Some remarks on the description of earth rotation according to the IAU 2000 resolutions

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Abstract. The current version of the transformation used by IERS for the conversion from the ICRF to the ITRF, involves the Celestial Intermediate Pole (CIP), which is a smoothed version of the instantaneous rotation axis. An instantaneous rotation axis mathematically compatible with the same transformation is introduced as a Compatible Celestial Pole (CCP). Three necessary conditions are derived for a rotation axis, appearing in a transformation having the same analytical form as the one given by IERS, to be the compatible instantaneous rotation axis (CCP). It is shown that the CIP satisfies only one of the three conditions and that the two NRO conditions (for CEO and TEO) are only sufficient conditions for the satisfied condition. In addition the means are provided for calculating the direction of the CCP in the celestial system. Finally an alternative definition to UT1 based on the rate of rotation around the CCP is proposed.

Keywords. Reference systems, earth rotation, precession nutation, polar motion, Universal Time.

1 Introduction

The classical description, in the era of astrogeodetic position determination (Mueller, 1969), involved the two required fundamental reference systems, a celestial one fixed with respect to the stellar background and realized through the (directional) coordinates of stars included in the astronomical catalogues (e.g. the FK5), and a terrestrial one fixed with respect to the practically rigid earth and realized by the astronomic longitude and latitude of the fundamental observatories of the International Polar Motion Service, now replaced by the IERS. The 3rd axis of the terrestrial system was pointing at a selected direction corresponding roughly to the mean position of the direction of rotation as seen from the earth, while its 1st axis was defined by the selection of a particular meridian (plane of 1st and 3rd axis), the Greenwich meridian.

We will specify the direction of the axes of the celestial system after introducing some relevant auxiliary concepts.

In addition to the above two systems, three intermediate reference systems were introduced. Two of them had their common 3rd axis (at least in theory) along the instantaneous axis of rotation and were hence characterized as “true” systems. Their 1st and 2nd axes lied on the true equator, i.e. the plane perpendicular to the instantaneous axis of rotation. The true celestial system was in a certain (though not satisfactory) sense a non-rotating system close to the celestial one, a property which followed from the selection of its 1st axis in the direction of the (true) vernal equinox, i.e. the intersection of the instantaneous equatorial plane with the instantaneous ecliptic plane.

The true terrestrial system was a rotating one, which closely followed the terrestrial system in its rotational motion (thus non-rotating with respect to the earth), a fact which was secured by selecting as its 1st axis the intersection of the instantaneous equator with the plane formed by the instantaneous direction of rotation and the 1st axis of the terrestrial system.

The introduction of the true systems separated the total earth rotation in three parts: (1) precession-nutation corresponding to the variation of the direction of rotation with respect to the inertial stellar background and expressed by the transformation from the celestial to the true celestial system, (2) diurnal rotation corresponding to the variation of the angular velocity of rotation or length-of-the-day
variation and expressed by the transformation from the true celestial to the true terrestrial system and (3) polar motion corresponding to the variation of the direction of rotation with respect to the earth and expressed by the transformation from the true terrestrial to the terrestrial system.

The non-rotating character of the true systems has been a relative one since both the instantaneous vernal equinox (1st axis of the true celestial) was moving with respect to the stellar background due to precession-nutation as well as the variation of the ecliptic plane and the 1st axis of the true celestial system was moving with respect to the earth due to polar motion.

Another “peculiarity” of the classical description of earth rotation was the separation of the unique phenomenon of precession-nutation into two parts, the dominant precession of the direction of rotation tracing the surface of a cone in a period of about 25800 years and the remaining fluctuations (nutation) from the mean direction defined by precession. In correspondence to this separation, still another intermediate system was introduced, the mean celestial system with its 3rd axis in the mean direction of rotation defined by considering precession only and its 1st and 2nd axes on the mean equator (plane perpendicular to the mean direction of rotation). The 1st axis was selected in the direction of the mean vernal equinox, i.e. the intersection of the instantaneous mean equator with the instantaneous ecliptic. Thus precession was expressed by the transformation from the celestial system to the mean celestial one, while nutation was expressed by the transformation from the mean celestial system to the true celestial one. Furthermore the celestial frame itself was simply selected to be identical with the mean celestial system of a particular fixed reference epoch.

A characteristic of the traditional description, which also survived in the modern description in a slightly different context, is the discrepancy between theory and observation, which is of course connected to the accuracy of the observations at the particular stage of technological development.

In the first period precession and nutation was determined from the theoretical solution of the differential equations of the rigid earth using the known applied lunar solar and planetary torques computed from the positions of the attracting celestial bodies provided by ephemerides constructed from observations. The low accuracy of observations did not allow the detection of discrepancies between theory and observation, while polar motion remained undetected. The advances in the accuracy of astrogeodetic observations revealed polar motion with a behavior too erratic to be predicted by theoretical means. Thus the problem of earth rotation determination became a mixed theoretical – observational problem. Modern space techniques and in particular Very Long Baseline Interferometry (VLBI) led to very high accuracies at the order of 1 mas (milli-arc-second) corresponding to a centimeter level on the earth surface. Such observations are capable of determining earth rotation a posteriori independently of any theory. Nevertheless theoretical work made advances in solving the problem for a deformable earth that allow the prediction of precession-nutation at an internal accuracy comparable to that of observations. However discrepancies between predicted and deduced from observations are still present and they may be due not only in shortcomings in the theory but also in differences in the definition of the terrestrial reference system.

Theory incorporates a terrestrial system with Tisserand axes (Munk & MacDonald, 1960, Moritz & Mueller, 1987), i.e. such that the motion of the earth masses with respect to the terrestrial system does not contribute to the angular momentum of the earth (vanishing relative angular momentum). Such a definition incorporates all earth masses in contrast to the definition of the operational terrestrial system, realized by the International Terrestrial Reference Frame (ITRF), which implicitly incorporates a discrete Tisserand principle applying to a discrete global network of points considered as unit mass points (Dermanis 2003. See also Boucher et al., 1999, Altamini et al., 2002, Sillard & Boucher, 2001, Dermanis, 1995, 2000, 2001). The discrepancy between these differently defined systems can only be estimated as shown by Dermanis (2002). The ITRF provides coordinates of the deformable network points at any epoch through their coordinates at a reference epoch and their velocities by assuming a linear variation of coordinates with respect to time.

On the other hand the definition of the celestial reference system, realized by the International Celestial Reference Frame (ICRF) has considerably improved, since it is based on the (directional) coordinates of stable extragalactic radio sources observed by VLBI, which do not show any proper motion as did the stars, previously used for the realization of the celestial frame by their coordinates in astronomical catalogues.

The situation in the new era of space techniques may be summarized as follows: Theory, based on very eloquent solutions of the equations of motion of a deformable earth (see Dehant et. al, 1999, for relevant references), provides accurate results for the precession-nutation part, which augmented by observational data for polar motion provides a “theoretical” version of the ICRF to ITRF transformation. Observations are capable of providing an “operational” version of the same transformation,
which is usually presented as an additional transformation to be applied to the theoretical one. Theoretical work aims at explaining these discrepancies by refining the solution and “tuning” to more appropriate values of parameters related to the deformational behavior of the earth.

An essential discrepancy between theory and observations lies in their different spectral aspects. Solutions to the precession-nutation problem can be spectrally refined at will by simply adding more (higher frequency) terms in the incorporated Fourier series expansions. On the other hand the spectral resolution of observations are limited by the time period required for co-observing VLBI stations to collect sufficient data for their efficient correlation, which will result in observations (time differences in arrival of the signal from the radio source at two stations) meeting the prescribed centimeter level accuracy requirements. Any detail below the Nyquist frequency corresponding to this required time interval are lost by aliasing.

A fundamental difference between theory and observation is the use of the rotation vector components as unknown functions in the theoretical de-
scription of earth rotation. These parameters have the character of “derivatives”, which can not be observed as they rather belong to the category of “mathematical fiction”. Observations can only provide time-discrete values of Eulerian angles (or other parameters) defining the relative orientation of the terrestrial system with respect to the celestial, from which the rotation vector (more precisely the instantaneous vector of angular velocity) can be estimated by interpolation in time. This fundamental aspect is not altered by the fact that the interpolation may not be explicit (a posteriori applied) but rather implicit in the models, used for the Eulerian angles as functions of time within the analysis of the observations.

These problems have been identified a long time ago, as a “non-observability” problem. In simple words, it does not make sense to include in a theoretical solution, even if available, high frequency terms not detectable by the spectral resolution of the observations. A decisive step in the new IAU 2000 resolutions, was the replacement of the instantaneous rotation axis (or an estimate of it) by a “smoothed” version called the Celestial Intermediate Pole (CIP). The CIP is defined by removing from the theoretically provided precession-nutation the terms with frequencies higher than half cycle per day (period lower that 2 days). In order to maintain the same total transformation, balancing additional terms have to be added to the observed (IERS-provided) polar motion, which refers to the instantaneous rotation axis. It is not clear whether the cutoff frequency in the CIP definition refers to the present spectral resolution capabilities of VLBI observations, or simply meets the spectral requirements of current astronomical work. For the record it must be clarified that the CIP has not replaced the (estimated) rotation axis but the somewhat different Celestial Ephemeris Pole (CEP).

In accordance with the introduction of the CIP the traditional “true” systems have been replaced by two “intermediate” systems having their common 3rd axis in the direction of the CIP. A considerable concepnitional theoretical advance refers to the choice of the 1st axes of the intermediate systems. The Celestial Ephemeris Origin (CEO), being the 1st axis of the intermediate celestial system, and the Terrestrial Ephemeris Origin (TEO), being the 1st axis of the intermediate terrestrial system, are optimally defined according to the Non Rotating Origin (NRO) principle, which asserts that nor the intermediate celestial system neither the intermediate terrestrial system rotate with respect to the celestial and the terrestrial system, respectively. The NRO principle will be explained in a following section.

Another improvement is the abandonment of the mean celestial system and hence of the unjustified obsolete separation of nutation from precession.

Despite the use of the CIP as a means of separating precession-nutation from (accordingly modified) diurnal rotation and polar motion the three parts add up to a total transformation (possibly with the further addition of observational corrections) from the celestial to the terrestrial system. To this transformation a mathematically defined axis of rotation can be derived which we might call the Compatible Celestial Pole (CCP) to emphasize the fact that it is an “instantaneous” pole compatible with the earth rotation transformation provided by the IAU 2000 resolutions as implemented by the IERS.

The counterpoint between the CIP and the CCP concepts is the subject of this work. We will derive the three necessary conditions for the rotation axis, defined by a transformation having the same analytical form as the IERS earth rotation (ICRF to ITRF) transformation, to be the compatible instantaneous rotation axis (CCP). We will show that the CIP satisfies only one of the three conditions and we will show that the two NRO conditions (for CEO and TEO) are only sufficient conditions for the satisfied condition, thus offering an independent alternative justification for their adoption. In addition we will provide the means for calculating the direction of the CCP with respect to the celestial system. Finally an alternative definition to UT1 based on the CCP concept will be proposed.

We close this introduction with some clarifications concerning the notation used.

We represent the orthonormal triad of a reference system by a row-matrix $\mathbf{e} = [\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3]$, and vector components by column-matrices $\mathbf{v} = [v^1 v^2 v^3]^T$, so that any vector can be written in the form $\mathbf{v} = \mathbf{e}_1 v^1 + \mathbf{e}_2 v^2 + \mathbf{e}_3 v^3 = \mathbf{e} \mathbf{v}$.

For the exterior product of two vectors $\mathbf{u} = \mathbf{e} \mathbf{w} = \mathbf{u} \times \mathbf{v} = (\mathbf{e} \mathbf{u}) \times (\mathbf{e} \mathbf{v})$ we have the component representation $\mathbf{w} = [\mathbf{u} \times] \mathbf{v}$ with

$$[\mathbf{u} \times] = \begin{bmatrix} 0 & -u^3 & u^2 \\ u^3 & 0 & -u^1 \\ -u^2 & u^1 & 0 \end{bmatrix}$$

(1)

If $\mathbf{e}^0 = [\mathbf{e}_1^0 \mathbf{e}_2^0 \mathbf{e}_3^0]$ and $\mathbf{e} = [\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3]$ are two different reference systems they are connected by $\mathbf{e} = \mathbf{e}^0 \mathbf{R}^T$, where $\mathbf{R}$ is an orthogonal rotation matrix. The components in the two systems of any vector $\mathbf{v} = \mathbf{e}^0 \mathbf{v}^0 = \mathbf{e} \mathbf{v}$ are connected by $\mathbf{v} = \mathbf{R} \mathbf{v}^0$. The rotation vector (instantaneous angular velocity
vector) of \( \mathbf{e} \) with respect to \( \mathbf{e}^0 \) has components
with components \( \omega = \mathbf{R} \omega_0 \), which can be determined by

\[
[\omega \times] = \mathbf{R} \frac{d\mathbf{R}^T}{dt} + \frac{d\mathbf{R}}{dt} \mathbf{R}^T,
\]

(2)

\[
[\omega_0 \times] = \frac{d\mathbf{R}^T}{dt} \mathbf{R} - \mathbf{R}^T \frac{d\mathbf{R}}{dt}.
\]

(3)

We use a dot to denote differentiation with respect to time: \( \dot{f} = df/dt \) for any function \( f(t) \).

2 General representation of earth rotation

Referring to the earth rotation we need two reference systems, both geocentric: The celestial reference system \( \mathbf{e}^C \), which is quasi-inertial having the direction of its axes fixed with respect to extra-galactic radio sources and the terrestrial frame \( \mathbf{e}^T = \mathbf{e}^C \mathbf{R}^T \), defined so that it best represents the deformable earth as a whole. The earth rotation vector \( \omega = \mathbf{e}^T \mathbf{\omega} \), follows from the earth rotation matrix \( \mathbf{R} \) according to (2) \( [\omega \times] = \mathbf{R} \frac{d\mathbf{R}^T}{dt} \).

Earth rotation is traditionally separated into precession-nutation \( \mathbf{Q} \) (variation of the direction of \( \omega \) with respect to inertial space), diurnal rotation \( \mathbf{D} \) (rotation around \( \omega \)) and polar motion \( \mathbf{W} \) (variation of the direction of \( \omega \) with respect to the earth), according to

\[
\mathbf{R} = \mathbf{WDQ}.
\]

(4)

This calls for the introduction of two intermediate reference systems with third axis in the direction of \( \dot{\omega} \). The intermediate celestial system \( \mathbf{e}^{IC} \) with \( \mathbf{e}_3^{IC} = |\dot{\omega}|^{-1} \dot{\omega} \), which is “close” to \( \mathbf{e}^C \), and the intermediate terrestrial frame \( \mathbf{e}^{IT} \) with \( \mathbf{e}_3^{IT} = \omega \), “close” to \( \mathbf{e}^T \). Thus we have 4 systems connected by

\[
\mathbf{e}^{IC} = \mathbf{e}^C \mathbf{Q}^T,
\]

(5)

\[
\mathbf{e}^{IT} = \mathbf{e}^{IC} \mathbf{D}^T = \mathbf{e}^C \mathbf{Q}^T \mathbf{D}^T
\]

(6)

\[
\mathbf{e}^T = \mathbf{e}^{IT} \mathbf{W}^T = \mathbf{e}^{IC} \mathbf{D}^T \mathbf{W}^T = \mathbf{e}^C \mathbf{Q}^T \mathbf{D}^T \mathbf{W}^T = \mathbf{e}^C \mathbf{R}^T
\]

(7)

Three new “relative” rotation vectors may be introduced: the rotation vector of \( \mathbf{e}^{IC} \) with respect to \( \mathbf{e}^C \)

\[
\dot{\omega}_Q = \mathbf{e}^{IC} \mathbf{\omega}_Q = \mathbf{e}^{IC} (\mathbf{\omega}_Q)_{IC},
\]

(8)

with components \( (\mathbf{\omega}_Q)_{IC} = \omega_0 Q \) defined by

\[
[\omega_0 Q] = \frac{d\mathbf{Q}^T}{dt},
\]

(9)

the rotation vector of \( \mathbf{e}^{IT} \) with respect to \( \mathbf{e}^{IC} \)

\[
\dot{\omega}_D = \mathbf{e}^{IT} \mathbf{\omega}_D = \mathbf{e}^{IT} (\mathbf{\omega}_D)_{IT},
\]

(10)

with components \( (\mathbf{\omega}_D)_{IT} = \omega_D \) defined by

\[
[\omega_D] = \frac{d\mathbf{D}^T}{dt},
\]

(11)

and the rotation vector of \( \mathbf{e}^T \) with respect to \( \mathbf{e}^{IT} \)

\[
\dot{\omega}_W = \mathbf{e}^T \mathbf{\omega}_W = \mathbf{e}^T (\mathbf{\omega}_W)_{T},
\]

(12)

with components \( (\mathbf{\omega}_W)_{T} = \omega_W \) defined by

\[
[\omega_W] = \mathbf{W} \frac{d\mathbf{W}^T}{dt}.
\]

(13)

Since

\[
[\omega \times] = \mathbf{R} \frac{d\mathbf{R}^T}{dt} = \mathbf{WDQ} \frac{d\mathbf{Q}^T}{dt} (\mathbf{D}^T \mathbf{W}^T) =
\]

\[
= \mathbf{WDQ} \frac{d\mathbf{Q}^T}{dt} \mathbf{D}^T \mathbf{W}^T + \mathbf{Q}^T \frac{d\mathbf{D}^T}{dt} \mathbf{W}^T + \mathbf{Q}^T \mathbf{D} \frac{d\mathbf{W}^T}{dt} =
\]

\[
= \mathbf{WDQ} \frac{d\mathbf{Q}^T}{dt} \mathbf{D}^T \mathbf{W}^T + \mathbf{WD} \frac{d\mathbf{D}^T}{dt} \mathbf{W}^T + \mathbf{W} \frac{d\mathbf{W}^T}{dt} =
\]

\[
= \mathbf{WD}[\omega_N] \mathbf{D}^T \mathbf{W}^T + \mathbf{W}[\omega_D] \mathbf{W}^T + [\omega_W] \mathbf{W}^T
\]

(14)

the rotation vector components are related through

\[
\mathbf{\omega} = \mathbf{W} \omega_Q + \mathbf{\omega}_D + \mathbf{\omega}_W.
\]

(15)

Since \( \dot{\omega} = \mathbf{e}^T \mathbf{\omega} = \mathbf{e}^T \mathbf{W} \mathbf{\omega}_Q + \mathbf{e}^T \mathbf{W} \mathbf{\omega}_D + \mathbf{e}^T \mathbf{W} \mathbf{\omega}_W =
\]

\[
= \mathbf{e}^{IC} \mathbf{\omega}_Q + \mathbf{e}^{IT} \mathbf{\omega}_D + \mathbf{e}^T \mathbf{\omega}_W, \text{ we have the decomposition of the rotation vector}
\]
\[ \ddot{\omega} = \ddot{\omega}_Q + \ddot{\omega}_D + \ddot{\omega}_W \]  

(16)

into a precession-nutation, a diurnal and a polar motion part.

We will also need the components of \( \dot{\omega} = e^T \omega = e^T W^T \omega = e^T \omega_{IT} \) in \( e^T \), which are

\[ \omega_{IT} = W^T \omega = D \omega_Q + \omega_D + W^T \omega_W = \\
= (\omega_Q)_{IT} + (\omega_D)_{IT} + (\omega_W)_{IT}. \]  

(17)

3 The IERS representation of earth rotation

We will examine next the specific parametrization of the matrices \( Q \), \( D \) and \( W \) used by the IERS conventions. These are

\[ Q = R_3(-s)R_2(-E)R_2(d)R_3(E) = \\
= R_3(-S)R_2(d)R_3(E), \]  

(18)

\[ D = R_3(\theta), \]  

(19)

\[ W = R_3(-F)R_2(-g)R_3(F)R_3(s') = \\
= R_3(-F)R_2(-g)R_3(S'). \]  

(20)

\[ R = R_3(-F)R_2(-g)R_3(F)R_3(s' + \theta - s)R_2(-E)R_2(d)R_3(E) = \\
= R_3(-F)R_2(-g)R_3(S' + \theta - S)R_2(d)R_3(E) \]  

(21)

The superfluous elementary rotations \( R_3(-E) \) in \( Q \) and \( R_3(F) \) in \( W \) have been included in order to bring the axes of \( e^IC \) close to those of \( e^C \) and the axes of \( e^IT \) close to those of \( e^T \), respectively. This is necessary, because, due to the fact that the axes \( \tilde{e}_1^C \), \( \tilde{e}_1^IC \) and \( \tilde{e}_1^T \) are close to each other, the angles \( d \) and \( g \) are small, while the angles \( E \) and \( F \) may take any value. We have also introduced, for the sake of convenience the following auxiliary angles

\[ S = s + E, \]  

(22)

\[ S' = s' + F. \]  

(23)

The above representations involve 7 parameters (functions of time) \( d \), \( E \), \( s \) (or \( S \)), \( \theta \), \( g \), \( F \), \( s' \) (or \( S' \)) for the description of the rotation matrix \( R \) instead of the minimal required 3 parameters. Therefore there must exist certain compatibility conditions, at first sight 7–3=4 of them. Nevertheless the true number of conditions is 3 since the angles \( s \) and \( s' \) cannot be separated in the sum \( s + \theta - s' \) appearing in the representation (21) of the rotation matrix \( R \). Any choice of \( s \) and \( s' \) is indistinguishable from the choice \( \tilde{s} = s + \Delta s \) and \( \tilde{s}' = s' - \Delta s \), for any \( \Delta s \), since \( \tilde{s} + \theta - \tilde{s}' = s + \theta - s' \) and they both give the same rotation matrix \( R \).

The IAU-IERS representation reduces the number of independent parameters to 5 (\( d \), \( E \), \( \theta \), \( g \), \( F \)) through relations of the form \( s = s(E,d) \), \( s' = s'(F,g) \) derived on the basis of two conditions

\[ (\omega_Q)_{IC} = I_3^1 \omega_Q = 0 \]  

(24)

and

\[ (\omega_W)_{IT} = I_3^1 (\omega_W)_{IT} = I_3^1 W^T \omega_W = 0. \]  

(25)

This means that the relative rotation vectors \( \ddot{\omega}_Q \) of \( \tilde{e}^IC \) with respect to \( \tilde{e}^C \) and \( \ddot{\omega}_W \) of \( \tilde{e}^IT \) with respect to \( \tilde{e}^T \) have no component along the axis of diurnal rotation \( \tilde{e}_3^IC = \tilde{e}_3^IT \). Thus \( \tilde{e}^IC \) does not rotate with respect to \( \tilde{e}^C \) nor does \( \tilde{e}^IT \) with respect to \( \tilde{e}^T \). In this way the positions of the first axes \( e_1^IC \) and \( \tilde{e}_1^IC \) are determined according to the principle of “Non Rotating Origin” (NRO). \( e_1^IC \) is the “Celestial Ephemeris Origin” (CEO) and \( \tilde{e}_1^IT \) the “Terrestrial Ephemeris Origin” (TEO).

4 Conditions for the representation of the earth rotation by diurnal rotation

Ignoring the 2 NRO conditions for the time being we will investigate the conditions under which the 7 angle representation (21) of the rotation matrix \( R \), contains a diurnal rotation \( D = R_3(\theta) \) which represents a rotation around the direction of the instantaneous rotation axis \( \dot{\omega} \) with an angular velocity equal to the magnitude \( \omega = |\dot{\omega}|. \) This requirement can be expressed by the 3 conditions

\[ \omega_{IT} = \omega_D = \dot{\theta} I_3 = [0 \ 0 \ \dot{\theta}]^T. \]  

(26)

We emphasize that \( \ddot{\omega} \) is not the true instantaneous rotation vector (whatever this means) but merely the
vector \( \vec{\omega} = \vec{e}^T \vec{w} \) compatible with the given rotation matrix \( \mathbf{R} \), via eq. (2) \( [\vec{\omega} \times] = \mathbf{R} (d\mathbf{R} / dt)^T \).

In order to compute the components \( \vec{\omega}_T \), we will first compute the components \( \vec{\omega}_Q \), \( \vec{\omega}_D \), \( \vec{\omega}_W \) of the respective vectors \( \vec{\omega}_Q \), \( \vec{\omega}_D \), \( \vec{\omega}_W \), which constitute the rotation vector \( \vec{\omega} \). To this aim we shall utilize the properties

\[
\frac{\partial}{\partial \alpha} \mathbf{R}_k(\alpha) = -[i_k \times] \mathbf{R}_k(\alpha) = -\mathbf{R}_k(\alpha)[i_k \times]
\]

where \( i_k \) is the kth row of the \( 3 \times 3 \) identity matrix \( \mathbf{I} = [i_1, i_2, i_3] \) and for any orthogonal matrix \( \mathbf{M} \)

\[
[(\mathbf{Ma}) \times] = \mathbf{M}[(\mathbf{a} \times) \mathbf{M}^T].
\]

The precession-nutation part \( \vec{\omega}_Q = \vec{e}^T \vec{w}_Q \) is computed from (9)

\[
[\vec{\omega}_Q \times] = \mathbf{Q} \frac{d\mathbf{Q}^T}{dt} = \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-E)\mathbf{R}_2(\alpha)\mathbf{R}_3(S) \} = 0
\]

\[
\cdot \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \}
\]

\[
\cdot \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \}
\]

\[
\cdot \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \}
\]

\[
\mathbf{Q} = \mathbf{Q}_3(-S)\mathbf{R}_3(E) = \mathbf{Q}_3(-S)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-S)\mathbf{R}_3(E) \}
\]

The diurnal rotation part \( \vec{\omega}_D = \vec{e}^T \vec{w}_D \) is computed from (11)

\[
[\vec{\omega}_D \times] = \mathbf{D} \frac{d\mathbf{D}^T}{dt} = \mathbf{R}_3(\theta) \frac{d}{dt} \{ \mathbf{R}_3(-\theta) \} = \mathbf{R}_3(\theta)\mathbf{R}_3(-\theta) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-\theta) \}
\]

\[
\mathbf{D} = \mathbf{Q} \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \}
\]

The polar motion part \( \vec{\omega}_W = \vec{e}^T \vec{w}_W \) is computed from (13)

\[
[\vec{\omega}_W \times] = \mathbf{W} \frac{d\mathbf{W}^T}{dt} = \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \}
\]

The components of \( \vec{\omega}_T = \vec{e}^T \vec{w}_T \) in the intermediate terrestrial frame are

\[
(\vec{\omega}_T) = \mathbf{W}^T \mathbf{w}_T = \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \}
\]

\[
\cdot \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \}
\]

\[
\cdot \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \}
\]

\[
\cdot \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \}
\]

\[
\cdot \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \}
\]

\[
\cdot \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \}
\]

The components of \( \vec{\omega}_T = \vec{e}^T \vec{w}_T \) in the intermediate terrestrial frame can now be computed by applying (17)

\[
\vec{\omega}_T = \mathbf{D} \vec{\omega}_Q + \vec{\omega}_D + \mathbf{W}^T \vec{\omega}_W = \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \}
\]

\[
\cdot \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \}
\]

\[
\cdot \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \}
\]

\[
\cdot \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \}
\]

\[
\cdot \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \}
\]

\[
\cdot \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \}
\]

\[
\cdot \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \}
\]

\[
\cdot \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \}
\]

\[
\cdot \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \cdot \frac{d}{dt} \{ \mathbf{R}_3(-S)\mathbf{R}_2(\alpha)\mathbf{R}_3(E) \}
\]
\[ = \dot{E} \mathbf{R}_3(\theta - S) \mathbf{R}_2(d) \mathbf{i}_3 + \dot{d} \mathbf{R}_3(\theta - S) \mathbf{i}_2 + \\
+ (\dot{\theta} + \dot{S}' - \dot{S}) \mathbf{i}_3 - \\
- \dot{\mathbf{g}} \mathbf{R}_3(-S') \mathbf{i}_2 - \mathbf{F} \mathbf{R}_3(-S') \mathbf{R}_2(\mathbf{g}) \mathbf{i}_3 \] (36)

Carrying out the necessary matrix computations in the above relation we finally obtain

\[
\omega_{IT}^1 = - \cos(\theta - S) \sin d \dot{E} + \sin(\theta - S) \dot{d} + \\
+ \sin S' \dot{\mathbf{g}} + \cos S' \sin g \dot{F} \]
(37)

\[
\omega_{IT}^2 = \sin(\theta - S) \sin d \dot{E} + \cos(\theta - S) \dot{d} - \\
- \cos S' \dot{\mathbf{g}} + \sin S' \sin g \dot{F} \]
(38)

\[
\omega_{IT}^3 = \cos d \dot{E} + \dot{\theta} + \dot{S}' - \dot{S} - \cos g \dot{F} \]
(39)

and the 3 conditions

\[
\omega_{IT} = \omega_D = \dot{\theta} \mathbf{i}_3 = [0 \ 0 \ \dot{\theta}]^T \]
(40)

become

\[
\begin{align*}
\cos(\theta - S) & \sin d \dot{E} - \sin(\theta - S) \dot{d} - \\
- \sin S' \dot{g} & - \cos S' \sin g \dot{F} = 0 
\end{align*} \]
(41)

\[
\begin{align*}
\sin(\theta - S) & \sin d \dot{E} + \cos(\theta - S) \dot{d} - \\
- \cos S' \dot{g} & + \sin S' \sin g \dot{F} = 0 
\end{align*} \]
(42)

\[
\cos d \dot{E} + \dot{S}' - \dot{S} - \cos g \dot{F} = 0 \]
(43)

The first two conditions (41) and (42) may be characterized as “direction conditions” since they guarantee the alignment of the rotation vector with the common 3rd axis of the intermediate celestial and intermediate terrestrial systems (axis of diurnal rotation), \( \hat{e}_3^{IC} = \hat{e}_3^{IT} = [\hat{\omega}]^{-1} \hat{\omega} \). Provided that the direction conditions hold, the third condition (43) may be characterized as the “magnitude or universal time condition” since it guarantees that the angular velocity of diurnal rotation is equal to that of earth rotation, \( \dot{\omega} = [\hat{\omega}] \) and thus it can serve as the rate of universal time (UT1).

5 Implementation of the NRO conditions

In order to implement the two NRO conditions (24) and (25) we need we need the components \((\omega_Q)_{IC}\) from (30) and \((\omega)_{IT}\) from (35). Carrying out the necessary calculations we arrive at

\[
(\omega_Q)_{IC} = \\
\begin{bmatrix}
- \cos S \sin d \dot{E} - \sin S \dot{d} \\
- \sin S \sin d \dot{E} + \cos S \dot{d} \\
\cos d \dot{E} - \dot{S}
\end{bmatrix} \]
(44)

and

\[
(\omega_W)_{IT} = \mathbf{R}_3(-S') \\
\begin{bmatrix}
\sin g \dot{F} \\
- \dot{g} \\
\dot{S}' - \cos g \dot{F}
\end{bmatrix} \]
(45)

Consequently

\[
(\omega_Q)_{IC}^3 = \cos d \dot{E} - \dot{S} \]
(46)

\[
(\omega_W)_{IT}^3 = \dot{S}' - \cos g \dot{F} \]
(47)

and the NRO conditions \((\omega_Q)_{IC}^3 = 0\), \((\omega_W)_{IT}^3 = 0\) give the equation

\[
\dot{S} = \cos d \dot{E} \]
(48)

with solution of the form \( S = S(d, E) \) defining the position of the CEO \((= \hat{e}_1^{IC})\) and the equation

\[
\dot{S}' = \cos g \dot{F} \]
(49)

with solution of the form \( S' = S'(g, F) \) defining the position of the TEO \((= \hat{e}_1^{IT})\). In terms of the original IERS angles \( s = S - E \) and \( s' = S' - F \) we have instead

\[
\dot{s} = (\cos d - 1) \dot{E} \]
(50)

\[
\dot{s}' = (\cos g - 1) \dot{F} \]
(51)

with solutions of the forms \( s = s(d, E) \) and \( s' = s'(g, F) \), defining the CEO and TEO, respectively. In the IERS conventions (McCarthy, 2003) these conditions are given in alternative forms, where \( d, E \) are replaced by \( X, Y \) and \( g, F \) by \( x_p, y_p \). The alternative parameters are simply the 1st and 2nd components of the unit vector, in the direction of the “diurnal rotation axis” \( \hat{e}_3^{IC} = \hat{e}_3^{IT} \), with respect to the celestial and terrestrial system, respectively, defined by \( \hat{e}_3^{IC} = \hat{e}^C [X \ Y \ Z]^T \) and \( \hat{e}_3^{IT} = \hat{e}^T [x_p \ y_p \ z_p]^T \).
The two NRO conditions can be immediately identified as sufficient conditions for the “magnitude condition” (43) to hold. On the other hand, the direction conditions (41), (42) do not hold. This implies that

the axis of the IERS diurnal rotation is not in the same direction as the rotation axis implied by the transformation from the celestial to the terrestrial system provided by IERS.

This discrepancy is due to the IAU decision to remove high frequency terms from precession-nutation and thus use the Celestial Intermediate Pole (CIP) instead of the Compatible Celestial Pole (CCP). We may express this fact as

\[
\text{CIP} = \tilde{e}_3^{IC} = \tilde{e}_3^{IT} \neq \frac{1}{\omega} \tilde{\omega} = \text{CCP}.
\] (52)

The violation of the direction conditions does not allow the magnitude condition, although it holds, to provide a rate of diurnal rotation equal to the magnitude of the rotation vector implied by the celestial-to-terrestrial system transformation, as provided by IERS. We may express this fact as

\[
\theta \neq \omega = | \tilde{\omega} |.
\] (53)

UT1 is linked by IERS to the stellar angle \( \theta \) by a linear relation of the form \( \text{UT1} = a \theta + b \), with constant \( a \) and \( b \) (McCarthy, 2003). Thus it is not the “primary” Universal Time compatible with the adopted description of earth rotation, which should rather be defined by a relation of the form

\[
\text{UT1} = a \int \omega(t) dt + b.
\] (54)

The adopted smoothing of the CIP results in a consequent smoothing of the resulting UT1.

We must clarify that the fact that the NRO conditions are only sufficient and not necessary conditions for the magnitude condition, does not imply that the NRO conditions are weaker than the magnitude condition derived here. On the contrary they are stronger conditions because they do not only relate to the “correct” rate of diurnal rotation but they secure the non-rotating character of both the intermediate celestial and intermediate terrestrial system. This means that the total rotation around the CIP by an angle \( \Psi = S' + \theta - S \) is properly divided, so that precession-nutation, diurnal rotation and polar motion are properly defined parts of the total earth rotation. On the contrary the magnitude condition secures only the proper separation of \( \Psi \) into the parts \( S' - S \) and \( \theta \), while leaving undefined the separation of \( S' - S \) into its proper parts \( S' \) and \( S \), which must go into polar motion and precession-nutation, respectively.

6 Computation of the direction of the CCP and the rate of rotation

The role of space geodesy is to provide independent from theory estimates of earth rotation parameters, based solely upon observations, which will serve for comparison with theoretical results. Such an analysis will provide an ICRF-to-ITRF transformation matrix \( \mathbf{R}_{\text{obs}} \) and a compatible rotation vector \( \tilde{\omega}_{\text{obs}} = \tilde{e}_3^T \mathbf{e}_{\text{obs}} \) through \[ \mathbf{e}_{\text{obs}} = \mathbf{R}_{\text{obs}} \tilde{\mathbf{e}}^I_{\text{obs}} \]. This vector will give a corresponding rate of rotation \( \tilde{\omega}_{\text{obs}} = | \tilde{\omega}_{\text{obs}} | \) and a direction of rotation \( \mathbf{e}_{\text{obs}} \tilde{\omega}_{\text{obs}} \). These should not be compared with the direction of the CIP and the diurnal rotation rate \( \dot{\theta} \), provided by IERS, but rather with the direction of the IERS-based CCP and rate \( \omega \). For this reason we shall give next the necessary relations for computing the latter from IERS-provided rotation parameters.

We may rewrite eq. (36) in the form

\[
\mathbf{e}_{\text{IT}} = \mathbf{R}_3(\theta - S)\{ \dot{E} \mathbf{R}_2(d) \mathbf{i}_3 + d \mathbf{i}_2 - \dot{S} \mathbf{i}_3 \} + \dot{\theta} \mathbf{i}_3 + \\
+ \mathbf{R}_3(-S')\{-\dot{g} \mathbf{i}_2 - \dot{F} \mathbf{R}_2(g) \mathbf{i}_3 + \dot{S'} \mathbf{i}_3 \} = \\
= \mathbf{R}_3(\theta - S)a_O + \dot{\theta} \mathbf{i}_3 + \mathbf{R}_3(-S')a_W
\] (55)

where we have set

\[
\mathbf{a}_O = \dot{E} \mathbf{R}_2(d) \mathbf{i}_3 + d \mathbf{i}_2 - \dot{S} \mathbf{i}_3 = \begin{bmatrix} -\sin dE \\ d \\ \cos dE - \dot{S} \end{bmatrix}
\] (56)

\[
\mathbf{a}_W = -\dot{g} \mathbf{i}_2 - \dot{F} \mathbf{R}_2(g) \mathbf{i}_3 + \dot{S'} \mathbf{i}_3 = \begin{bmatrix} \sin \dot{g} \dot{F} \\ -\dot{g} \\ \dot{S'} - \cos \dot{g} \dot{F} \end{bmatrix}
\] (57)

or after the implementation of the NRO conditions

\[
\mathbf{a}_O = \begin{bmatrix} -\sin dE \\ d \\ 0 \end{bmatrix}, \quad \mathbf{a}_W = \begin{bmatrix} \sin \dot{g} \dot{F} \\ -\dot{g} \\ 0 \end{bmatrix}.
\] (58)

The components of the unit vector in the CCP direction in the celestial frame are given by
\[
\begin{bmatrix}
X_{CCP} \\
Y_{CCP} \\
Z_{CCP}
\end{bmatrix} = \frac{1}{\omega} \mathbf{C}
\]

where
\[
\mathbf{C} = \mathbf{Q}^T \mathbf{D}^T \mathbf{C}
\]

The matrix \( \mathbf{R}_3(-E) \mathbf{R}_2(-d) \mathbf{R}_3(E) \) is given by (McCarthy, 2003)
\[
\mathbf{R}_3(-E) \mathbf{R}_2(-d) \mathbf{R}_3(E) =
\begin{bmatrix}
1 - aX^2 & -aXY & X \\
-aXY & 1 - aY^2 & Y \\
-X & -Y & 1 - a(X^2 + Y^2)
\end{bmatrix}
\]

where
\[
a = \frac{1}{1 + \cos d} = \frac{1}{1 + \sqrt{1 - (X^2 + Y^2)^2}}.
\]

Conversion to the IERS angles
\[
\psi = -s + \theta - s' \\
x_p = \cos F \sin g \\
y_p = \sin F \sin g
\]

The last form (61) is the most appropriate for calculation using the IERS provided parameters
\[
X_{CCP} = \frac{1}{\omega} \left[ (1 - aX^2)(b_{1Q}^1 + \cos \psi b_{1w}^1 - \sin \psi b_{1w}^2) - aXY(b_{1Q}^2 + \sin \psi b_{1w}^1 + \cos \psi b_{1w}^2) + X\dot{\theta} \right] \\
Y_{CCP} = \frac{1}{\omega} \left[ -aXY(b_{1Q}^1 + \cos \psi b_{1w}^1 - \sin \psi b_{1w}^2) + -(a - Y^2)(b_{1Q}^2 + \sin \psi b_{1w}^1 + \cos \psi b_{1w}^2) + Y\dot{\theta} \right].
\]

It remains only to give the appropriate expressions for \( b_{1Q}^1, b_{1Q}^2, b_{1w}^1, b_{1w}^2 \) and \( \omega \). Differentiation of (65) and (66) gives
\[
\dot{X} = -\sin E \sin d \dot{E} + \cos E \cos d \dot{d} \\
\dot{Y} = \cos E \sin d \dot{E} + \sin E \cos d \dot{d}
\]

and taking into account that
\[
\cos E = \frac{X}{\sqrt{X^2 + Y^2}}, \quad \sin E = \frac{Y}{\sqrt{X^2 + Y^2}}
\]

it follows that
\[
\sin d \dot{E} = -\sin E \dot{X} + \cos E \dot{Y} = \frac{XY - YX}{\sqrt{X^2 + Y^2}}
\]

\[
\cos d \dot{d} = \cos E \dot{X} + \sin E \dot{Y} = \frac{XX + YY}{\sqrt{X^2 + Y^2}}
\]
\[ \cos E \sin d\dot{E} = X \frac{XY - YY}{X^2 + Y^2} \]  
(79)

\[ \sin E \sin d\dot{E} = Y \frac{XY - YY}{X^2 + Y^2} \]  
(80)

\[ \sin E d\dot{E} = \frac{aY(XX + YY)}{(1-a)(X^2 + Y^2)} \]  
(81)

\[ \cos E d\dot{E} = \frac{aX(XX + YY)}{(1-a)(X^2 + Y^2)} \]  
(82)

and therefore

\[ b_Q^1 = -\cos E \sin d \dot{E} - \sin E \dot{d} = \]  
(83)

\[ = \left[ \frac{(1-a)X^2 + aY^2}{(1-a)(X^2 + Y^2)} \right] \dot{X} - \left[ \frac{(1-2a)X}{(1-a)(X^2 + Y^2)} \right] \dot{Y} \]  
(84)

By complete analogy

\[ \dot{b}_W^1 = -\left[ \frac{(1-a_p)x_P^2 + ay_P^2}{(1-a_p)(x_P^2 + y_P^2)} \right] \dot{x}_P + \left[ \frac{(1-2a_p)x_P}{(1-a_p)(x_P^2 + y_P^2)} \right] \dot{y}_P \]  
(85)

\[ b_W^2 = \left[ \frac{a_p x_P^2 + (1-a_p)y_P^2}{(1-a_p)(x_P^2 + y_P^2)} \right] \dot{x}_P - \left[ \frac{(1-2a_p)x_P}{(1-a_p)(x_P^2 + y_P^2)} \right] \dot{y}_P \]  
(86)

where

\[ a_p = \frac{1}{1 + \cos g} = \frac{1}{1 + \sqrt{1 - (x_P^2 + y_P^2)}} \]  
(87)

We may now compute the rotation rate \( \omega = |\hat{\omega}| \), from

\[ \omega^2 = \hat{\omega} \cdot \hat{\omega} = \omega_C^T \omega_C = \]  
(88)

Expressions for the computation of the stellar angle \( \theta \), \( s = s(X, Y) \) and \( s' = s'(x_P, y_P) \), present in the angle \( \psi = s + \theta - s' \), can be found in the IERS Conventions (McCarthy, 2003).

References


