $I$ the unit matrix.

In the Eulerian approach the strain tensor $E_E$ is a function of the surface on which the objective surface is mapped, $E_E(\beta)$, defined by

$$ds^2_\alpha - ds^2_\beta \overset{\text{def}}{=} 2 d\beta^T E_E d\beta$$  \hspace{1cm} (5)

and similarly

$$E_E = \frac{1}{2} [(G^{-1})^T G^{-1} - I].$$  \hspace{1cm} (6)

Dealing with the mapping of a curved space onto another curved space, we can pass to the corresponding local tangent spaces and operate as above, since the strains involved are differential quantities.
3. — CARTOGRAPHIC APPLICATION

Considering the mapping

\[ \text{sphere} (\lambda, \varphi) \rightarrow \text{plane} (x, y) \]

\[ \text{tangent plane} (s_\lambda, s_\varphi) \]

\[ \text{(azimuthal plane)} \]

where \((\lambda, \varphi)\) orthogonal spherical coordinates on a unit sphere, \((x, y)\) cartesian coordinates on the map plane and \((s_\lambda, s_\varphi)\) coordinates on the local tangent plane, i.e., \(s_\lambda\) and \(s_\varphi\) are lengths along the local parallel and meridian respectively. The corresponding differential mappings are

\[ (d\lambda, d\varphi) \rightarrow J \rightarrow (dx, dy) \]

\[ Q^{-1} \rightarrow S = JQ \]

\[ (ds_\lambda, ds_\varphi) \]

where

\[ S = JQ \]

\[ J = \frac{\partial (x, y)}{\partial (\lambda, \varphi)} \]

\[ Q = \frac{\partial (\lambda, \varphi)}{\partial (s_\lambda, s_\varphi)} \]

\[ S = \frac{\partial (x, y)}{\partial (s_\lambda, s_\varphi)} \]
From the well known relations

\[
\begin{align*}
\frac{ds_\phi}{ds_\lambda} &= d\phi \\
\frac{ds_\lambda}{ds_\phi} &= \cos \phi 
\end{align*}
\]

it follows that

\[
Q = \begin{pmatrix}
1 & 0 \\
\frac{1}{\cos \phi} & 0 \\
0 & 1 
\end{pmatrix}
\]

\(Q\) being diagonal due to the orthogonality of the geographical grid on the sphere.

3.1. - **LAGRANGIAN CARTOGRAPHIC STRAIN**

The mapping \((ds_\lambda, ds_\phi) \rightarrow (dx, dy)\) involves the Jacobian \(S\), which introduced into (4) provides the Lagrangian strain of map projections

\[
E_L = \frac{1}{2} (S^T S - I)
\]

or

\[
E_L = \frac{1}{2} (T - I)
\]

where

\[
T = S^T S.
\]

Replacing (7) into (13) we obtain

\[
T = Q^T J^T J Q
\]
or

\[
T = \begin{bmatrix}
\frac{g}{\cos^2 \varphi} & \frac{f}{\cos \varphi} \\
\frac{f}{\cos \varphi} & e
\end{bmatrix}
\]

(15)

\(g, f, e\) being the Gauss forms defined by

\[
\begin{bmatrix}
g & f \\
f & e
\end{bmatrix} \overset{\text{def}}{=} \bar{J}^T J.
\]

(16)

The matrix \(T\) is nothing but the Tissot tensor, from which the Tissot ellipse is computed. Comparing (15) with (12) we observe that the Tissot tensor is not a pure strain tensor but only a part of the Lagrangian strain tensor.

From (15) the maximum and minimum semi-axes of the Tissot ellipse, \(a_T\) and \(b_T\), are computed, as well as the direction \(\psi_T\) of \(a_T\)

\[
\begin{bmatrix}
a_T^2 \\
b_T^2
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
g \cos^2 \varphi + e \\
(g - e \cos^2 \varphi)^2 + (2 f \cos \varphi)^2
\end{bmatrix}
\]

(17)

\[
\psi_T = \frac{1}{2} \arctan \frac{-2 f \cos \varphi}{g - e \cos^2 \varphi}.
\]

By definition

\[
2a_T^2 = \Delta_T + \gamma_T
\]

(18)

\[
2b_T^2 = \Delta_T - \gamma_T
\]

where \(\Delta_T\) and \(\gamma_T\) could be called the pseudo-dilatation and the pseudo-maximum shear strain respectively, or the Tissot dilatation and the Tissot maximum shear strain.
The Lagrangian strain ellipse is computed from (12)

\[ 2a_L = \Delta_L + \gamma_L \]  

\[ 2b_L = \Delta_L - \gamma_L \]  

where

\[ \Delta_L = \frac{1}{2} \left( \frac{g}{\cos^2 \varphi} + e \right) - 1 \]  

\[ \gamma_L = \sqrt{\left( g - e \cos^2 \varphi \right)^2 + \left( 2f \cos \varphi \right)^2} \]  

\[ 2 \cos^2 \varphi \]  

and the direction of \( a_L \)

\[ \psi_L = \psi_T \]  

Comparing the above expressions we obtain the relations

\[ \Delta_L = \frac{1}{2} \Delta_T - 1 \]  

\[ \gamma_L = \frac{1}{2} \gamma_T \]  

Note that the real dilatation \( \Delta_L \) and the maximum shear strain \( \gamma_L \) are systematically smaller than the ones referred to the Tissot tensor.

The Lagrangian strain analysis described above is referred to the sphere. For a similar analysis on the plane, which is of more practical interest the Eulerian approach can be followed.
3.2. - Eulerian Cartographic Strain

The mapping \((ds_x, ds_y) \rightarrow (dx, dy)\) is done through the Jacobian \(S\), which introduced in (6) provides the Eulerian strain tensor

\[
E_e = \frac{1}{2} \left[(S^{-1})^T S^{-1} - I\right]
\]

(23)

where \(S\) is a matrix that represents the deformation of the domain. The matrix \(T\) is defined in (15)

\[
T = \begin{bmatrix}
\frac{1}{2} & g' \\
-f' & e'
\end{bmatrix}
\]

(05)

or

\[
E_e = \frac{1}{2} (H - I)
\]

(24)

where \(H = (S^{-1})^T S^{-1}\) is computed. Comparing (15) with (12) we observe that the Tissot tensor is a pure strain tensor but only a part of the Lagrangian strain tensor.

From (15) the maximum and minimum semi-axes of the Tissot ellipse, and \(b_p\), are computed, as well as the directions \(\psi_T\) of the Tissot ellipse.

or

\[
H = \begin{bmatrix}
g' & f' \\
f' & e'
\end{bmatrix}
\]

(26)

with

\[
g' = \cos^2 \varphi \left( \frac{\partial \lambda}{\partial x} \right)^2 + \left( \frac{\partial \varphi}{\partial x} \right)^2
\]

(18)

\[
f' = \cos^2 \varphi \left( \frac{\partial \lambda}{\partial x} \right) \left( \frac{\partial \lambda}{\partial y} \right) + \frac{\partial \varphi}{\partial x} \frac{\partial \varphi}{\partial y}
\]

\[
e' = \cos^2 \varphi \left( \frac{\partial \lambda}{\partial y} \right)^2 + \left( \frac{\partial \varphi}{\partial y} \right)^2
\]

(27)
The axes of the strain ellipse, in this case, will be

\[ 2a_E = \Delta_E + \gamma_E \]  

(28)

\[ 2b_E = \Delta_E - \gamma_E \]

\[ \Delta_E = \frac{g' + e'}{2} - 1 \]  

(37)

\[ \gamma_E = \frac{1}{2} \sqrt{(g' - e')^2 + 4f'^2} \]  

(29)

and the direction of \( a_E \)

\[ \psi_E = \frac{-2f'}{g' - e'} . \]  

(30)

3.3. - Relative Cartographic Strain

It is of cartographic interest to analyze the strain involved when comparing two different maps. In such a case the mapping will be of the type

plane \((x, y)\) \(\rightarrow\) plane \((x', y')\)

plane carrée \((\lambda, \varphi)\)

\[ G = \begin{pmatrix} 1 + \varepsilon & \xi \varepsilon & \eta \varepsilon \\ 0 & 1 + \eta & -\xi \\ 0 & 0 & 1 \end{pmatrix} \]

\[ R = \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ U = \begin{pmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ \varepsilon = \frac{1}{2} \sqrt{(\xi^2 + \eta^2)} \]

\[ \eta = \frac{1}{2} (g' - e') \]

\[ \xi = \frac{1}{2} (g' + e') \]

\[ r = \frac{1}{2} \sqrt{\frac{g'^2}{e'^2}} \]

\[ u = \frac{1}{2} \sqrt{\frac{e'^2}{g'^2}} \]

\[ \Delta_E = \frac{1}{2} (g' + e') - 1 \]

\[ \gamma_E = \frac{1}{2} \sqrt{(g' - e')^2 + 4f'^2} \]
where the displacements \((u, v)\) are defined by the differences of the corresponding sets \((x, y)\) and \((x', y')\)

\[
\begin{align*}
u &= x' - x \\
v &= y' - y
\end{align*}
\]

The mapping \((d\mathbf{x}, d\mathbf{y}) \rightarrow (d\mathbf{x}', d\mathbf{y}')\) is done through the Jacobian \(S\), which introduced in (3) provides the Eulerian strain tensor

\[
E_{\mathbf{S}} = \frac{1}{2} \left( S^T S - I \right)
\]

and the Jacobian of the displacements is given by

\[
D = J_{\mathbf{R}} - I
\]

where

\[
J_{\mathbf{R}} = J' J^{-1}
\]

The corresponding relative strain tensor is given by the well known relation

\[
E_{\mathbf{R}} = \frac{1}{2} \left( D + D^T + D^T D \right)
\]

which is obviously of Lagrangian type. Replacing (32) into (34), we obtain

\[
E_{\mathbf{R}} = \frac{1}{2} \left( J_{\mathbf{R}}^T J_{\mathbf{R}} - I \right)
\]
from which the strain ellipse is easily computed:

\[ 2a_R = \Delta_R + \gamma_R \]
\[ 2b_R = \Delta_R - \gamma_R \]

(36)

\( \Delta_R \) and \( \gamma_R \) being the relative dilatation and the relative maximum shear strain respectively.

For the case of very small displacements (infinitesimal deformation) the product \( D^TD \) in (34) can be neglected, so that

\[ E_R = \frac{1}{2} (D + D^T) \]

(37)

Combining with (32)

\[ E_R = \frac{1}{2} (J_R + J_R^T) - I \]

(38)

and the corresponding approximate relative strains can be computed.

3.4. - Rotation

When the strain tensor \( E \) is given in the form

\[ E = \frac{1}{2} (G^T G - I) \]

(39)

\( G \) being the Jacobian of the relevant plane to plane mapping, an orthogonal rotation matrix \( R \) can be extracted from \( G \) by means of the unique decomposition

\[ G = R U = VR \]

(40)

where \( U \) and \( V \) are called the right and left stretch tensors respectively. It can be shown that

\[ R = G(G^T G)^{-\frac{1}{2}} = (G^T G)^{-\frac{1}{2}} G \]

(41)
The difficulties in the numerical treatment of (41) can be bypassed in the case of infinitesimal strain, where an infinitesimal rotation is approximately computed from the antisymmetric matrix

\[ \Omega \sim \frac{1}{2} (D - D^T) \]  

(42)

where \( D \) is the Jacobian of the displacements, as in the previous chapter, related to \( G \) by

\[ G = I + D \]  

(43)

For the Lagrangian description of strain, \( D = S - I \), and (42) becomes

\[ \Omega = \frac{1}{2} (S - S^T) \]  

(44)

Denoting

\[ \Omega = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \]  

(45)

and combining with (44), we obtain

\[ \omega = \frac{\partial x}{\partial \varphi} - \frac{1}{\cos \varphi} \frac{\partial y}{\partial \lambda}. \]  

(46)

Replacing (43) into (41), the orthogonal matrix \( R \) is written

\[ R = (I + D) \left( I + D + D^T + D^T D \right)^{-\frac{1}{2}} \]  

(47)

Neglecting \( D^T D \), (47) becomes

\[ R \sim (I + D) \left( I + D + D^T \right)^{-\frac{1}{2}} \]  

(48)

and observing that

\[ (I + D + D^T)^{-\frac{1}{2}} \sim \left( I - \frac{1}{2} D - \frac{1}{2} D^T \right), \]  

(49)
(48) is written

\[ R \sim (I + D) (I - \frac{1}{2} D - \frac{1}{2} D^T) \sim I + \frac{1}{2} (D - D^T) \quad (50) \]

or in view of (42)

\[ R \sim I + \Omega \quad (51) \]

which is the well known decomposition of orthogonal matrices close to the identity.

When the orthogonal matrix \( R \) is rigorously computed from (41) the corresponding angle of rotation \( \theta \) is given by

\[
R = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}
\quad (52)
\]

e.g.,

\[ \theta = \arctan \left( \frac{R_{21}}{R_{11}} \right) \quad (53) \]

3.5. - Strain Energy

Another interesting measure of deformation is the strain energy considered as dissipated when the cartographic mapping is visualized as the process of deforming an objective surface of fictitious material into the map plane. In order to have a local measure, we consider the required energy per unit area at any point.

For the two-dimensional surface the physical dimensions of stresses are \([\text{force/length}]\), or equivalent \([\text{work/area}]\). Stresses \( \Sigma_{ij} \) are related to strains \( E_{ij} \) (see, e.g., Love, 1927), by the relation

\[ \sigma = C \varepsilon \quad (54) \]

where

\[ \sigma = [\Sigma_{11} \Sigma_{22} \Sigma_{12}]^T \quad (55) \]

\[ \varepsilon = [E_{11} E_{22} E_{12}]^T \quad (56) \]
and difficulties in the numerical treatment of (41) can be bypassed with (42) or infinitesimal strain, where an infinitesimal rotation is approximately computed from the antisymmetric matrix

$$ C = \begin{vmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{vmatrix} \tag{57} $$

\( \lambda \) and \( \mu \) are the Lamé constants which characterize the material. It is well known that the corresponding strain energy \( W \) is given by

$$ W = \frac{1}{2} \varepsilon^t \sigma \tag{58} $$

Combining with (54)

$$ W = \frac{1}{2} \varepsilon^t C \varepsilon \tag{59} $$

follows. Replacement of

$$ \Delta = E_{11} + E_{22} \tag{60} $$

$$ \gamma_2 = 2E_{12} $$

in (59), where \( \Delta \) and \( \gamma_2 \) the dilatation and one of the maximum shear strain components respectively, yields

$$ W = \frac{\lambda + 2\mu}{2} \Delta^2 - 2\mu E_{11} E_{22} + \frac{\mu}{2} \gamma_2^2 \tag{61} $$

To express \( W \) in terms of \( \Delta \) and \( \gamma = (\gamma_1^2 + \gamma_2^2)^{1/2} \), where \( \gamma_1 = E_{11} - E_{22} \) the other component of maximum shear strain, we note that

$$ E_{11} E_{22} = \frac{1}{4} (E_{11} - E_{22})^2 - (E_{11} + E_{22})^2 \tag{62} $$

or

$$ E_{11} E_{22} = \frac{1}{4} (\gamma_1^2 - \Delta^2) \tag{63} $$
Thus, (61) becomes

$$W = \frac{\lambda + \mu}{2} \Delta^2 + \frac{\mu}{2} \gamma^2$$  \hspace{1cm} (64)

a function of dilatation and maximum shear strain only.

For comparison purposes, we use a material characterized by the Poisson’s ratio

$$\gamma = \frac{\lambda}{2(\lambda + \mu)} = \frac{1}{4}$$  \hspace{1cm} (65)

This choice, already suggested by Poisson in 1829, does not correspond to any actual material, but leads to

$$\lambda = \mu$$  \hspace{1cm} (66)

Since any choice of the value $\lambda = \mu$ changes the value of $W$ by a scalar factor, we set

$$\lambda = \mu = 1$$  \hspace{1cm} (67)

without loss of generality and (64) becomes

$$W = \Delta^2 + \frac{1}{2} \gamma^2$$  \hspace{1cm} (68)

Replacing into (68) the appropriate values of $\Delta$ and $\gamma$ referred to the Lagrangian, Eulerian and Relative approaches, the respective energies are obtained.
4. — CONCLUDING REMARKS

The introduction of strain criteria, i.e., dilatation, maximum shear strain, rotation and strain energy, in studying map projections is proposed, for a better understanding of deformations implied when mapping a surface into another. This analysis allows a more detailed classification of maps, even within families of maps with the same traditional foundamental properties, i.e., conformality, equivalence, etc.

The main concern is focused on the determination of dilatation and maximum shear strain, since these parameters reflect respectively the isotropic and anitropic characteristics of map deformation. Expressions are given in both Lagrangian and Eulerian terms, in order to study strains either on the sphere or on the plane of representation. An expression for strain energy is derived as a function of dilatation and maximum shear strain, which allows another insight into the map distortion with respect to the corresponding fictitious stress field.

Interesting applications can be carried out, not only in Mathematical Cartography, but also in modern Thematic Cartography and pattern analyses of "themes", when the themes under consideration can be defined as functions of the above scalar strain quantities.

REFERENCES


