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APPLICATIONS OF STRAIN CRITERIA IN CARTOGRAPHY

Summary

Some of the most commonly used map projections, are analysed from the viewpoint of the deformation introduced, when the objective spherical surface is mapped into a plane. After a short theoretical account, formulae for the computation of dilatation and maximum shear strain are given for various projections. These two invariant quantities may be used as criteria for the choice of a map projection in addition to the traditional criteria of conformality, equivalence, etc .

Introduction

Applications of the Theory of Elasticity in Cartography appear in the literature since the late 19th Century (Tissot 1881, Fiorini 1881). Interesting work has been carried out up to now, utilizing in abstraction the tools of mechanics for the study of deformations induced when original figures on a Riemmanian space, (sphere or ellipsoid) considered as the objective space, are mapped by a one–to–one correspondance on a Euclidean space (see, e.g., Marussi 1951, 1959,Swonarew 1953,Bernasconi 1953, 1956, Caputo 1956, Biernacki 1965, Bocchio 1969, Chovitz 1979, Hojovec and Jokl 1981, Dermanis and Livieratos 1981a, 1981b).

In this paper we are concerned with the representation of the invariant criteria of dilatation and maximum shearing strain, on some known cylindrical, conic and azimuthal cartographic projections. This is done using the local strain tensors derived by mapping differentially the local azimuthal plane onto the plane of representation. The analysis follows the Lagrangian approach, thus the deformation maps are referred to the objective surface of a sphere. The detailed theory and the mathematical tools are given in previous works by Dermanis and Livieratos [1981a, 1981b, 1983] .

Lagrangian cartographic strain

Considering the vectors of coordinates

$$\underline{\varphi} = (\lambda, \varphi)^T \tag{1}$$

$$\underline{x} = (x, y)^T \tag{2}$$

$$\underline{s} = (s_\lambda . s_\varphi)^T \tag{3}$$

where λ ,φ the orthogonal (longitude and latitude) spherical coordinates on a unit

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sphere, x, y the cartesian coordinates on the map plane (projection plane) and s_λ, s_φ the coordinates on the local tangent plane (azimuthal plane) which are lengths along the local parallel and meridian respectively. The differential mappings of $\underline{\varphi}$ onto $\underline{x}$, of $\underline{\varphi}$ onto $\underline{s}$ and of $\underline{s}$ onto $\underline{x}$, will be respectively

$$d\underline{x} = \underline{J}\ d\underline{\varphi} \quad (4$$

$$d\underline{s} = \underline{Q}\ d\underline{\varphi} \quad (5$$

$$d\underline{x} = \underline{S}\, d\underline{s} \quad (6$$

where $\underline{J}$, $\underline{Q}$, $\underline{S}$ the Jacobians

$$\underline{J} = \frac{\partial (x, y)}{\partial (\lambda, \varphi)}, \qquad \underline{Q} = \frac{\partial (s_\lambda, s_\varphi)}{\partial (\lambda, \varphi)}, \qquad \underline{S} = \frac{\partial (x, y)}{\partial (s_\lambda, s_\varphi)} . \quad (7$$

It follows that

$$\underline{S} = \underline{J}\ \underline{Q}^{-1} \quad (8$$

when $\det(\underline{Q}) \neq 0$.

Due to the orthogonality of the λ, φ geographical grid the matrix Q is diagona

$$\underline{Q} = \begin{bmatrix} \cos\varphi & 0 \\ 0 & 1 \end{bmatrix} \quad (9$$

since

$$\begin{aligned} ds_\lambda &= \cos\varphi\ d\lambda \\ ds_\varphi &= d\varphi . \end{aligned} \quad (10$$

The Lagrangian strain tensor $\underline{E}_L$, which is associated with the mappin $(ds_\lambda, ds_\varphi) \rightarrow (dx, dy)$ involves the Jacobian $\underline{S}$ (Dermanis and Livieratos, 1981a), anc it is written

$$\underline{E}_L = \frac{1}{2} (\underline{S}^T \underline{S} - \underline{I}) \quad (11$$

where $\underline{I}$ the identity matrix.

Replacing (8) into (11) we obtain

$$\underline{E}_L = \frac{1}{2} (\underline{Q}^{-1} \underline{J}^T \underline{J}\ \underline{Q}^{-1} - \underline{I}) \quad (12)$$

where $\underline{J}^T \underline{J}$ is the well-known matrix of the Gauss differential forms.

$$\underline{J}^T \underline{J} = \begin{bmatrix} g & f \\ f & e \end{bmatrix} \quad (13)$$

with

$$g = \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2$$

$$f = \frac{\partial x}{\partial \lambda}\frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial \lambda}\frac{\partial y}{\partial \varphi} \qquad (14)$$

$$e = \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 .$$

Cartographic dilatation and maximum shear strain

The Lagrangian strain ellipse is computed by applying the eigenvalue–eigenvector problem on (12). The maximum and minimum semi–axes a_L and b_L of the ellipse are given by

$$a_L = \frac{\Delta_L + \gamma_L}{2} \qquad (15)$$

$$b_L = \frac{\Delta_L - \gamma_L}{2} \qquad (16)$$

where

$$\Delta_L = \frac{1}{2}\left(\frac{g}{cos^2 \varphi} + e\right) - 1 \qquad (17)$$

$$\gamma_L = \frac{\sqrt{(g - e\, cos^2 \varphi)^2 + (2 f cos \varphi)^2}}{2\, cos^2 \varphi} \qquad (18)$$

and the direction ψ_L of a_L is given by

$$\psi_L = \frac{1}{2} arctan\left(\frac{-2 f cos \varphi}{g - e\, cos^2 \varphi}\right) \qquad (19)$$

Δ_L and γ_L in (15) and (16) as defined in (17) and (18) are invariant, unitless scalar quantities, called in continuum mechanics the dilatation and maximum shear strain respectively.

Dilatation represents the isotropic part of deformation. Δ_L is called dilatation following the current terminology in continuum mechanics. However it does not exactly correspond to change of area per unit area (true dilatation), but only approximately for infinitesimal deformations.

Maximum shear strain represents the anisotropic part of deformation and it is the shear across the direction of its maximum value (always positive). Its significance is alteration in shape independently of magnification or reduction.

From the well–known conformality conditions concerning the Gauss differential forms

$$f = 0 \qquad g = e \cos^2 \varphi \tag{20}$$

it follows that

$$\gamma_L = 0 \qquad \Delta_L = e - 1 \tag{21}$$

for conformal projections.

It is also known that if an area element dA on the unit sphere is mapped onto an element dA' on the plane, then

$$\frac{dA'}{dA} = \frac{\sqrt{ge - f^2}}{\cos\varphi} \tag{22}$$

For conformal projections conditions (20) give

$$\frac{dA'}{dA} = e \tag{23}$$

The change of area per unit area (true dilatation) is

$$\frac{dA' - dA}{dA} = e - 1 = \Delta_L \tag{24}$$

It follows that for conformal projections, Δ_L corresponds exactly to true dilatation.

Perhaps the use of the term dilatation for the isotropic part of deformation is somewhat misleading in the case of map deformation. Δ_L is called dilatation within the Theory of Linear Elasticity, where for infinitesimal deformations it is a first–order approximation to true dilatation. Here we call dilatation the isotropic part of deformation for a conceptual linkage, with the terminology in Elasticity.

Δ_L and γ_L are obviously related to the semiaxes a_T, b_T of the Tissot indicatrix. In fact it can be shown (Dermanis and Livieratos, 1981a) that

$$\Delta_L = \frac{a_T^2 + b_T^2}{2} - 1 \tag{25}$$

$$\gamma_L = \frac{a_T^2 - b_T^2}{2} \tag{26}$$

In *Table 1* expressions for Δ_L and γ_L are given concerning twelve map projections used in cartography. Some examples shown in *Figures 1, 2, 3* illustrate the isotropic and anisotropic properties of several map projections in terms of dilatation and maximum shear strain.

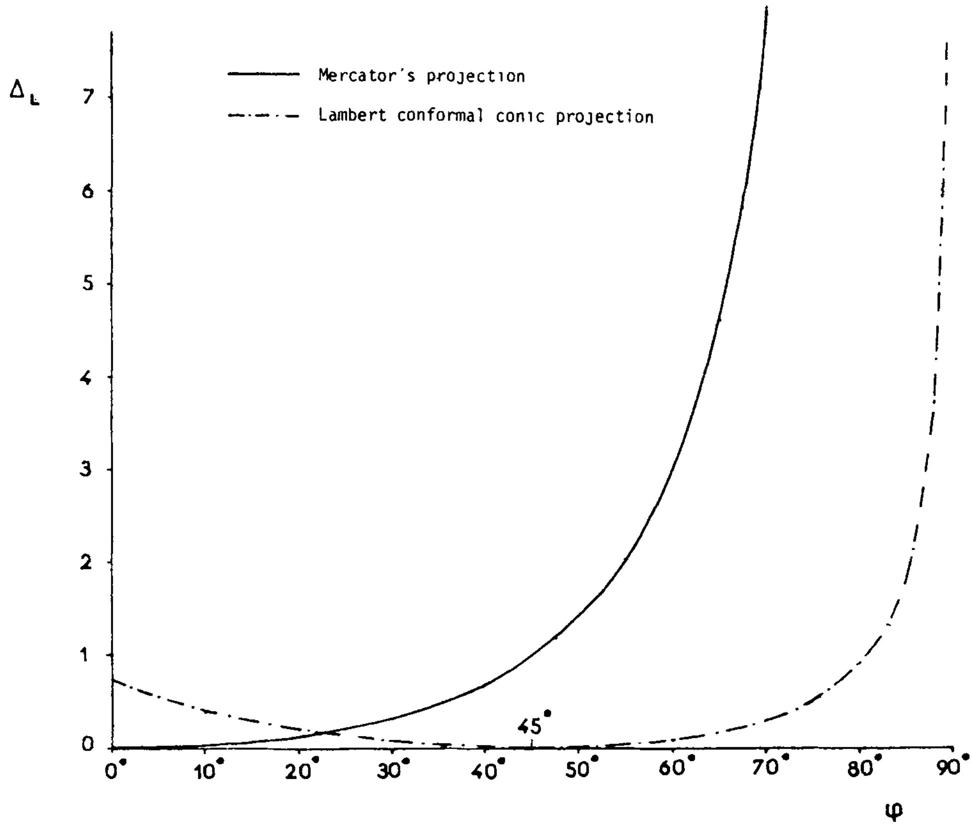

Fig. 1 — Mercator and Lambert conformal conic projection. Dilatation Δ_L is a function of latitude only.

Concluding Remarks

The study of the invariant strain parameters Δ_L (dilatation) and γ_L (maximum shear strain) has been carried out in their Lagrangian expression, for several known map projections. These parameters represent the isotropic and anisotropic behaviour of map deformation and allow deeper understanding of geometric alternations of the map, even when the maps under comparison belong to the same projectional family due to their traditional properties, i.e. conformality, equivalence, etc. This could be of interest not only in Mathematical but also in Thematic Cartography.

For the family conformal projections the anisotropic part of deformation vanishes ($\gamma_L = 0$) while the isotropic part Δ_L corresponds exactly to the change of area per unit area (true dilatation) and can be used as a criterion for the comparison between conformal projections. For the purpose of illustration *Fig. 1* shows Δ_L for two projections, where Δ_L is longitude independent. *Fig. 2* is a comparison between Δ_L for the Mercator and the Transverse Mercator projection.

For the family of equal–area projections, Δ_L does not vanish, while true

Table 1

Expressions for Δ_L and γ_L for several commonly used map projections.
(Unit sphere to plane.)

Projection	Transformation relations	Auxiliary expressions	Dilatation Δ_L	Maximum shear strain γ_L
Cylindrical equal - spaced	$x = \lambda$ $y = \varphi$		$\frac{\tan^2\varphi}{2}$	$\frac{\tan^2\varphi}{2}$
Mercator's	$x = \lambda$ $y = \ln(\tan(\frac{\pi}{4} + \frac{\varphi}{2}))$		$\tan^2\varphi$	0
Cylindrical equal - area	$x = \lambda$ $y = \sin\varphi$		$\frac{\tan^2\varphi \sin^2\varphi}{2}$	$\frac{\tan^2\varphi + \sin^2\varphi}{2}$
Transverse Mercator's	$x = \ln(\tan(\frac{\pi}{4} + \frac{v}{2}))$ $y = u$	$\sin v = \cos\varphi \sin\lambda$ $\cot u = \cot\varphi \cos\lambda$	$\frac{\cos^2\lambda + \sin^2\varphi \sin^2\lambda}{\cos^4\varphi(\cos^2\lambda + \tan^2\varphi)^2} - 1$	0
Simple conical with one standard parallel	$x = \rho \sin\theta$ $y = \rho_o - \rho \cos\theta$	$\rho_o = \cot\varphi_o$ $\rho = \rho_o + (\varphi_o - \varphi)$ $\theta = \sin\varphi_o \lambda$	$\frac{\sin^2\varphi_o \rho^2}{2 \cos^2\varphi} - \frac{1}{2}$	$\frac{\sin^2\varphi_o \rho^2}{2 \cos^2\varphi} - \frac{1}{2}$
Lambert conformal conic	$x = \rho \sin\theta$ $y = \rho_o - \rho \cos\theta$	$\rho_o = \cot\varphi_o$, $\kappa = \sin\varphi_o$ $\rho = \rho_o(\frac{\tan(\pi/4+\varphi_o/2)}{\tan(\pi/4+\varphi/2)})^\kappa$ $\theta = \sin\varphi \lambda / \rho$	$\frac{\sin^2\varphi_o \rho^2}{\cos^2\varphi} - 1$	0
Bonne equal - area	$x = \rho \sin\theta$ $y = \rho_o - \rho \cos\theta$	$\rho_o = \cot\varphi_o$ $\rho = \rho_o + (\varphi_o - \varphi)$ $\theta = \cos\varphi \lambda / \rho$	$\frac{(\theta - \sin\varphi \lambda)^2}{2}$	$A = \sqrt{(4+(\theta - \lambda \sin\varphi)^2)}$ $\frac{(\theta - \sin\varphi \lambda) A}{2}$

Table 1 *(continued)*

Projection	Transformation relations	Auxiliary expressions	Dilatation Δ_L	Maximum shear strain γ_L
Werner equal - area	$x = \rho \sin\theta$ $y = -\rho \cos\theta$	$\rho = \frac{\pi}{2} - \varphi$ $\theta = (\pi/2-\varphi)\,\lambda \cos\varphi$	$A = \lambda(\sin\varphi - (\cos\varphi/(\pi/2 - \varphi)))$ A^2	$(A/2)\sqrt{(4 + A^2)}$
Lambert conformal conic, with two standard parallels (φ_1, φ_2)	$x = \rho \sin\theta$ $y = \rho_o - \rho \cos\theta$	$\omega_1 = \pi/4 - \varphi_1/2$ $\omega_2 = \pi/4 - \varphi_2/2$ $\rho = C(\tan(\pi/4-\varphi/2))^{\sin\varphi_o}$ $\rho_o = \cot\varphi_o$ $\sin\varphi_o = \frac{\ln(\cos\varphi_1/\cos\varphi_2)}{\ln(\tan\omega_1/\tan\omega_2)}$ $C = \frac{\cos\varphi_1}{\sin\varphi_o(\tan\omega_1)^{\sin\varphi_o}} = \frac{\cos\varphi_2}{\sin\varphi_o(\tan\omega_2)^{\sin\varphi_o}}$ $\theta = \lambda \sin\varphi_o$	$\frac{\sin^2\varphi_o\, \rho^2}{\cos^2\varphi} - 1$	$\emptyset$
Sinusoidal equal area (Sanson - Flamsteed)	$x = \lambda \cos\varphi$ $y = \varphi$		$A = \lambda^2 \sin^2\varphi$ $A/2$	$(\sqrt{A}/2)\sqrt{(4+A)}$
Stereographic polar	$x = \rho \sin\lambda$ $y = -\rho \cos\lambda$	$\rho = 2 \tan(\pi/4 - \varphi/2)$	$\frac{1}{2}(\frac{\rho^2}{\cos^2\varphi} + \frac{4}{(1+\sin\varphi)^2}) - 1$	$\frac{1}{2\cos^2\varphi}\sqrt{(\rho^2 - \frac{4\cos^2\varphi}{(1+\sin\varphi)^2})}$
Gall's stereographic	$x = \frac{\sqrt{2}}{2}\lambda$ $y = \frac{\sqrt{2}+2}{2} \tan\frac{\varphi}{2}$		$A = \frac{3+2\sqrt{2}}{4(1+\cos\varphi)^2}$ $1/(2\cos^2\varphi)+A-1$	$1/(4\cos^2\varphi)-A$

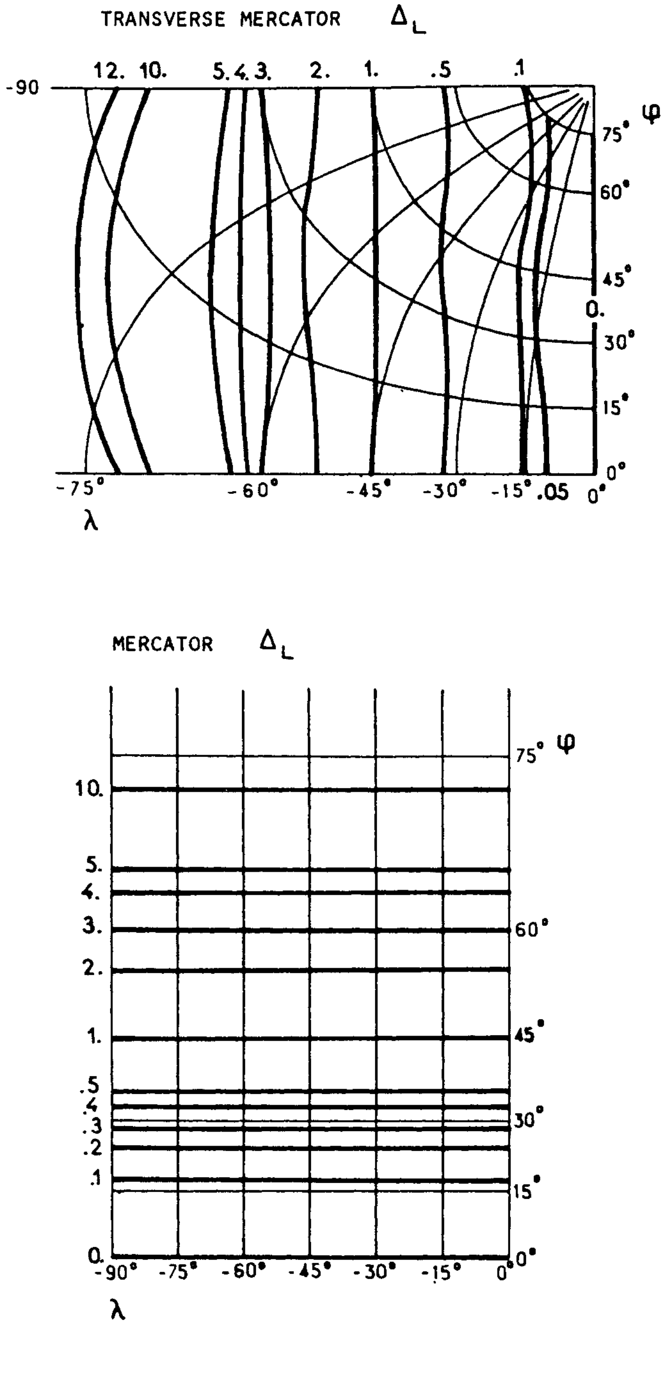

Fig. 2 – Transverse Mercator and Mercator projection. Dilatation Δ_L in Transverse Mercator is a function of latitude and longitude, while in Mercator is a function of latitude only.

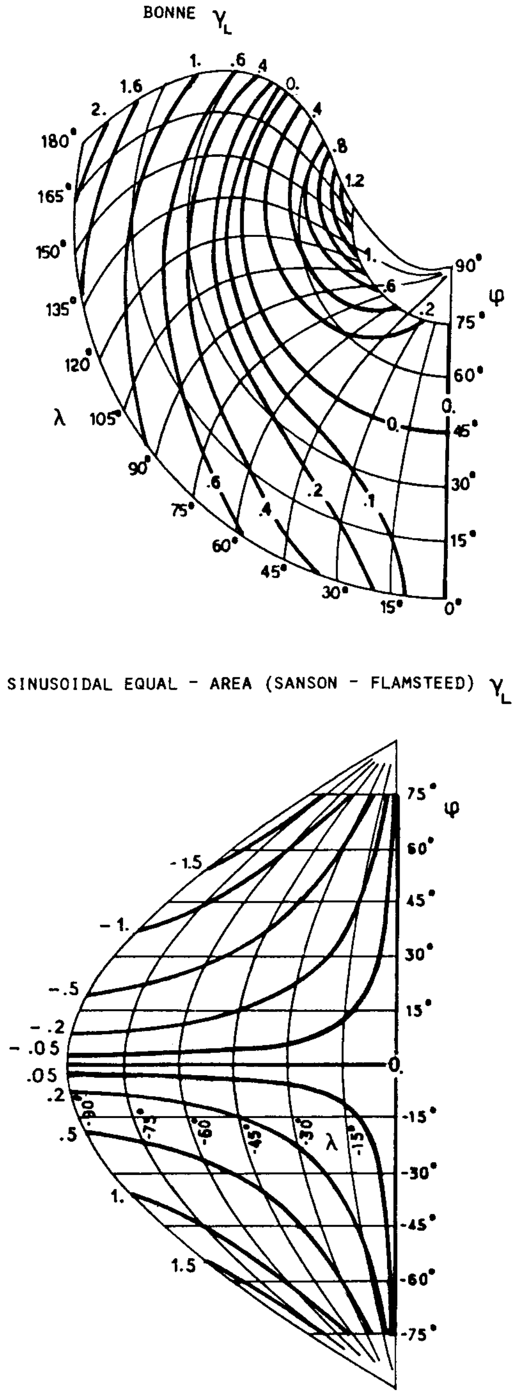

Fig. 3 — Sinusoidal equal—area (Sanson — Flamsteed) and Bonne projection. Maximum shear strain γ_L is a function of latitude and longitude.

dilatation does. In this case the anisotropic part γ_L giving a measure of the "change of shape" is a criterion for comparison between equal–area projection. In *Fig. 3* γ_L is presented for two equal–area projections.

For map projections outside the above main two families, both the isotropic part Δ_L and the anisotropic part γ_L of deformation may serve as criteria for comparison. However γ_L representing deformation of shape has a more obvious significance especially for maps, where preservation of shape is an important requirement. We note that for the two particular equidistant projections at *Table 1,* the isotropic and anisotropic parts of deformations coincide $(\Delta_L = \gamma_L)$.

In addition to the comparison of different map projections, Δ_L and γ_L may also be used for the selection of the appropriate standard parallel(s) in any specific projection, i.e., a conic projection depending on one or two such parallels. If the projection is to be used for the region $\lambda_1 \leqslant \lambda \leqslant \lambda_2$, $\varphi_1 < \varphi < \varphi_2$, optimality criteria may be introduced in the form,

$$\int_{\lambda_1}^{\lambda_2} \int_{\varphi_1}^{\varphi_2} |\Delta_L|^p \cos\varphi \, d\varphi \, d\lambda = min \tag{27}$$

e.g., for the Lambert conformal conic projection with one or two standard parallels or in the form,

$$\int_{\lambda_1}^{\lambda_2} \int_{\lambda_2}^{\varphi_2} | \gamma_L|^p \cos\varphi \, d\varphi \, d\lambda = min \tag{28}$$

e.g., for the Bonne equal–area projection, where p an integer, usually $p = 1$ or $p = 2$. The optimal values of the latitudes of the standard parallel(s) can be found either analytically utilizing techniques from the calculus of variations which is quite a difficult task, or numerically utilizing techniques from optimization theory.

Generally speaking, the strain criteria in map projections compared to the traditional global properties, of conformality, equivalence, etc., can be considered as intrinsic deformation properties of maps.

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