A differential geometric approach to the formulation of geodetic boundary conditions

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Summary

The differential geometric characteristics of the gravity potential of the earth are utilized for the derivation of rigorous boundary conditions for three types of geodetic boundary value problems: the vectorial free, the scalar free and the fixed problem. The conditions are first derived for any type of normal field and are further simplified with the consideration of rotationally symmetric ones. For the free problems the conditions are given on the telluroid (Molodensky's problem) and on the reference ellipsoid (Stokes' problem). All boundary conditions are analytically expressed for three coordinate systems: geodetic, spherical and ellipsoidal. General expressions are also given for the second order terms of the residuals neglected in the linearization.

1. Introduction

The determination of the gravity field of the earth by means of the formulation of geodetic boundary value problems has always been a subject of great interest, both theoretical and practical. From our particular point of view here, which is the precise formulation of the boundary conditions, the most important contributions are, Krapup's investigations on the vectorial boundary value problem (Krapup, 1973) and the formulation of the more realistic scalar boundary value problem (Sacerdote & Sansò, 1986). In addition to the previous free problems, where the shape of the boundary surface is also unknown, the possibility of independent position determination with satellite techniques (GPS) made relevant the formulation of the simpler fixed boundary value problem (Koch & Pope, 1972, Bjerhammar & Svensson, 1983). A review of the current status of our knowledge about theoretical questions can be found in Sacerdote & Sansò (1991), while the practical problems are discussed in depth in Heck (1991) (see also Heck, 1988, 1989a, 1989b, Grafarend, 1989).

One of the aspects of the various geodetic boundary value problems is the derivation of the linearized boundary condition on a known surface. This surface is the telluroid for the Molodensky-type problems, where the original data refer to the earth surface, the reference ellipsoid for Stokes-type problems with data reduced to the geoid, and the earth surface itself for the fixed problem.

Formulation of the linearized boundary condition in various degrees of approximation resulting from the truncation of series expansions for the normal potential can be found in Jekeli (1981), Rapp (1981), Cruz (1986), Rapp & Cruz (1986), Holota (1986), Bjerhammar (1987), Holota (1988) and especially in Heck (1991), where the different concepts of vectorial and scalar free problems are emphasized and clarified.

Here, following Dermanis (1984, 1991), the linearized condition is derived in a rigorous way, i.e., without any approximations other than those of the linearization. This is realized with the implementation of some simple characteristics of the differential geometry of the normal gravity potential, originally introduced by Marussi (1951), and further elaborated by Hotine (1969).

The boundary conditions for the vectorial free, scalar free and the fixed geodetic boundary value problems are first derived in their more general form, for any type of normal potential. These are further specialized for the simpler case where a rotationally symmetric normal field is used. For the free problems, the boundary conditions are also given in their simpler forms on the reference ellipsoid, when the original data refer to the geoid and the normal field used is the Somigliana-Pizzetti field.

Matrix notation is used and vectors are represented by column-matrices of their components with respect to a global fixed cartesian frame. The ith column of the 3x3 identity matrix is denoted by $e_i$. 
2. Linearization of the gravity potential and vector

The free nonlinear geodetic boundary value problem can be transformed into a fixed linearized one with the help of a known approximation \( U \) (normal potential) to the unknown gravity potential \( W \), and a known surface (telluroid) approximating the unknown physical surface of the earth where the original data are given. The telluroid is realized by mapping each point \( P \) of the earth surface with unknown position vector \( x_p \) to a point \( Q \) on the telluroid with known position \( x_Q \). As a result, the new unknowns are the harmonic disturbing potential \( T = W - U \) and the position anomaly vector \( \xi = x_p - x_Q \). Using the standard Taylor series expansions, retaining up to second order terms, it follows that

\[
U_p = U_Q + \frac{\partial U}{\partial x} \bigg|_Q (x_p - x_Q) + \frac{1}{2} (x_p - x_Q)^T \left( \frac{\partial^2 U}{\partial x \partial x} \right)^T \bigg|_Q (x_p - x_Q) = U_Q + \gamma_Q^T \xi + \frac{1}{2} \xi^T M_Q \xi \quad (M = \frac{\partial \gamma}{\partial x})
\]  

(1)

\[
T_p = T_Q + \frac{\partial T}{\partial x} \bigg|_Q (x_p - x_Q) = T_Q + \delta g_Q^T \xi \quad (\delta g = \text{grad} T)
\]  

(2)

With the help of the above equations the potential anomaly \( \Delta W \) becomes

\[
\Delta W = W_p - U_Q = U_p - U_Q + T_p = \gamma_Q^T \xi + \frac{1}{2} \xi^T M_Q \xi + T_Q + \delta g_Q^T \xi
\]  

(3)

Separating linear from second order terms and dropping the subscript \( Q \), the following relation holds for every telluroid point

\[
\Delta W = T + \gamma^T \xi + S_{\Delta W}
\]  

(4)

with

\[
S_{\Delta W} = \xi^T \text{grad} T + \frac{1}{2} \xi^T M \xi.
\]  

(5)

Every component \( \gamma_i \) of the normal gravity vector \( \gamma = \text{grad} U \) can also be expanded to a Taylor series

\[
\gamma_{i,p} = \gamma_{i,Q} + \frac{\partial \gamma_i}{\partial x} \bigg|_Q (x_p - x_Q) + \frac{1}{2} (x_p - x_Q)^T \frac{\partial^2 \gamma_i}{\partial x \partial x} \bigg|_Q (x_p - x_Q) = \gamma_{i,Q} + \frac{\partial \gamma_i}{\partial x} \bigg|_Q \xi + k_i, Q
\]  

(6)

where

\[
k_i = \frac{1}{2} \xi^T H_i \xi, \quad H_i = \frac{\partial}{\partial x} \left( \frac{\partial \gamma_i}{\partial x} \right)^T
\]  

(7)

For the whole vector

\[
\gamma_p = \gamma_Q + M_Q \xi + k_Q
\]  

(8)

and for the gravity disturbance vector

\[
\delta g_p = \delta g_Q + \frac{\partial \delta g}{\partial x} \bigg|_Q (x_p - x_Q) = \delta g_Q + \delta M_Q \xi
\]  

(9)

\[
\delta M = \frac{\partial}{\partial x} \left( \frac{\partial T}{\partial x} \right)^T
\]  

(10)

With the help of the above relations the gravity anomaly vector becomes

\[
\Delta g = g_p - \gamma_Q = g_p - \gamma_Q + \delta g_p = M_Q \xi + k_Q + \delta g_Q + \delta M_Q \xi
\]  

(11)

or dropping the subscript \( Q \)

\[
\Delta g = \text{grad} T + M \xi + s_{\Delta g}
\]  

(12)

\[
s_{\Delta g} = k + \delta M \xi.
\]  

(13)

Three directions are involved in the above equations: the direction of the gravity vector \( g \) defined by the astronomic longitude \( \lambda \) and latitude \( \phi \), the direction of the normal gravity vector \( \gamma \) defined by the normal longitude \( \lambda' \) and latitude \( \phi' \), and the direction of the position anomaly vector \( \xi \) defined by two similar angles \( \lambda_\xi' \) and \( \phi_\xi \). If the corresponding unit vectors are denoted by \( n \), \( n_\xi \) and \( v \), the following relations hold

\[
n = -\frac{1}{g} g = A^T e_3
\]  

(14)

\[
n_\xi = -\frac{1}{\gamma} \gamma = A^T e_3
\]  

(15)

\[
v = \frac{1}{\xi} \xi = A^T e_3
\]  

(16)

with \( e_3 = [0 \ 0 \ 1]^T \) (\( e_i \) being in general the \( i \)th row of \( I_3 \))

and
\[ A = R_i (90^\circ - \Phi) R_3 (90^\circ + \Lambda) \] (17)

\[ A_0 = R_i (90^\circ - \Phi) R_3 (90^\circ + \lambda) \] (18)

\[ A_\gamma = R_i (90^\circ - \phi) R_3 (90^\circ + \lambda \gamma) \] (19)

When \( \Lambda, \Phi \) and \( W \) or \( g \), are observed, the direction \( n \) is known and so is the whole gravity vector \( g \). In this case the vectorial geodetic boundary value problem is formulated. The mapping from \( P \) to \( Q \) is usually realized by the Marussi mapping \((W_p = U_Q, \mathbf{n}_p = \mathbf{n}_o_Q)\), but in the more general case \( Q \) is defined by a known approximation \( x_Q \) to the unknown position vector \( x_P \) of \( P \). Completely unknown is the position anomaly vector \( \xi \) and consequently also its direction \( v \).

When only \( g \) is observed the scalar geodetic boundary value problem is formulated, assuming that the direction \( v \) of the position anomaly vector \( \xi \) is known. The unknowns of the problem in this case are \( T \) and \( \xi \).

### 3. The vectorial problem

#### 3.1. The vectorial boundary condition

In the vectorial problem \( \xi \) is unknown and must be eliminated. Following Krarup (1973) (see also Hörmander, 1976) equation (12) is solved for \( \xi \):

\[ \xi = M^{-1} \Delta g - M^{-1} \text{grad} T - M^{-1} s_{\Delta g} \] (20)

and replacement in (4) leads to the boundary condition:

\[ \gamma^T M^{-1} \Delta g - \Delta W = \]
\[ = -T + \gamma^T M^{-1} \text{grad} T + \gamma^T M^{-1} s_{\Delta g} - S_{\Delta W} \] (21)

Since \( g = -g n, \gamma = -\gamma n_o \) and \( \Delta g = g - \gamma = \gamma n_o \) and \( \Delta g = s_{\Delta g} \) is the scalar gravity anomaly, the vector gravity anomaly becomes

\[ \Delta g = g - \gamma = -g n + \gamma n_o = -\Delta g n + \gamma (n_o - n) \] (22)

and the boundary condition (21) is transformed into

\[ \gamma (n_o^T M^{-1} n) \Delta g - \Delta W = \]
\[ = \gamma^2 (n_o^T M^{-1} n - n_o^T M^{-1} n) - T - \gamma n_o^T M^{-1} \text{grad} T - \]
\[ - \gamma n_o^T M^{-1} s_{\Delta g} - S_{\Delta W} \] (23)

or

\[ \Delta g = \frac{1}{n_o^T M^{-1} n} \left[ \gamma (n_o^T M^{-1} n - n_o^T M^{-1} n) - \frac{T - \Delta W}{\gamma} - n_o^T M^{-1} \text{grad} T \right] + S_{\Delta g} \] (24)

where

\[ S_{\Delta g} = -\frac{1}{n_o^T M^{-1} n} \left[ n_o^T M^{-1} s_{\Delta g} + \frac{1}{\gamma} S_{\Delta W} \right] \] (25)

and with the help of (5) and (13)

\[ S_{\Delta g} = -\frac{1}{n_o^T M^{-1} n} \left[ n_o^T M^{-1} k + n_o^T M^{-1} \delta M \xi + \right. \]
\[ + \left. \frac{1}{\gamma} \xi^T M^{-1} \text{grad} T + \frac{1}{\gamma^2} \xi^T M^{-1} \text{grad} T \right] \] (26)

### 3.2. The boundary condition in terms of differential parameters of the normal gravity field

Introducing the auxiliary parameter

\[ \tau = \frac{n_o^T M^{-1} n}{n_o^T M^{-1} n} \] (27)

and neglecting the second order terms, the linearized boundary condition takes the more convenient form

\[ \tau \Delta g + \gamma (\tau - 1) = \]
\[ = -\frac{1}{n_o^T M^{-1} n} \left[ \frac{T - \Delta W}{\gamma} + n_o^T M^{-1} \text{grad} T \right] \] (28)

The Marussi matrix \( M \) (matrix of second order derivatives of the normal potential \( U \)) can be expressed in terms of the normal Eötvös matrix \( E \) (matrix of the second derivatives in the local normal astronomic frame) according to

\[ M = \frac{\partial \gamma}{\partial x} = \frac{\partial \gamma}{\partial x} \frac{\partial \gamma}{\partial x} = A_o^T E A_o \] (29)

where \( x^* \) and \( \gamma^* \) are the counterparts of \( x \) and \( \gamma \), in the local normal astronomic frame. We also introduce the vector of the components of the direction of gravity vector \( n \) in the local normal astronomic frame

\[ c = n^* = A_o n = A_o A^T e_3 = \]
\[ = R_i (90^\circ - \Phi) R_3 (\lambda - \Lambda) R_i (\Phi - 90^\circ) e_3 \] (30)

With the help of (15), (29), (30) the terms present in the boundary condition become

\[ n_o^T M^{-1} n_o = e_3^T A_o A_o^T E^T A_o A_o^T e_3 = \{ E^T \}_{33} \] (31)
\[ n_o^T M^{-1} n_o = e_3^T A_o A_o^T E^{-1} A_o A^T e_3 = e_3^T E^{-1} c = \]
\[ \{E^{-1} c\}_3 \]  
\[ \tau = \frac{\{E^{-1} c\}_3}{\{E^{-1}\}_3} \] (33)

The Eötvös matrix and its inverse are given by (Marussi, 1951, Hotine, 1969, eq. 12.162)

\[ E = -\gamma 
\begin{bmatrix}
  k_E & t_E & \kappa_E \\
  t_E & k_N & \kappa_N \\
  \kappa_E & \kappa_N & \gamma 
\end{bmatrix} 
\] (34)

\[ E^{-1} = -\frac{1}{\gamma \Delta} 
\begin{bmatrix}
  k_N \frac{\gamma}{\gamma} - k_N^2 & k_E \kappa_N - t_E \frac{\gamma}{\gamma} & \gamma K H_1 \\
  k_E \kappa_N - t_E \frac{\gamma}{\gamma} & k_E \kappa_E - k_E^2 & \gamma K H_2 \\
  \gamma K H_1 & \gamma K H_2 & K 
\end{bmatrix} 
\] (35)

where \( k_E, k_N \) are the curvatures of the east and north normal sections, respectively, of the normal equipotential surface, \( t_E \) is its geodetic torsion, \( \kappa_E, \kappa_N \) are the east and north curvature components, respectively, of the normal plumb line, \( \gamma \) is the derivative of normal gravity \( \gamma \) in the zenith direction,

\[ K = k_E k_N - \frac{\gamma}{\gamma_E} \] (36)

is the Gaussian curvature of the normal equipotential surface,

\[ H_1 = \frac{t_E k_N - k_N \kappa_E}{\gamma K} = -\frac{1}{\gamma^2 \cos \phi} \frac{\partial \gamma}{\partial \lambda} \] (37)

\[ H_2 = \frac{t_E k_E - k_E \kappa_N}{\gamma K} = -\frac{1}{\gamma^2 \frac{\partial \gamma}{\partial \phi}} \] (38)

\[ \Delta = K \left[ \frac{\gamma}{\gamma} + \gamma (\kappa_E H_1 + \kappa_N H_2) \right] \] (39)

Here the notation of Marussi (1951) has been used. \( H_1, H_2 \) are defined by his eq. (18.9) and related to derivatives of gravity in eq. (25.1). Also the elements of the Eötvös matrix are given by Marussi in his equations (9.2), (12.12), (13.2). With the above representation of \( E^{-1} \) it follows that

\[ n_o^T M^{-1} n_o = \{E^{-1}\}_3 = -\frac{K}{\gamma \Delta} \]

\[ \tau = -\frac{1}{\gamma} \gamma^2 \left( \kappa_E H_1 + \kappa_N H_2 \right) \]

\[ \tau = -\frac{1}{\gamma} \gamma^2 \left( \kappa_E H_1 + \kappa_N H_2 \right) \]

where

\[ a_o = \frac{\Lambda}{K} = \frac{\gamma}{\gamma} + \gamma \kappa_E H_1 + \gamma \kappa_N H_2 = \]

\[ = \frac{\gamma}{\gamma} - \frac{\kappa_E}{\cos \phi} \frac{\gamma}{\gamma} \kappa_N \gamma \] (41)

\[ n_o^T M^{-1} n_o = \{E^{-1} c\}_3 = -\frac{\gamma H_1 c_1 + \gamma H_2 c_2 + c_3}{\gamma} \]

\[ = -\frac{\gamma H_1 c_1 + \gamma H_2 c_2 + c_3}{\gamma a_o} \] (42)

\[ \tau = \gamma H_1 c_1 + \gamma H_2 c_2 + c_3 = -\frac{c_1}{\cos \phi} \frac{\gamma}{\gamma} - \frac{c_2}{\gamma} \frac{\gamma}{\gamma} + c_3 \] (43)

The gradT term in the boundary condition, for any system of curvilinear coordinates \( y = [y_1 y_2 y_3]^T \), becomes

\[ -\frac{1}{(n_o^T M^{-1} n_o)} n_o^T M^{-1} \text{grad} T = \gamma a_o \frac{\partial T}{\partial x} M^{-1} n_o = \]

\[ = \gamma a_o \frac{\partial T}{\partial y} \frac{\partial y}{\partial x} A_o^T E^{-1} e_3 \]

\[ = \frac{\partial T}{\partial y} a_y = a_{y_1} \frac{\partial T}{\partial y_1} + a_{y_2} \frac{\partial T}{\partial y_2} + a_{y_3} \frac{\partial T}{\partial y_3} \] (44)

where

\[ a_y = \gamma a_o \frac{\partial y}{\partial x} A_o^T E^{-1} e_3 \] (45)

Considering only orthogonal curvilinear coordinates (geodetic \( \phi, \lambda, h \), spherical \( \phi, \lambda, r \), ellipsoidal \( \beta, \lambda, u \)) it holds that

\[ \frac{\partial y}{\partial x} = G^{-1/2}_y A_y \] (46)

\( A_y \) being an orthogonal matrix and \( G_y \) the diagonal metric matrix. A similar decomposition with \( G^{-1/2}_y \) only symmetric, is possible for any matrix (polar decomposition theorem of Cauchy, see e.g. Halmos, 1958, §83). The diagonality of \( G^{-1/2}_y \) is a consequence of the orthogonality of the coordinate system. Since

\[ E^{-1} e_3 = -\frac{1}{\gamma a_o} \left[ \begin{array}{c} \gamma H_1 \\ \gamma H_2 \\ 1 \end{array} \right] = \frac{1}{\gamma a_o} \left[ \begin{array}{c} \gamma \lambda \\ \gamma \phi \\ -1 \end{array} \right] \] (47)
the boundary condition takes the form
\[ \tau \Delta g + (\tau - 1) \gamma + a_o \Delta W = \]
\[ = a_o T + a_{y_1} \frac{\partial T}{\partial y_1} + a_{y_2} \frac{\partial T}{\partial y_2} + a_{y_3} \frac{\partial T}{\partial y_3} \]
\[ \hspace{1cm} (48) \]

where \( \tau \) is given by (43), \( a_o \) by (41) and the coefficients \( a_{y_1}, a_{y_2}, a_{y_3}, \) from
\[ a_y = \gamma a_o G_y^{-1/2} A_y A_o^T E^{-1} e_3 \]
\[ \hspace{1cm} (49) \]

which in view of (47) becomes
\[ \begin{bmatrix} a_{y_1} \\ a_{y_2} \\ a_{y_3} \end{bmatrix} = -G_y^{-1/2} A_y A_o^T \begin{bmatrix} \gamma H_1 \\ \gamma H_2 \\ 1 \end{bmatrix} = G_y^{-1/2} A_y A_o^T \begin{bmatrix} \gamma \lambda \\ \gamma \cos \phi \\ -1 \end{bmatrix} \]
\[ \hspace{1cm} (50) \]

The parameters \( \tau \), \( \Delta g \) and \( \Delta W \) depend on which particular telluroid mapping is used (Grafarend, 1978). For the usual choice of mappings with \( \lambda_p = \lambda_Q, \Phi_p = \Phi_Q \), it follows that \( n_o = n \), so that eq. (30) gives \( c = e_3 \) and from eq. (33) it follows that in this case \( \tau = 1 \). For the Marussi mapping, where \( W_p = U_Q, \Delta W = 0 \), or more generally \( \Delta W = \text{const} \). For the gravimetric mapping, where \( g_p = \gamma Q, \Delta g = 0 \).

It must be also noted that all the information about the observed parameters \( \Lambda \) and \( \Phi \), which in fact differentiate the vectorial from the scalar formulation, is contained in the single parameter \( \tau \).

The matrix \( G_y^{-1/2} A_y A_o^T \) required for the computation of the coefficients \( a_{y_1}, a_{y_2}, a_{y_3} \), can be computed by differentiation of the relations between cartesian and curvilinear coordinates. For the specific coordinate systems used here, it takes the following forms:

Geodetic coordinates \( \tilde{\lambda}, \tilde{\phi}, h \):
\[ G_y^{-1/2} A_y A_o^T = \]
\[ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_1(90^\circ - \phi_y) R_3(\tilde{\lambda} - \lambda) R_1(\phi - 90^\circ) \]
\[ \hspace{1cm} (51) \]

Spherical coordinates \( \lambda, \phi, r \):
\[ G_y^{-1/2} A_y A_o^T = \]
\[ = \begin{bmatrix} 1 \cos \phi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_1(90^\circ - \phi_y) R_3(\tilde{\lambda} - \lambda) R_1(\phi - 90^\circ) \]
\[ \hspace{1cm} (52) \]

Ellipsoidal coordinates \( \lambda, \beta, u \):
\[ G_y^{-1/2} A_y A_o^T = \]
\[ = \begin{bmatrix} 1 \cos \beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_1(90^\circ - \beta) R_3(\tilde{\lambda} - \lambda) R_1(\phi - 90^\circ) \]
\[ \hspace{1cm} (53) \]

The angle \( \tilde{\beta} \) is the latitude of the perpendicular to the local normal equipotential surface, defined by
\[ \sin \tilde{\beta} = \frac{\nu}{L} \sin \beta, \quad \cos \tilde{\beta} = \frac{u}{L} \cos \beta, \]
\[ \hspace{1cm} (54) \]
\[ v^2 = u^2 + E^2, \quad L^2 = u^2 + E^2 \sin^2 \beta, \quad E^2 = a^2 - b^2, \]
\[ \hspace{1cm} (55) \]

a and b being the semi-axes of the reference ellipsoid.

The boundary conditions can be easily obtained by replacing the term \( G_y^{-1/2} A_y A_o^T \) from (51), (52) or (53) into (50), and setting the resulting values of \( a_{y_1}, a_{y_2}, a_{y_3} \) in eq. (48). The obtained rather lengthy expressions are omitted here. An outline of the computational procedure for the parameters appearing in the boundary condition is given in Appendix A.

The vectorial formulation must be complemented with a three-dimensional version of Bruns' formula, which gives \( \xi \) as a function of the solution \( T \) of the geodetic boundary value problem and its derivatives (Dermanis, 1987 and Heck, 1991, pp. 8-14). A derivation of Bruns' formula is sketched in Appendix B.

3.3. The vectorial boundary condition for a rotationally symmetric normal field

The boundary condition on the telluroid is significantly simplified when a rotationally symmetric normal field is used, such as the standard Somigliana-Pizzetti field, which is \( \lambda \)-invariant. In this case the curvilinear coordinate \( \tilde{\lambda} \) is identical to the normal longitude \( \lambda \). Furthermore,
Ellipsoidal coordinates $\tilde{\lambda}, \beta, u$:

\[ \tau \Delta g + \gamma (\tau - 1) + \left( \frac{\gamma}{\gamma} - r_N \kappa^2 \right) \Delta W = \left( \frac{\gamma}{\gamma} - r_N \kappa^2 \right) T + \]
\[ + \frac{1}{M + h} \left[ \cos(\phi - \phi_0) \kappa r_N - \sin(\phi - \phi_0) \right] \frac{\partial T}{\partial \phi} + \]
\[ - \left[ \sin(\phi - \phi_0) \kappa r_N + \cos(\phi - \phi_0) \right] \frac{\partial T}{\partial h} \]  

where $\phi = \phi_0$ for the geodetic coordinates, $\bar{\psi} = \bar{\beta}$ for the spherical and $\bar{\psi} = \bar{\beta}$ for the ellipsoidal ones. Carrying out the necessary computations the boundary condition on the ellipsoid for a $\lambda$-invariant normal field takes the following forms for the various coordinate systems:

**Geodetic coordinates $\bar{\lambda}, \phi_0, h$**:

\[ \tau \Delta g + \gamma (\tau - 1) + \left( \frac{\gamma}{\gamma} - r_N \kappa^2 \right) \Delta W = \left( \frac{\gamma}{\gamma} - r_N \kappa^2 \right) T + \]
\[ + \frac{1}{M + h} \left[ \cos(\phi - \phi_0) \kappa r_N - \sin(\phi - \phi_0) \right] \frac{\partial T}{\partial \phi} + \]
\[ - \left[ \sin(\phi - \phi_0) \kappa r_N + \cos(\phi - \phi_0) \right] \frac{\partial T}{\partial h} \]  

**Spherical coordinates $\bar{\rho}, \bar{\phi}, \bar{r}$**:

\[ \tau \Delta g + \gamma (\tau - 1) + \left( \frac{\gamma}{\gamma} - r_N \kappa^2 \right) \Delta W = \left( \frac{\gamma}{\gamma} - r_N \kappa^2 \right) T + \]
\[ + \frac{1}{r} \left[ \cos(\phi - \bar{\phi}) \kappa r_N - \sin(\phi - \bar{\phi}) \right] \frac{\partial T}{\partial \phi} + \]
\[ - \left[ \sin(\phi - \bar{\phi}) \kappa r_N + \cos(\phi - \bar{\phi}) \right] \frac{\partial T}{\partial \phi} \]  

3.4 The vectorial boundary condition on the ellipsoid

When the data refer to points $P$ on the geoid, the Stokes problem is formulated instead of the Molodensky problem for the physical surface. With a mapping where $\Delta W = 0$, (or more generally $\Delta W = \text{const.}$, to account for uncertainties in horizontal datum definition) the geoid is mapped on the reference ellipsoid provided that the $\lambda$-invariant Somigliana-Pizzetti normal field is used. Several relations needed for the reduction to the ellipsoid have been derived in Appendix C. On the ellipsoid $\phi_0 = \phi$, $\bar{\beta} = \phi$, $r_N = M$, $r_g = N$, $u = b$, $v = a$ and $L = \frac{b N}{a}$.

From $\Delta U = \text{tr}(\mathbf{E}) = 2\omega^2$, Bruns' equation follows

\[ \gamma_N = -\left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} \right] \]  

where $\omega$ is the rotational velocity of the earth, while $M$ and $N$ are the radii of curvature of the normal sections of the reference ellipsoid in the prime vertical and the meridian direction respectively. The curvature of the plumb line on the ellipsoid is given by (Appendix C, see also Dermanis, 1984, 1991)

\[ \kappa = \frac{a \rho}{b N} \sin 2\phi = \frac{r^2 \rho}{a b M} \sin 2\phi = \frac{\rho}{M} \sin 2\beta \]  

with

\[ \rho = (\rho^* + f) \frac{N \gamma_N}{a} - \frac{e^2 a}{2 b} \]  

where $e^2$ is the first eccentricity, $f$ the flattening, $\rho^*$ the dynamic flattening and $\gamma_N$ the gravity at the equator. Equation (65) is an exact equation and it has been derived in Appendix C.
With the above values (58) gives
\[ \tau = c_3 - \frac{a M}{b N} \rho \sin 2\phi c_2 = c_3 - \frac{r^2}{a b} \rho \sin 2\phi c_2 = \]
\[ = c_3 - \rho \sin 2\beta c_2 \quad (66) \]
and equations (60), (61), (63) transform to the following boundary conditions on the ellipsoid:

**Geodetic coordinates**:
\[ \tau \Delta g + \gamma (\tau - 1) - \left( \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} + \frac{\rho^2 N}{a^2 \sin^2 2\phi} \right) \Delta W = \]
\[ = -\left( \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} + \frac{\rho^2 N}{a^2 \sin^2 2\phi} \right) T + \]
\[ + \frac{a}{b N} \sin 2\phi \left( \rho - \frac{e^2 a}{2 b} \frac{\partial T}{\partial \phi} \right) \]
\[ + \frac{a}{b N} \sin 2\phi \left( \frac{a}{r} + \frac{e^2 \rho r^3}{2 b^3 \sin^2 2\phi} \right) \frac{\partial T}{\partial r} \quad (67) \]

**Spherical coordinates**:
\[ \tau \Delta g + \gamma (\tau - 1) - \left( \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} + \frac{\rho^2 N}{a^2 b^2 M \sin^2 2\phi} \right) \Delta W = \]
\[ = -\left( \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} + \frac{\rho^2 N}{a^2 b^2 \sin^2 2\phi} \right) T + \]
\[ + \frac{a}{b N} \sin 2\phi \left( \rho - \frac{e^2 a}{2 b} \frac{\partial T}{\partial \phi} \right) \]
\[ - \frac{a}{N} \left( \frac{a}{r} + \frac{e^2 \rho r^3}{2 b^3 \sin^2 2\phi} \right) \frac{\partial T}{\partial r} \quad (68) \]

**Ellipsoidal coordinates**:
\[ \tau \Delta g + \gamma (\tau - 1) - \left( \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} + \frac{\rho^2 N}{M \sin^2 2\beta} \right) \Delta W = \]
\[ = -\left( \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} + \frac{\rho^2 N}{M \sin^2 2\beta} \right) T + \]
\[ + \frac{a}{b N} \sin 2\beta \left( \rho - \frac{a^2 \rho}{b N \sin 2\beta} \frac{\partial T}{\partial \beta} \right) \quad (69) \]

Again the left sides of eqs. (67), (68), (69) are simplified to \( \Delta g \) for the usual telluroid mappings. Approximations of the above conditions up to the order of \( e^2 \) can be found in Dermanis (1991), where they are compared with the similar results of Holota (1986, 1988) and are found to be in agreement. They are also compared with similar results from Jekeli (1981) and Cruz (1986) and they are found to be different as expected. The reason for this difference is that Dermanis and Holota follow the vectorial formulation while the derivations of Jekeli and Cruz are along the lines of the scalar formulation to be examined next.

### 3.5. The vectorial boundary condition in terms of the isozential

An isozential is defined as a line along which the direction of the normal gravity vector remains constant
\[ n_o = -\frac{1}{\gamma} \tau = \text{const.} \quad (70) \]

If \( x = x(s) \) is the natural parametric representation of an isozenthal line, where \( s \) is the length along the isozenthal, the derivative of \( n_o \) with respect to \( s \) must vanish as a consequence of (70), so that
\[ \frac{\partial n_o}{\partial s} = \frac{\partial}{\partial s} \left( -\frac{1}{\gamma} \gamma \right) = \]
\[ = \frac{1}{\gamma^2} \frac{\partial \gamma}{\partial s} - \frac{1}{\gamma} \frac{\partial \gamma}{\partial s} = \frac{1}{\gamma^2} \frac{\partial \gamma}{\partial s} - \frac{1}{\gamma} \frac{\partial \gamma}{\partial s} \frac{\partial x}{\partial s} = \]
\[ = \frac{1}{\gamma^2} \gamma_s \gamma - \frac{1}{\gamma} M i = 0 \quad , \quad (71) \]

where \( i = \frac{\partial x}{\partial s} \) is the unit vector tangent to the isozenthal, and \( \gamma_s = \frac{\partial \gamma}{\partial s} \). Solving for \( i \) it follows that
\[ i = -\gamma_s M^{-1} n_o = -\gamma_s A_o^T E^{-1} e_3 \quad (72) \]

while the derivative \( \gamma_s \) is determined from
\[ | i |^2 = i^T i = \gamma_s^2 n_o^T M^{-2} n_o = \]
\[ = \gamma_s^2 e_3^T A_o A_o^T E^{-1} A_o A_o^T E^{-1} A_o A_o^T e_3 = \]
\[ = \gamma_s^2 \{ E^{-2} \}_{33} = 1 \quad (73) \]

which gives
\[ \gamma_s = \frac{\partial \gamma}{\partial s} = \frac{1}{\sqrt{\{ E^{-2} \}_{33}}} \quad (74) \]

If \( \Theta_{iy} \) is the small angle between the isozenthal \( i \) and \(-\gamma\), while \( \Theta_{ig} \) is the small angle between \( i \) and \(-g\), it holds that
\[ n_o^T M^{-1} n_o = -\frac{1}{\gamma_s} n_o^T i = -\frac{1}{\gamma_s} \cos \Theta_{iy} \quad , \quad (75) \]
\[ n_o^T M^{-1} i = -\frac{1}{\gamma_s} i^T n = -\frac{1}{\gamma_s} \cos \Theta_{ig} \quad , \quad (76) \]

and
\[ n_o^T M^{-1} \text{grad} T = -\frac{1}{\gamma_s} i^T \text{grad} T = -\frac{1}{\gamma_s} \frac{\partial T}{\partial s} \quad , \quad (77) \]

where \( \frac{\partial T}{\partial s} \) is the derivative of \( T \) in the direction of the isozenthal. Replacing the above results in (24) the boundary
The starting point is equations (4) and (12) rewritten as
\[
\Delta W = T - \gamma \zeta n_o^T \nu + S_{\Delta W} \tag{83}
\]
\[
\Delta g = \text{grad} T + \zeta M \nu + s_{\Delta g} \tag{84}
\]
From (83) it follows that (Bruns' formula)
\[
\zeta = \frac{1}{\gamma (n_o^T \nu)} (T - \Delta W + S_{\Delta W}) \tag{85}
\]
Replacement in (84) and combination with (22) gives
\[
(\gamma + \Delta g) n = \gamma n_o - \text{grad} T - \frac{T - \Delta W}{\gamma (n_o^T \nu)} M \nu - s \tag{86}
\]
where
\[
s = -\frac{S_{\Delta W}}{\gamma (n_o^T \nu)} M \nu + s_{\Delta g} = \frac{1}{\gamma (n_o^T \nu)} \left[ \zeta \nu^T \text{grad} T + \frac{\zeta^2}{2} \nu^T M \nu \right] M \nu + \zeta^2 q + \zeta \delta M \nu \tag{87}
\]
and
\[
q_i = \frac{1}{2} \nu^T H_i \nu, \quad H_i = \frac{\partial}{\partial x} \left( \frac{\partial n_o}{\partial x} \right)^T \tag{88}
\]
The next step is the elimination of the unknown direction \( n \) from (86). Following Heck (1989a, Appendix) we take the square norm of both sides and retaining only up to second order terms, results in
\[
(\gamma + \Delta g)^2 = \gamma^2 - 2 \gamma n_o^T \text{grad} T - 2 \frac{T - \Delta W}{(n_o^T \nu)} (n_o^T M \nu) - 2 \gamma n_o^T s + (\text{grad} T)^T \text{grad} T + 2 \frac{T - \Delta W}{\gamma (n_o^T \nu)} \nu^T M \text{grad} T + \frac{(T - \Delta W)^2}{\gamma^2 (n_o^T \nu)^2} (\nu^T M \nu) \tag{89}
\]
Division by \( \gamma^2 \) gives
\[
\left(1 + \frac{\Delta g}{\gamma} \right)^2 = 1 - \frac{2}{\gamma} n_o^T \text{grad} T - \frac{2 (n_o^T M \nu)}{\gamma^2 (n_o^T \nu)} (T - \Delta W) + s_2 = 1 + X \tag{90}
\]
with an obvious definition of \( X \) and
\[
s_2 = -\frac{2}{\gamma} n_o^T s + \frac{1}{\gamma^2} | \text{grad} T |^2 + \]
\[ + \frac{2}{\gamma^3} \left( n_0^T \nabla T \right) (T - \Delta W) + \]
\[ + \frac{1}{\gamma^4} \left( \frac{v^T M^2 v}{(n_0^T v)^2} \right) (T - \Delta W)^2 \]  
\[ X^2 = \frac{4}{\gamma^2} \left( n_0^T \nabla T \right)^2 + \frac{8}{\gamma^3} \left( n_0^T M v \right) (n_0^T \nabla T) (T - \Delta W) + \]
\[ + \frac{4}{\gamma^2} \left( n_0^T M v \right)^2 \frac{1}{(n_0^T v)^2} (T - \Delta W)^2 \]  
(91)  
(92)  

The square root of (90) is

\[ 1 + \frac{\Delta g}{\gamma} = \sqrt{1 + X} = 1 + \frac{X}{2} - \frac{X^2}{8} \]

so that

\[ \Delta g = \frac{\gamma}{2} X - \frac{\gamma}{8} X^2 \]

and finally with the values of X and X^2 from (90) and (92), respectively,

\[ \Delta g = - n_0^T \nabla T - \frac{(n_0^T M v)}{\gamma (n_0^T v)} (T - \Delta W) + S_{\Delta g} \]

with

\[ S_{\Delta g} = - \zeta \frac{(n_0^T M v)}{\gamma (n_0^T v)} v^T \nabla T - \frac{\zeta^2 (v^T M v) (n_0^T \delta M v)}{2 \gamma (n_0^T v)} - \]
\[ - \zeta^2 (n_0^T q) - \zeta (n_0^T \delta M v) + \frac{1}{2 \gamma} \left| \nabla T \right|^2 + \]
\[ + \frac{1}{\gamma^2} \left( n_0^T \nabla T \right) (T - \Delta W) + \]
\[ + \frac{(v^T M^2 v)}{2 \gamma^3 (n_0^T v)^2} (T - \Delta W)^2 - \frac{1}{2 \gamma} (n_0^T \nabla T)^2 - \]
\[ - \frac{(n_0^T M v)}{\gamma^2} (n_0^T \nabla T) (T - \Delta W) - \]
\[ + \frac{(n_0^T M v)^2 (n_0^T \nabla T)^2}{2 \gamma^3 (n_0^T v)^2} (T - \Delta W)^2 \]  
(95)  
(96)  

As in (95), the linear part of equation

\[ \Delta g = \frac{1}{\gamma} \cos \Theta \nabla \theta \]

\[ - \frac{1}{\gamma} \cos \Theta \frac{\partial T}{\partial n_0} \frac{1}{\gamma} \cos \Theta \nabla \theta \]

(95*)

where \( \Theta \nabla \theta \) is the angle between the direction of the position anomaly vector and the direction opposite to the gradient of the normal gravity, while \( \Theta \nabla \theta \) is the angle between the direction of the position anomaly vector and the direction opposite to that of the normal gravity vector. The derivative of T is in the direction opposite to that of the normal gravity vector.

### 4.2. The scalar boundary condition in terms of differential parameters of the normal gravity field

The boundary condition (95) with respect to any system of curvilinear coordinates \( y = [y_1, y_2, y_3]^T \), can be written in the general form

\[ \Delta g + s_o \Delta W = a_o T + a_{y_1} \frac{\partial T}{\partial y_1} + a_{y_2} \frac{\partial T}{\partial y_2} + a_{y_3} \frac{\partial T}{\partial y_3} \]

(97)  

where

\[ a_o = - \frac{(n_0^T M v)}{\gamma (n_0^T v)} \]

(98)  

and

\[ a_y = [a_{y_1} a_{y_2} a_{y_3}]^T = - \frac{\delta v}{\delta x} n_0 \]

(99)  

For orthogonal curvilinear coordinates, equation (46) holds and

\[ a_y = [a_{y_1} a_{y_2} a_{y_3}]^T = - G_y^{-1/2} A_y A_o^T e_3 \]

(100)

It must be noticed that the coefficient \( a_o \) is independent of the particular coordinate system used and depends solely on the direction \( v \). On the contrary the coefficients \( a_{y_1}, a_{y_2}, a_{y_3} \), while they depend on the coordinate system used they are independent of the direction \( v \) of the position anomaly vector: Using the representation of (16) and (19) for \( v \), the vector of its components in the local normal astronomic frame is

\[ v^* = A_o v = A_o A_o^T e_3 = \]

\[ R_1(90^\circ - \phi) R_3(\lambda - \lambda_c) R_1(\phi - 90^\circ) e_3 = \]

\[ = \begin{bmatrix} 
-\cos \phi \sin (\lambda - \lambda_c) \\
- \sin \phi \cos \phi \sin (\lambda - \lambda_c) + \cos \phi \sin \phi \]
\[ \cos \phi \cos \phi \cos (\lambda - \lambda_c) + \sin \phi \sin \phi \]
\end{bmatrix} \]

(101)

From (34) it follows that
so that

\[ n_o^T M v = e_3^T E A_o A_\zeta e_3 = e_3^T E v^* = \]

\[ = \gamma_k E \cos \phi \sin (\lambda - \lambda_\zeta) + \]

\[ + \gamma_k N [ \sin \phi \cos \phi \cos (\lambda - \lambda_\zeta) - \cos \phi \sin \phi \sin \phi_\zeta ] + \]

\[ - \gamma_k [ \cos \phi \cos \phi_\zeta \cos (\lambda - \lambda_\zeta) + \sin \phi \sin \phi_\zeta ] . \]

(103)

Since \( n_o^T M v = e_3^T A_o v = e_3^T v^* = v_3^* \), it follows from (98) and (101) that

\[ a_o = \frac{\gamma_k}{\gamma} - \frac{\kappa E \cos \phi \sin (\lambda - \lambda_\zeta) + \kappa N [ \sin \phi \cos \phi \cos (\lambda - \lambda_\zeta) - \cos \phi \sin \phi \sin \phi_\zeta ]}{\cos \phi \cos \phi \zeta \cos (\lambda - \lambda_\zeta) + \sin \phi \sin \phi_\zeta} \]

(104)

For specific mappings the proper values of \( \phi_\zeta \) and \( \lambda_\zeta \) should be used in (104). When the mapping from the earth surface to the telluroid is performed along the normal to the reference ellipsoid (Helmert-Pizzetti mapping) then \( \phi_\zeta = \phi_g \), and \( \lambda_\zeta = \lambda \). For the mapping along the radial direction \( \phi_\zeta = \bar{\phi}, \) and \( \lambda_\zeta = \bar{\lambda} \). The coefficients \( a_{x_1}, a_{x_2}, a_{x_3} \) can be computed from (100) for geodetic, spherical and ellipsoidal coordinates using (51), (52) and (53) respectively:

**Geodetic coordinates \( \bar{\lambda}, \phi_\zeta, h \):**

\[ a_\lambda = \frac{\cos \phi \sin (\bar{\lambda} - \lambda)}{N + h} \cos \phi_g \]

(105)

\[ a_{\phi_\zeta} = \frac{\sin \phi_g \cos \phi \cos (\bar{\lambda} - \lambda) - \cos \phi_g \sin \phi}{M + h} \]

(106)

\[ a_h = - [ \cos \phi_g \cos \phi \cos (\bar{\lambda} - \lambda) + \sin \phi_g \sin \phi ] . \]

(107)

**Spherical coordinates \( \bar{\lambda}, \phi, r \):**

\[ a_\lambda = \frac{\cos \phi \sin (\bar{\lambda} - \lambda)}{r} \cos \phi \]

(108)

\[ a_{\phi} = \frac{\sin \phi \cos \phi \cos (\bar{\lambda} - \lambda) - \cos \phi \sin \phi}{r} \]

(109)

\[ a_\lambda = - [ \cos \phi \cos \phi \cos (\bar{\lambda} - \lambda) + \sin \phi \sin \phi ] . \]  

(110)

**Ellipsoidal coordinates \( \bar{\lambda}, \beta, u \):**

\[ a_\lambda = \frac{\cos \phi \sin (\bar{\lambda} - \lambda)}{\nu \cos \beta} \]

(111)

\[ a_\beta = \frac{\sin \beta \cos \phi \cos (\bar{\lambda} - \lambda) - \cos \beta \sin \phi}{L} = \]

\[ = \frac{\nu \sin \beta \cos \phi \cos (\bar{\lambda} - \lambda) - u \cos \beta \sin \phi}{L^2} \]

(112)

\[ a_u = - \frac{\nu}{L^2} [ \cos \beta \cos \phi \cos (\bar{\lambda} - \lambda) + \sin \beta \sin \phi ] = \]

\[ = - \frac{\nu}{L^2} [ u \cos \beta \cos \phi \cos (\bar{\lambda} - \lambda) + \nu \sin \beta \sin \phi ] . \]  

(113)

**4.3. The scalar boundary condition for a rotationally symmetric normal field**

When the normal potential \( U \) is rotationally symmetric it is independent of longitude \( \lambda \), and furthermore normal longitude \( \bar{\lambda} \) is identical with the geodetic-spherical-ellipsoidal longitude \( \bar{\lambda} \). In this case \( \kappa_E = 0 \) and \( \kappa_N = \kappa \). For any type of earth surface to telluroid mapping

\[ a_o = \frac{\gamma_k}{\gamma} - \frac{\sin \phi \cos \phi \cos (\bar{\lambda} - \lambda_\zeta) - \cos \phi \sin \phi_\zeta}{\cos \phi \cos \phi_\zeta \cos (\lambda - \lambda_\zeta) + \sin \phi \sin \phi_\zeta} \]

(114)

For the mapping along the normal to the reference ellipsoid \((\lambda_\zeta = \bar{\lambda} = \lambda, \phi_\zeta = \phi_g)\) equation (114) becomes

\[ a_o = \frac{\gamma_k}{\gamma} - \kappa \tan (\phi - \phi_g) . \]

(115)

For the mapping along the radial direction \((\lambda_\zeta = \bar{\lambda} = \lambda, \phi_\zeta = \bar{\phi})\) (114) becomes

\[ a_o = \frac{\gamma_k}{\gamma} - \kappa \tan (\phi - \bar{\phi}) . \]

(116)

The remaining coefficients according to the coordinate system used are:

**Geodetic coordinates \( \bar{\lambda}, \phi_g, h \):**

\[ a_\lambda = 0 , \quad a_{\phi_g} = - \frac{\sin (\phi - \phi_g)}{M + h} , \quad a_h = - \cos (\phi - \phi_g) . \]

(117)

**Spherical coordinates \( \bar{\lambda}, \phi, r \):**

\[ a_\lambda = \frac{\cos \phi \sin (\bar{\lambda} - \lambda)}{r} \cos \phi \]

(108)

\[ a_{\phi} = \frac{\sin \phi \cos \phi \cos (\bar{\lambda} - \lambda) - \cos \phi \sin \phi}{r} \]

(109)
\( a_\chi = 0, \quad a_s = -\frac{\sin(\phi - \bar{\phi})}{r}, \quad a_z = -\cos(\phi - \bar{\phi}). \) (118)

**Ellipsoidal coordinates** \( \bar{\lambda}, \beta, u : \)

\[
a_\chi = 0
\]

\[
a_s = -\frac{\sin(\phi - \bar{\phi})}{L} = \frac{\nu \sin \beta \cos \phi - u \cos \beta \sin \phi}{L^2}
\]

\[
a_u = -\frac{\nu}{L} \cos(\phi - \bar{\phi}) = -\frac{\nu}{L^2} \left[ u \cos \beta \cos \phi + \nu \sin \beta \sin \phi \right].
\] (119)

The boundary condition for a rotationally symmetric normal field, with mapping along the normal to the reference ellipsoid, expressed in geodetic coordinates takes the form

\[
\Delta g - \left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} \right] \Delta W = -\left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} \right] T - \frac{\partial T}{\partial \phi}
\]

\[
\Delta g - \left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} \right] T - \frac{\partial^2 T}{\partial \phi \partial \theta} - \frac{\partial^2 T}{\partial \phi} - \frac{\partial^2 T}{\partial \theta}
\] (123)

For the mapping along the radial direction, \( a_s \) is given by (116) in combination with (63), (64), (65) and the values of \( \sin(\phi - \bar{\phi}), \cos(\phi - \bar{\phi}) \) on the ellipsoid from Appendix C, equation (C7). The resulting boundary conditions are

\[
\Delta g - \left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} + \frac{e^2 \rho N}{2 a b} \sin^2 \phi \right] \Delta W =
\]

\[
= -\left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} + \frac{e^2 \rho N}{2 a b} \sin^2 \phi \right] T - \frac{\partial T}{\partial \phi}
\] (124a)

\[
\Delta g - \left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} + 2 a b^3 M \sin^2 \phi^2 \right] \Delta W =
\]

\[
= -\left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} + 2 a b^3 M \sin^2 \phi^2 \right] T -
\]

\[
- \frac{\partial^2 T}{\partial \phi} - \frac{\partial^2 T}{\partial \theta} - \frac{\partial^2 T}{\partial \phi \partial \theta}
\] (124b)

\[
\Delta g - \left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} + \frac{e^2 \rho a}{2 b M} \sin^2 \beta \right] \Delta W =
\]

\[
= -\left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} + \frac{e^2 \rho a}{2 b M} \sin^2 \beta \right] T - \frac{\partial^2 T}{\partial \phi \partial \theta}
\] (124c)

The boundary condition for a rotationally symmetric normal field, with mapping along the radial direction, expressed in spherical coordinates takes the form

\[
\Delta g - \left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} \right] \Delta W =
\]

\[
= -\left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} \right] T - \frac{\partial T}{\partial \phi}
\] (125)

\[
\Delta g - \left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} \right] T - \frac{\partial^2 T}{\partial \phi} - \frac{\partial^2 T}{\partial \theta}
\] (126)

\[
\Delta g - \left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} \right] T - \frac{\partial^2 T}{\partial \phi \partial \theta}
\] (127)

\[
= -\left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} \right] T - \frac{\partial^2 T}{\partial \phi \partial \theta}
\] (128)

4.4. **The scalar boundary condition on the ellipsoid**

When the original data refer to the geoid and the mapping used is such that \( \Delta W = \text{const.} \), the geoid is mapped to the reference ellipsoid. On the ellipsoid (see Appendix C) \( \phi_g = \phi, \bar{\beta} = \phi, r_N = M, \quad u = b, \quad v = a \) and \( L = \frac{b N}{a} \).

Furthermore \( \frac{\gamma}{\gamma} \) is given by Bruns' equation (63) and the \( \kappa \) from (64) and (65).

Furthermore \( \frac{\gamma}{\gamma} \) is given by Bruns' equation (63) and the \( \kappa \) from (64) and (65).

For the above values and the mapping along the normal to the ellipsoid

\[
a_s = -\left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} \right]
\] (122)

and the corresponding boundary condition on the ellipsoid, for geodetic, spherical and ellipsoidal coordinates, becomes

\[
\Delta g - \left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} \right] \Delta W = -\left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} \right] T - \frac{\partial T}{\partial \phi}
\]

\[
= -\left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} \right] T - \frac{\partial^2 T}{\partial \phi} - \frac{\partial^2 T}{\partial \theta} - \frac{\partial^2 T}{\partial \phi \partial \theta}
\] (123)

For the mapping along the radial direction, \( a_s \) is given by (116) in combination with (63), (64), (65) and the values of \( \sin(\phi - \bar{\phi}), \cos(\phi - \bar{\phi}) \) on the ellipsoid from Appendix C, equation (C7). The resulting boundary conditions are

\[
\Delta g - \left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} + \frac{e^2 \rho N}{2 a b} \sin^2 \phi \right] \Delta W =
\]

\[
= -\left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} + \frac{e^2 \rho N}{2 a b} \sin^2 \phi \right] T - \frac{\partial T}{\partial \phi}
\] (124a)

\[
\Delta g - \left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} + 2 a b^3 M \sin^2 \phi^2 \right] \Delta W =
\]

\[
= -\left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} + 2 a b^3 M \sin^2 \phi^2 \right] T -
\]

\[
- \frac{\partial^2 T}{\partial \phi} - \frac{\partial^2 T}{\partial \theta} - \frac{\partial^2 T}{\partial \phi \partial \theta}
\] (124b)

\[
\Delta g - \left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} + \frac{e^2 \rho a}{2 b M} \sin^2 \beta \right] \Delta W =
\]

\[
= -\left[ \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} + \frac{e^2 \rho a}{2 b M} \sin^2 \beta \right] T - \frac{\partial^2 T}{\partial \phi \partial \theta}
\] (124c)

It must be pointed out, that the formulation of boundary conditions on the ellipsoid, for both the scalar and the vectorial problem, requires the reduction of the original data obtained on the surface of the earth to the geoid. The magnitude of the approximations involved in such a reduction limits the importance of a more precise formulation of the boundary condition.

5. The fixed boundary value problem

The fixed boundary value problem, where the boundary surface is assumed to be known, has been studied by Koch (1971), Koch & Pope (1972), Bjerhammar & Svensson (1983), Heck (1989b, 1991), Engels (1991). When gravity \( g \) is measured on the earth surface with known geometry, i.e. with known position \( x_p \) for all of its points \( P \), linearization takes place only with respect to a known
normal potential $U$:

$$g^2 = g^T g = (\gamma + \delta g)^T (\gamma + \delta g) = \gamma^T \gamma + 2 \gamma^T \delta g + \delta g^T \delta g =$$

$$= \gamma^2 - 2 \gamma n_o^T \delta g + |\delta g|^2 \quad (125)$$

$$g = \gamma \sqrt{1 - 2 \frac{1}{\gamma} n_o^T \delta g + \frac{1}{\gamma^2} \delta g^T \delta g} \quad (126)$$

which upon using the expansion of eq. (93) becomes

$$g = \gamma - n_o^T \delta g + \frac{1}{2\gamma} |\delta g|^2 - \frac{1}{2\gamma} (n_o^T \delta g)^2 \quad (127)$$

and in terms of the gravity disturbance $\delta g = g - \gamma$, the boundary condition on the earth surface takes the form

$$\delta g = -n_o^T \text{grad} T + S_{\delta g} \quad (128)$$

with second order term

$$S_{\delta g} = \frac{1}{2\gamma^2} |\text{grad} T|^2 - \frac{1}{2\gamma} (n_o^T \text{grad} T)^2 \quad (129)$$

For any set of orthogonal curvilinear coordinates the linearized boundary condition becomes

$$\delta g = -\frac{\partial T}{\partial y} \frac{\partial y}{\partial x} A_y^T e_3 = -\frac{\partial T}{\partial y} G_y^{-1/2} A_y A_0^T e_3$$

$$= \frac{\partial T}{\partial y} a_y = a_{y1} \frac{\partial T}{\partial y_1} + a_{y2} \frac{\partial T}{\partial y_2} + a_{y3} \frac{\partial T}{\partial y_3} \quad , \quad (130)$$

where

$$a_y = -G_y^{-1/2} A_y A_0^T e_3 \quad . \quad (131)$$

The matrices $G_y^{-1/2} A_y A_0^T$ are given by eqs. (51), (52) and (53) for geodetic, spherical and ellipsoidal coordinates, respectively.

For a rotationally symmetric normal field the boundary condition takes the following forms:

**Geodetic coordinates**:

$$\delta g = -\frac{\sin(\phi - \phi_0)}{M+h} \frac{\partial T}{\partial \phi} - \cos(\phi - \phi_0) \frac{\partial T}{\partial h} \quad (132)$$

**Spherical coordinates**:

$$\delta g = -\frac{\sin(\phi - \phi_0)}{r} \frac{\partial T}{\partial \phi} - \cos(\phi - \phi_0) \frac{\partial T}{\partial r} \quad (133)$$

**Ellipsoidal coordinates**:

$$\delta g = -\frac{\sin(\phi - \phi_0)}{L} \frac{\partial T}{\partial \beta} - \frac{\nu}{L} \cos(\phi - \phi_0) \frac{\partial T}{\partial u} \quad (134)$$

The boundary condition on the ellipsoid has no practical relevance, since the boundary surface for the fixed problem is the surface of the earth.

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**References**


**Appendix A:**

The computation of parameters appearing in the boundary conditions

The parameters which appear in the boundary conditions are differential parameters which are elements of the Eötvös matrix and trigonometric functions of the angles (λ−λ), φ and ψ or ψ, where ψ stands for one of the angles β, φ, or, φ, depending on the coordinate system, while ψ stands for β, φ, or φ, and thus differs from ψ only when ellipsoidal coordinates are used. The reference potential to be used is in general a function of curvilinear coordinates U = U(y) = U(λ,ψ,t), where t is one of h, r, u, depending on the coordinate system. The normal gravity vector γ in curvilinear coordinates becomes

\[
\gamma = \left( \frac{\partial U}{\partial x} \right)^T = \left( \frac{\partial y}{\partial x} \right)^T \frac{\partial U}{\partial y} = A_y^T G_y^{-1/2} \left( \frac{\partial U}{\partial y} \right)^T = \left( \frac{\partial U}{\partial y} \right)^T
\]

where the decomposition \( \frac{\partial y}{\partial x} = G_y^{-1/2} A_y \) has been used, with \( A_y = R_1(90°-\lambda) R_1(\psi-90°) G_y^{-1/2} A_y \).

Comparison with the alternative form

\[
\gamma = -\gamma A_y e_3 = -\gamma R_1(90°-\lambda) R_1(\phi-90°) e_3 = -\gamma R_1(\psi-90°) R_1(\phi-90°) e_3
\]

directly gives

\[
R_3(\lambda-\lambda) R_1(\phi-90°) e_3 = R_1(\psi-90°) \left\{ -\frac{1}{\gamma} G_y^{-1/2} \left( \frac{\partial U}{\partial y} \right)^T \right\} \equiv R_1(\psi-90°) z
\]

(A3)
which after performing the necessary computations gives
\[
\tan(\lambda - \lambda) = \frac{z_1}{\sin \Psi z_2 - \cos \Psi z_3}
\]
(A4)
\[
\tan \phi = \frac{\cos \Psi z_3 + \sin \Psi z_2}{\sqrt{z_1^2 + (\sin \Psi z_2 - \cos \Psi z_3)^2}}
\]
where for orthogonal coordinates
\[
z_1 = -\frac{1}{\gamma} w_\lambda \frac{\partial U}{\partial \lambda}, \quad z_2 = -\frac{1}{\gamma} w_\Psi \frac{\partial U}{\partial \Psi}, \quad z_3 = -\frac{1}{\gamma} w_t \frac{\partial U}{\partial t}
\]
(A5)
and the following abbreviation has been used
\[
w_\lambda = (G_{\Psi}^{-1/2})_{11}, \quad w_\Psi = (G_{\Psi}^{-1/2})_{22}, \quad w_t = (G_{\Psi}^{-1/2})_{33}
\]
(A6)

The required value of the normal gravity can be obtained from (A1) which gives
\[
\gamma^2 = \gamma^T \gamma = \frac{\partial U}{\partial \Psi} G_{\Psi}^{-1} \left( \frac{\partial U}{\partial \Psi} \right)^T = w_\lambda^2 U_\lambda^2 + w_\Psi^2 U_\Psi^2 + w_t^2 U_t^2
\]
(A7)

In the particular case of a rotationally symmetric normal potential \( U \) (with \( \bar{\lambda} = \lambda \)) it holds that \( R_i(\Phi - \Psi) = z \) and the required trigonometric functions are given by
\[
\sin(\Phi - \Psi) = z_2 = -\frac{1}{\gamma} w_\Psi \frac{\partial U}{\partial \Psi}
\]
\[
\cos(\Phi - \Psi) = z_3 = -\frac{1}{\gamma} w_t \frac{\partial U}{\partial t}
\]
(A8)

For the computation of the Eötvös matrix the derivatives of \( \gamma \) are required, which using the abbreviation \( U_\gamma = \left( \frac{\partial U}{\partial \gamma} \right)^T \) become
\[
\frac{\partial \gamma}{\partial y_i} = \frac{\partial}{\partial y_i} \left( A_\gamma^T G_{\gamma}^{-1/2} U_\gamma \right) = \\
= \frac{\partial}{\partial y_i} \left( A_\gamma^T G_{\gamma}^{-1/2} \right) U_\gamma + A_\gamma^T G_{\gamma}^{-1/2} \frac{\partial}{\partial y_i} U_\gamma = \\
= -A_\gamma^T G_{\gamma}^{-1/2} \Gamma_\gamma^T U_\gamma + A_\gamma^T G_{\gamma}^{-1/2} \frac{\partial U}{\partial y_i}
\]
(A9)

\[
\left( \Gamma_\gamma^T = -G_{\gamma}^{-1/2} A_\gamma \frac{\partial}{\partial y_i} \left( A_\gamma^T G_{\gamma}^{-1/2} \right) \right)
\]

where \( \Gamma_i \) are matrices of Christoffel symbols. The Marussi matrix is given by
\[
M = \frac{\partial \gamma}{\partial x} \frac{\partial \gamma}{\partial y} \frac{\partial \gamma}{\partial x} = \\
= -A_\gamma^T G_{\gamma}^{-1/2} \left[ \Gamma_\gamma^T U_\gamma \Gamma_\gamma^T U_\gamma \Gamma_\gamma^T U_\gamma \right] G_{\gamma}^{-1/2} A_\gamma + \\
+ A_\gamma^T G_{\gamma}^{-1/2} \frac{\partial}{\partial y} \left( \frac{\partial U}{\partial y} \right)^T G_{\gamma}^{-1/2} A_\gamma = A_\gamma^T H A_\gamma
\]
(A10)

where
\[
H = -G_{\gamma}^{-1/2} \left[ \Gamma_\gamma^T U_\gamma \Gamma_\gamma^T U_\gamma \Gamma_\gamma^T U_\gamma \right] G_{\gamma}^{-1/2} + \\
+ G_{\gamma}^{-1/2} \frac{\partial}{\partial y} \left( \frac{\partial U}{\partial y} \right)^T G_{\gamma}^{-1/2}
\]
(A11)

and the Eötvös matrix is finally computed from
\[
E = A_\gamma M A_\gamma^T = A_\gamma A_\gamma^T H A_\gamma A_\gamma^T
\]
(A12)

For the particular types of orthogonal coordinates used here we have \( A_\gamma \) independent of \( t \) and \( G_{\gamma}^{-1/2} \) independent of \( \bar{\lambda} \), which gives
\[
- G_{\gamma}^{-1/2} \Gamma_\gamma^T = [\omega_\Psi \times] G_{\gamma}^{-1/2}, \quad \omega_\Psi = R_i(90^\circ - \Psi) e_3,
\]
(A13)

\[
[\omega_\Psi \times] = \\
\begin{bmatrix}
0 & -\sin \Psi \cos \Psi \\
\sin \Psi & 0 & 0 \\
-\cos \Psi & 0 & 0
\end{bmatrix}
\]

\[- G_{\gamma}^{-1/2} \Gamma_\gamma^T = -\frac{\partial \Psi}{\partial y} [e_3 \times] G_{\gamma}^{-1/2} + \frac{\partial}{\partial y} \left( G_{\gamma}^{-1/2} \right),
\]
(A14)

\[
[e_3 \times] = \\
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{bmatrix}
\]

\[- G_{\gamma}^{-1/2} \Gamma_\gamma^T = -\frac{\partial \Psi}{\partial t} [e_3 \times] G_{\gamma}^{-1/2} + \frac{\partial}{\partial t} \left( G_{\gamma}^{-1/2} \right)
\]
(A15)

For the special case of a rotationally symmetric potential \( U \), the above relations can be further simplified due to the fact that the derivatives of \( U \) with respect to \( \bar{\lambda} \) vanish. Letting the partial derivatives of \( U \) be denoted by subscripts
\[
U_\gamma = \begin{bmatrix} 0 & U_{\gamma y} & U_{\gamma t} \end{bmatrix}, \quad \frac{\partial}{\partial y} \left( \frac{\partial U}{\partial y} \right)^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & U_{\gamma\gamma} & U_{\gamma t} \\ 0 & U_{\gamma t} & U_{tt} \end{bmatrix}
\]
(A16)

we have
\[
H = \begin{bmatrix}
H_{\gamma\gamma} & 0 & 0 \\
0 & H_{\gamma\eta} & H_{\gamma t} \\
0 & H_{\gamma t} & H_{tt}
\end{bmatrix}
\]
(A17)
\[ H_{\varphi\varphi} = w_{\varphi} \left( \frac{\partial w_{\varphi}}{\partial \varphi} U_{\varphi} + \frac{\partial w_{\varphi}}{\partial \varphi} U_{\varphi} \right) + w_{\varphi}^2 U_{\varphi\varphi} \]  
(A19)

\[ H_{\varphi\eta} = w_{\varphi} \left( \frac{\partial w_{\varphi}}{\partial \eta} U_{\varphi} + \frac{\partial w_{\varphi}}{\partial \eta} U_{\varphi} \right) + w_{\varphi}^2 U_{\varphi\eta} \]  
(A20)

\[ H_{\eta\eta} = w_{\eta} \left( \frac{\partial w_{\eta}}{\partial \eta} U_{\eta} + \frac{\partial w_{\eta}}{\partial \eta} U_{\eta} \right) + w_{\eta}^2 U_{\eta\eta} \]  
(A21)

Finally the Eötvös matrix can be computed from

\[ E = R_{1}(\psi - \phi) H R_{1}(\phi - \psi) \]  
(A22)

where the trigonometric functions of \((\phi - \psi)\) are given by (A8), with \(\gamma\) from equation (A7), which for a rotationally symmetric field simplifies to

\[ \gamma^2 = w_{\varphi}^2 U_{\varphi}^2 + w_{\eta}^2 U_{\eta}^2 \]  
(A23)

**Appendix B:**

**The Bruns formula for the vectorial problem**

The Bruns formula expresses the position anomaly vector \(\zeta\) as a function of the disturbing potential \(T\). It follows directly from equation (12) written in the form

\[ \zeta = M^{-1} \Delta g - M^{-1} \text{grad} T \]  
(B1)

which upon using \(\Delta g = -(\gamma + \Delta g) n + \gamma n_o\), \(n = A^T e_3\), \(n_o = A_o^T e_3\), \(M^{-1} = A_o^T E^{-1} A_o\),

\[ c = n^* = A_o n = A_o A^T e_3, \]

\[ \text{grad} T = \left( \frac{\partial T}{\partial y} \right)^T = A_y^T G_y^{-1/2} \left( \frac{\partial T}{\partial y} \right)^T \]

becomes

\[ \zeta^* = \begin{bmatrix} \zeta_{E} \\ \zeta_{N} \end{bmatrix} = A_o \zeta = \begin{bmatrix} \gamma H_1 \\ \gamma H_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \gamma H_1 \\ \gamma H_2 \end{bmatrix} \]  
(B2)

where \(\zeta^*\) is the position anomaly vector expressed in the local normal astronomic frame. Equation (B2) is not yet Bruns formula since it depends not only on \(T\) but also on \(\Delta g\). However, when a gravimetric type of telluroid mapping is used where \(\Delta g = 0\), or more generally \(\Delta g = \text{constant}\), equation (B2) becomes automatically Bruns formula. In the different case where a Marussi type of telluroid mapping is used with \(\Delta W = 0\), or \(\Delta W = \text{constant}, \Delta g\) must be eliminated by using equation (48), which with the help of (49) can be written in the form

\[ \tau (\gamma + \Delta g) = \gamma a_o (T - \Delta W) + \gamma e_3 e_3^T E^{-1} A_o A_y^T G_y^{-1/2} \left( \frac{\partial T}{\partial y} \right)^T \]  
(B3)

Elimination of \(\gamma + \Delta g\) results in the Bruns formula

\[ \zeta^* = \begin{bmatrix} \frac{T}{\gamma} + \frac{a_o}{\gamma} (T - \Delta W) \end{bmatrix} E^{-1} c + \gamma E^{-1} e_3 + \begin{bmatrix} 1 + \frac{a_o}{\gamma} \end{bmatrix} E^{-1} A_o A_y^T G_y^{-1/2} \left( \frac{\partial T}{\partial y} \right)^T \]  
(B4)

When a telluroid mapping with \(n_p = n_o q\) (i.e., \(g_p // \gamma q\)) is used, such as the classical Marussi mapping, it holds that \(c = e_3, \tau = 1\) and the Bruns formula is simplified to

\[ \zeta^* = \frac{T - \Delta W}{\gamma} (-\gamma a_o E^{-1} e_3) - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \gamma H_1 \\ \gamma H_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} T - \Delta W \gamma \\ 1 \end{bmatrix} = \begin{bmatrix} \gamma H_1 \\ \gamma H_2 \end{bmatrix} \]  
(B5)

where \(a_o\) is given by equation (41), \(E^{-1}\) by equation (35) and \(A_o A_y^T G_y^{-1/2}\) by equations (51), (52), (53), according to the coordinate system used.

In the particular case of a rotationally symmetric field, \(a_o\) is given by equation (57), \(E^{-1}\) by equation (80) and the Bruns formula (B5) takes the form

\[ \zeta^* = \frac{T - \Delta W}{\gamma} \begin{bmatrix} 0 & \frac{r_t}{\gamma} \\ \frac{r_t}{\gamma} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi - \phi \end{bmatrix} G_y^{-1/2} \left( \frac{\partial T}{\partial y} \right)^T \]  
(B6)

where \(\psi\) is one of \(\beta, \phi \) and \(\delta\). In the three particular orthogonal coordinate systems used the components of \(\zeta^*\) become

\[ \zeta_{E} = \frac{r_t}{\gamma (N + h) \cos \phi} \frac{\partial T}{\partial \lambda} = \frac{r_t}{\gamma \tau \cos \phi} \frac{\partial T}{\partial \lambda} = \frac{r_t}{\gamma \cos \beta} \frac{\partial T}{\partial \lambda} \]  
(B7a)

\[ \zeta_{N} = -\frac{r_t}{\gamma} \frac{T - \Delta W}{\gamma} + \frac{r_t}{\gamma} \left[ \frac{\cos(\phi - \delta) \frac{\partial T}{\partial \phi} - \sin(\phi - \delta) \frac{\partial T}{\partial \delta}}{r} \right] = \]  
(B7b)

\[ \zeta_{Z} = \frac{T - \Delta W}{\gamma} \]  
(B7c)
On the ellipsoid \( r = N \), \( r_N = M \), \( \phi = \beta \), \( x \) is given by equation (A34) and \( \cos(\phi - \beta) \), \( \sin(\phi - \beta) \) from equation (A7). The Brun's formula takes on the reference ellipsoid the form

\[
\zeta = \frac{1}{\gamma N} \frac{\partial T}{\partial \lambda} = \frac{N \partial T}{\gamma r \cos \phi} = \frac{N \partial T}{\gamma a \cos \phi}
\]

(B8a)

\[
\zeta_N = -\frac{a M}{b N} \partial \phi \sin \phi \frac{T - AW}{\gamma} + \frac{1}{\gamma \phi} \partial T = -\frac{r^2}{a b} \partial \phi \sin \phi \frac{T - AW}{\gamma} + \frac{M a^2}{N} \left( \frac{1}{r^2 \phi} - \frac{e^2 r a^2 \partial T}{2 N b^2 \partial r} \right) =
\]

(B8b)

\[
\zeta = T - AW \gamma \cdot \phi \frac{N}{\gamma}
\]

(B8c)

\[
\sin 2\beta = \frac{N^2 b}{a^3} \sin 2\phi = \frac{r^2}{a b} \sin 2\phi
\]

(C5)

From the definitions of \( v = \sqrt{u^2 + v^2} \) and \( L = \sqrt{v^2 + E^2 \sin^2 \beta} \) it is obvious that on the ellipsoid where \( u = b \), it holds that \( v = a \) and

\[
L = \sqrt{b^2 + (a^2 - \gamma^2) \sin^2 \beta} = \frac{b N}{a}
\]

(C6)

From (C1) it is obvious that \( \cos \phi = \frac{r}{N} \cos \phi \), and

\[
\sin \phi = \frac{a^2 r}{b^2 N} \sin \phi
\]

which can be used to obtain

\[
\sin(\phi - \beta) = \frac{e^2 r a^2}{2 N b^2} \sin 2\phi \quad \cos(\phi - \beta) = \frac{a^2}{r \phi}
\]

(C7)

Using the Somigliana-Pizzetti field (Heiskanen and Moritz, 1967)

\[
U = \frac{kM}{E} \arctan \left( \frac{E}{u} \right) + \frac{\omega^2 a^2 q}{2 q_o} \left( \sin^2 \beta - \frac{1}{3} \right) + \frac{\omega^2}{2} v^2 \cos^2 \beta
\]

(C8)

direct differentiation and restriction to the geoid (u-b) gives

\[
U_u = \frac{\partial U}{\partial u} = -\frac{kM}{v^2} + \frac{\omega^2 a^2 q}{2 q_o} \left( \sin^2 \beta - \frac{1}{3} \right) + \omega^2 u \cos^2 \beta
\]

(C9)

\[
U_\beta = \frac{\partial U}{\partial \beta} = \frac{\omega^2 a^2 q}{2 q_o} \sin 2\beta - \frac{\omega^2}{2} v^2 \sin 2\beta =
\]

\[
= \frac{\omega^2 a^2 q}{2 q_o} \sin 2\beta - \frac{\omega^2}{2} a^2 \sin 2\beta = 0
\]

(C10)

\[
U_q = \frac{\partial^2 U}{\partial q \partial u} = \omega^2 \sin 2\beta \left( \frac{a^2 q}{2 q_o} - u \right) =
\]

\[
= \omega^2 \sin 2\beta \left( \frac{a^2 q}{2 q_o} - b \right)
\]

(C11)

\[
U_{uu} = \frac{\partial^2 U}{\partial u^2} = 2k \frac{kM}{v^4} + \frac{\omega^2 a^2 q}{2 q_o} \left( \sin^2 \beta - \frac{1}{3} \right) + \omega^2 v^2 \cos^2 \beta =
\]

\[
= 2b \frac{kM}{a^4} + \frac{\omega^2 a^2 q'}{2 q_o} \left( \sin^2 \beta - \frac{1}{3} \right) + \omega^2 v^2 \cos^2 \beta
\]

(C12)

\[
U_{\beta\beta} = \frac{\partial^2 U}{\partial \beta^2} = \omega^2 a^2 q_o \cos 2\beta - \frac{\omega^2}{2} v^2 \cos 2\beta =
\]

\[
= \omega^2 a^2 q_o \cos 2\beta - \frac{\omega^2}{2} a^2 \cos 2\beta = 0
\]

(C13)

In the above equations \( q = q(u) \), \( q_o = q(b) \), \( q' = \frac{dq}{du} \) and \( q' = q'(b) \). The gravity vector in ellipsoidal coordinates becomes
\[
\gamma = \left( \frac{\partial U}{\partial x} \right)^T = \left( \frac{\partial U}{\partial y} \right)^T \left( \frac{\partial U}{\partial y} \right)^T = \\
R_3(-90^\circ-\lambda) R_1(\beta-90^\circ) G^{-1/2}_{\chi} \left( \frac{\partial U}{\partial y} \right)^T = \\
R_3(-90^\circ-\lambda) R_1(\beta-90^\circ) \begin{bmatrix}
\frac{1}{\sqrt{\nu \cos \beta}} & 0 & 0 \\
0 & \frac{1}{L} & 0 \\
0 & 0 & \frac{v}{L}
\end{bmatrix} \begin{bmatrix}
0 \\
U_b \\
U_u
\end{bmatrix}
\]  
(C14)

which used in \( \gamma^2 = \gamma^T \gamma \) gives on the ellipsoid (\( U_b = 0 \))

\[
U_u = -\gamma \frac{L}{v} = -\gamma \frac{L}{a}
\]  
(C15)

From (C15) and (C9) we can derive \( \gamma \) on the ellipsoid and in particular at the equator (\( \beta=0, \alpha=0 \))

\[
\gamma_e = \frac{a^2 M}{b} + \frac{\omega^2 a^2 b}{6 b} q_0 - a \omega^2,
\]  
(C16)

and the pole (\( \beta=0, \alpha=0 \))

\[
\gamma_p = \frac{a^2 M}{b^2} - \frac{\omega^2 a^2 b}{3 b} q_0,
\]  
(C17)

Using the dynamic flattening \( f^* = \frac{\gamma_p - \gamma_e}{\gamma_e} \) and the geometric one \( f = \frac{a-b}{a} \) we have

\[
f^* + f = \frac{a \gamma_p - b \gamma_e}{a \gamma_e} = \omega^2 \left( \frac{b-a^2 q'_0}{2 q_0} \right),
\]  
(C18)

which can be used in (C11) to obtain

\[
U_{ub} = -\sin 2\beta \gamma_e (f^* + f).
\]  
(C19)

For the computation of the Eötvös matrix on the ellipsoid equations (A17) - (A22) should be applied with auxiliary parameters provided by equations (A6) and A(8). In this case \( \Psi = \beta, \bar{\Psi} = \bar{\beta}, t = u, w_y = w_3 = \frac{L}{1}, w_z = w_u = \frac{v}{L} \)

\[
\bar{a} = \frac{a}{L}, \bar{w}_\chi = \frac{1}{\nu \cos \beta} = \frac{1}{a \cos \beta}, \text{ and equation (A8) gives}
\]

\[
\sin(\phi-\bar{\beta}) = -\frac{1}{\gamma} \frac{1}{L} U_b = 0
\]  
(C20)

\[
\cos(\phi-\bar{\beta}) = -\frac{1}{\gamma} \frac{v}{L} U_u = 1
\]

where equation (C15) has been taken into account. This means that \( \bar{\beta} = \phi \) on the ellipsoid, \( R_1(\phi-\bar{\beta}) = I \) and the Eötvös matrix becomes according to (A17)-A(22)

\[
E = H = \begin{bmatrix}
E_{\chi \chi} & 0 & 0 \\
0 & E_{\phi \phi} & E_{\phi \chi} \\
0 & E_{\phi \chi} & E_{\chi \chi}
\end{bmatrix} = \begin{bmatrix}
\frac{-\gamma}{N} & 0 & 0 \\
0 & \frac{-\gamma}{M} & \gamma \chi \\
0 & \gamma \chi & \gamma \chi
\end{bmatrix}
\]  
(C21)

the last term coming from the representation of \( E \) (eq. 34) with \( k_e = \frac{1}{N}, k_n = \frac{1}{M} \) on the ellipsoid. Obviously \( \nu_n = \nu_e = 0 \) on the ellipsoid, while \( \nu_n = \chi \) has been set. The non-zero elements of the Eötvös matrix are given from equations (A18)-(A21):

\[
E_{\chi \chi} = w_\chi (-\sin \bar{\beta} w_\beta U_b + \cos \bar{\beta} w_\alpha U_u) = \bar{a} \bar{L} w_\alpha U_u = \frac{-\gamma}{N}
\]  
(C22)

\[
E_{\phi \phi} = w_\phi \frac{\partial \bar{\beta}}{\partial \phi} w_\alpha U_u + \frac{\partial w_\phi}{\partial \bar{\beta}} U_u + w_\phi^2 U_{\phi \phi} = \\
-\frac{a^2 b}{L^4} u = -\frac{\gamma}{M}
\]  
(C23)

\[
E_{\phi \chi} = w_\phi \left( \frac{\partial \bar{\beta}}{\partial \phi} w_\phi U_u + \frac{\partial w_\phi}{\partial \bar{\beta}} U_u \right) + w_\phi^2 U_{\phi \chi} = \\
-\frac{\gamma k}{L^4} u + \frac{a}{2L} u^2 = -\gamma \chi
\]  
(C24)

\[
E_{\chi \chi} = w_\chi \left( \frac{-\bar{\beta}}{L^2} w_\beta U_b + \frac{\partial w_\chi}{\partial \phi} U_u \right) + \bar{a}^2 \bar{L}^2 U_{\chi \chi} = \\
-\frac{b (L^2 - a^2)}{L^4} U_u = -\gamma \chi
\]  
(C25)

where use has been made of the derivatives

\[
\frac{\partial \bar{\beta}}{\partial \beta} = \frac{u v}{L^2} = \frac{a b}{L^2},
\]  
(C26)

\[
\frac{\partial \bar{\beta}}{\partial \phi} = -\frac{\chi}{\nu L^2} = -\frac{\chi}{a L^2} \quad \left( \chi = \frac{1}{2} E^2 \sin 2\beta \right)
\]  
(C27)

\[
\frac{\partial w_\phi}{\partial \phi} = -\frac{\chi v}{L^2} = -\frac{\chi v}{a L^2}
\]  
(C28)

Utilization of the values of \( U_u \) and \( U_{\phi \beta} \) from equations (C15) and (C19), respectively, gives

\[
N = \frac{a L}{b} \frac{a^2 b}{b} \sqrt{1 - e^2 \cos^2 \bar{\beta}} \quad \Rightarrow \quad L = \frac{b N}{a}
\]  
(C29)
\[ M = \frac{L^3}{a b} = \frac{b^2 N^3}{a^4} \]

\[ \chi = -\frac{X}{L^3} + \frac{a}{L^2} \sin 2\beta \gamma_c (f^* + f) = \frac{\Omega}{M} \sin 2\beta = \frac{a \Omega}{b N} \sin 2\phi_c = \frac{r^2 \Omega}{a b M} \sin 2\phi \]  

where the parameter

\[ Q = \frac{N \gamma_x}{a} \left( f^* + f \right) - \frac{e^2}{2} \frac{a}{b} \]

has been introduced, and the relations between the three angles from equation (1) have been used. For the term \( \gamma_x \) it is more convenient not to evaluate \( U_{uu} \) but rather to take advantage of the fact that \( \Delta U = \text{trace}(E) = 2\omega^2 \) to obtain

\[ \frac{\gamma_x}{\gamma} = -\left( \frac{1}{M} + \frac{1}{N} + \frac{2\omega^2}{\gamma} \right). \]