Global reference frames: Connecting observation to theory and geodesy to geophysics

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Abstract. Starting from a global geodetic reference frame, such as the ITRF, a methodology is developed for its conversion to an estimate of a geophysically meaningful earth reference frame, such as the Tisserand axes frame employed in theories of the rotation of the earth. This conversion allows a more straightforward comparison of the observed rotation of the frame with respect to an inertial celestial frame, with the one predicted from various theories of earth rotation. Assuming that the geodetic frame is realized by the time-dependent coordinates of a set of stations distributed over the various tectonic plates, the parameters of rigid motion for each separate plate are estimated through the solution of relevant differential equations. The contribution to the angular momentum relative to the frame is computed for each plate, with the help of adopted terrain-bathymetry, lithospheric depth and density models. Finally a set of differential equations is derived, the solution of which provides the time-dependent transformation parameters from the original geodetic frame to an estimated Tisserand frame of the earth, with vanishing relative angular momentum.

1 Introduction

The establishment of a global geodetic network based on the analysis of various geodetic observations is aiming at providing a unified means for describing position by means of coordinates of discrete points on the earth surface. As such it is an object of geodetic work, both applied and theoretical. Theory is mostly involved in providing the tools for an optimal choice among the various possible choices of a reference frame (Dermanis 2001). These choices are equivalent as far as observational evidence is concerned, since they all provide at any epoch the same "observed" (i.e., estimated) shape for the adopted *International Terrestrial Reference Frame* (ITRF).

On the other hand there is a need in geophysics for an *earth reference frame*, i.e. a frame that refers to the behavior of the deforming earth as a whole, involving all its mass points and not just a set of discrete points confined on its surface.

More specifically such reference frames are indispensable in theories of the earth rotation (Munk and MacDonald, 1960), where two frame types appear: the "axes of figure" frame and the "Tisserand axes" frames. They are both defined in ways, which introduce simplifications in the "Liouville equations" describing earth rotation.

The frame of the figure axes is the particular frame with respect to which the inertia tensor

$$\mathbf{C} = -\int_E [\mathbf{x}\times][\mathbf{x}\times]\,dm = \int_E [(\mathbf{x}^T\mathbf{x})\mathbf{I} - \mathbf{x}\mathbf{x}^T]\,dm \qquad (1)$$

becomes diagonal ($C_{ij}(t) = 0$, $i \neq j$). Its time varying orientation is well defined as far as the three moments of inertia $C_{11}(t)$, $C_{22}(t)$, $C_{33}(t)$ maintain distinct values. On the other hand the Tisserand axes, which are based on the elimination of the relative angular momentum of the whole earth

$$\mathbf{h} = \int_E [\mathbf{x}\times]\frac{d\mathbf{x}}{dt}\,dm = \mathbf{0}\,, \qquad (2)$$

are not uniquely defined. If a Tisserand frame $\mathbf{x}(t)$ is rotated by a time independent rotation matrix ($d\mathbf{Q}/dt = 0$) the resulting frame $\mathbf{x}'(t) = \mathbf{Q}\,\mathbf{x}(t)$ is also a Tisserand frame. It is therefore necessary to conventionally select a particular Tisserand frame by adopting its orientation at some particular epoch. Both the Tisserand and the figure axes frames are taken to be geocentric. By the way, the fact that $C_{11}(t) \approx C_{22}(t)$ does not allow the operationally efficient definition of the orientation of the first and second figure axes and the adoption of a zero meridian is necessary in practice, despite the fact that it is not required in theory.

The diurnal yield of the earth to tidal and rotational effect causes a variation of the position of the third figure axes with respect to the earth masses of the order of 60 m (Moritz and Mueller, 1987). For this reason theories of the rotation of a deforming earth make use of Tisserand axes.

Our starting point is a global reference frame established by geodetic work and realized by a time series of coordinates for each network point i , dense enough to be considered as representing a set of continuous time functions $\mathbf{x}_i(t)$. Our theoretical analysis is necessarily based on continuous functions, as typically done in applied sciences, despite the fact that only discrete observations are available in practice. We assume that the geodetic frame has been chosen by applying an arbitrary optimality criterion, which furthermore is applied to a selected small subset of reliable fundamental stations, where different observations (VLBI, SLR, GPS, etc.) are collocated. Therefore the geodetic frame is generally different from the Tisserand axes of the earth rotation theories. A comparison of the observed rotation of the geodetic network frame with the rotation of the Tisserand earth frame foreseen by any specific rotation theory is not a straightforward matter, since a relative rotation of the these two frames results from their different definitions and realizations. It is therefore necessary to convert the geodetic frame to an "estimate" of the Tisserand frame by introducing some additional geophysical hypotheses.

The conversion of the geodetic to a Tisserand frame estimate, which establishes a connection between geodetic practice and geophysical theory, strongly depends on the adopted hypotheses about earth deformation. The lack of observational evidence of mass motion below the crust (e.g. convection currents, rotation of a non-spherical core with respect to the lithosphere) poses a certain limitation on the validity of the obtained estimate. Nevertheless more of the relative mass motions, which contribute to the relative angular momentum, take place close to the surface: plate tectonics, seismic events, atmospheric and oceanic circulation.

The basic idea behind the frame conversion is simple: use information about mass motions, compute their contribution to the relative angular momentum with respect to the geodetic frame and finally modify the geodetic frame until the modified relative angular momentum becomes zero. Thus the new resulting geodetic frame is an estimate of the earth Tisserand frame (apart from a time independent constant rotation).

It must be emphasized that what is important here is not the choice of reference frame at any particular epoch, but its dynamical evolution in time. Two frames connected by a time-independent rotation are equivalent since they give rise to the same rotation (instantaneous angular velocity) vector $\vec{\omega}$, with the same magnitude and direction with respect to both the inertial space and the earth masses. Different time evolutions of the frame give rise to a different frame rotation vector, which is not an invariant as in the case of a rigid network or earth. In fact it is the chosen frame which rotates and not the network or the earth. In other words the choice of frame is a means of separating the total motion in two parts: a common frame rotation and a residual deformation with respect to the frame.

To overcome the limitations from the fact that only surface information is available one might try to use an earth frame definition, which involves only the earth surface and not the whole earth (Engels and Grafarend, 1999). However this choice does not solve the problem of observation to theory connection, since earth rotation theories refer to the dynamical behavior of the earth as a whole.

Another assumption inherent in the observation-theory comparison is the inertial character of the established celestial frame, which today is realized by the celestial coordinates (direction angles) of radio sources observed by VLBI. Possible variations in the center of radiation of such non-point sources could be an additional problem to the direct comparison.

As far as the origin of the frames is concerned, we assume here for the sake of simplicity that all frames involved are geocentric, since origin variations do not play a role in the study of earth rotation.

2 Incorporating plate tectonic motion in the definition of the geodetic global network frame (ITRF)

We will demonstrate the conversion of an arbitrary geodetic reference frame to an estimate of an earth Tisserand frame by incorporating a particular type of geophysical hypothesis about tectonic plate motion. We assume that each plate P_K undergoes an independent rigid motion with respect to other plates and the inner part of the earth, which will be determined from the time variation of the coordinates of stations lying on the plate. This hypothesis can be easily weakened or strengthened. In the first case we might allow in addition a simple deformation of each plate described by a small number of parameters (e.g. infinitesimal strain parameters). A much stronger but perhaps more realistic hypothesis could involve only rotation of the plate around the geocenter, which means that the plate moves

"floating" on the earth surface without change of horizontal inclination or vertical motion. Intermediate hypotheses could allow vertical motion in agreement with observational evidence. In such a case it will be more realistic to allow not a common vertical motion for the whole plate but rather a varying one in both a horizontal and radial sense, thus returning to the previous case of rigid motion plus a prescribed form of deformation.
The following steps shortly describe the suggested procedure:

(1) Start from the coordinates $\mathbf{x}_i$ of a global geodetic network points in a given frame "$\mathbf{x}$".

(2) For each plate P_K: Establish a coordinate frame "$\mathbf{x}'(P_K)$", attached to the plate, by introducing a "motion minimization" principle for the subnetwork D_K of network points i on the plate, i.e. determine $\mathbf{R}_K(t)$, $\mathbf{b}_K(t)$ so that the coordinates $\mathbf{x}'_i = \mathbf{R}_K \mathbf{x}_i + \mathbf{b}_K$ satisfy the relevant optimality principle.

(3) Assume that each plate moves in a rigid way so that for any of its points it holds that $\mathbf{x}' = \mathbf{R}_K \mathbf{x} + \mathbf{b}_K =$ constant.

(4) Compute the relative angular momentum $\mathbf{h}_{P_K}$ of each plate P_K and the total relative momentum $\mathbf{h}_T = \sum_K \mathbf{h}_{P_K}$, with respect to the frame "$\mathbf{x}$".

(5) Introduce a transformation $\tilde{\mathbf{x}} = \mathbf{R}\mathbf{x}$ from the frame "$\mathbf{x}$" to a new frame "$\tilde{\mathbf{x}}$" and compute the new relative angular momentum $\tilde{\mathbf{h}}_T = \tilde{\mathbf{h}}_T(\mathbf{R})$.

(6) Chose $\mathbf{R}$ so that $\tilde{\mathbf{h}}_T = \mathbf{0}$, i.e. the new frame "$\tilde{\mathbf{x}}$" is a "Tisserand" frame.

We proceed with the derivation of the frame conversion algorithm according to the above scheme. For each plate we seek a frame transformation from the original global geodetic frame "$\mathbf{x}$" to a plate frame "$\mathbf{x}'(P_K)$" with respect to which the individual plate station motions are minimized in a particular sense. The transformation for each plate is described by the rotation matrix $\mathbf{R}_K(t)$ and the displacement vector $\mathbf{b}_K(t)$, which appear in the relevant coordinate transformation $\mathbf{x}' = \mathbf{R}_K \mathbf{x} + \mathbf{b}_K$.
The rotation and translation parameters can be obtained by applying optimality criteria of best fitting a frame to the subnetwork of each plate, as shown in Dermanis (1995, 1999, 2001). A particular choice and the relevant solutions are presented next.

3 Computation of plate rigid motion transformations

For any plate P_K a frame "$\mathbf{x}'$" will be best fitted to the subnetwork D_K of the global geodetic frame consisting of n_K points lying on the plate. With respect to rotation the best fitting is achieved by selecting the transformation

$$\mathbf{x}'_i = \mathbf{R}_K \mathbf{x}_i + \mathbf{b}_K , \qquad i \in D_K , \tag{3}$$

which transforms the original relative angular momentum of the plate subnetwork

$$\mathbf{h}_{D_K} = \sum_{i \in D_K} [(\mathbf{x}_i - \mathbf{m}_K)\times] \frac{d}{dt} (\mathbf{x}_i - \mathbf{m}_K) \tag{4}$$

to a vanishing relative angular momentum

$$\mathbf{h}'_{D_K} = \sum_{i \in D_K} [(\mathbf{x}'_i - \mathbf{m}'_K)\times] \frac{d}{dt} (\mathbf{x}'_i - \mathbf{m}'_K) , \tag{5}$$

where

$$\mathbf{m}_K \equiv \frac{1}{n_K} \sum_{i \in D_K} \mathbf{x}_i \tag{6}$$

and

$$\mathbf{m}'_K \equiv \frac{1}{n_K} \sum_{i \in D_K} \mathbf{x}'_i \tag{7}$$

are the coordinates of the center of mass of the plate subnetwork in the respective frames.
For a best fitting with respect to translation we require that the center of mass of the plate subnetwork remains constant and in particular zero without loss of generality, i.e.,

$$\mathbf{m}'_K = \mathbf{0} . \tag{8}$$

From (3) it follows that

$$\mathbf{m}'_K = \mathbf{R}_K \mathbf{m}_K + \mathbf{b}_K = \mathbf{0} \tag{9}$$

so that

$$\mathbf{b}_K = -\mathbf{R}_K \mathbf{m}_K \tag{10}$$

and the transformation (3) takes the form

$$\mathbf{x}'_i = \mathbf{R}_K(\mathbf{x}_i - \mathbf{m}_K) \,. \tag{11}$$

Consequently

$$[\mathbf{x}'_i\times] = \mathbf{R}_K[(\mathbf{x}_i - \mathbf{m}_K)\times]\mathbf{R}_K^T \,. \tag{12}$$

Introducing the rotation vector $\boldsymbol{\omega}_K$ of the frame "$\mathbf{x}'$" with respect to the frame "$\mathbf{x}$", defined by

$$[\boldsymbol{\omega}_K\times] = \mathbf{R}_K^T \frac{d\mathbf{R}_K}{dt} \,, \tag{13}$$

the velocity vector in the frame "$\mathbf{x}'$" becomes

$$\frac{d\mathbf{x}'_i}{dt} = \frac{d\mathbf{R}_K}{dt}(\mathbf{x}_i - \mathbf{m}_K) + \mathbf{R}_K \frac{d}{dt}(\mathbf{x}_i - \mathbf{m}_K) =$$

$$= \mathbf{R}_K[\boldsymbol{\omega}_K\times](\mathbf{x}_i - \mathbf{m}_K) + \mathbf{R}_K \frac{d}{dt}(\mathbf{x}_i - \mathbf{m}_K) =$$

$$= \mathbf{R}_K\left(-[(\mathbf{x}_i - \mathbf{m}_K)\times]\boldsymbol{\omega}_K + \frac{d}{dt}(\mathbf{x}_i - \mathbf{m}_K)\right) \tag{14}$$

and since $\mathbf{m}'_K = \mathbf{0}$

$$\mathbf{h}'_{D_K} \equiv \sum_{i\in D_K}[(\mathbf{x}'_i - \mathbf{m}'_K)\times]\frac{d}{dt}(\mathbf{x}'_i - \mathbf{m}'_K) =$$

$$= \sum_{i\in D_K}[\mathbf{x}'_i\times]\frac{d\mathbf{x}'_i}{dt} =$$

$$= \sum_{i\in D_K}\mathbf{R}_K[(\mathbf{x}_i - \mathbf{m}_K)\times]\mathbf{R}_K^T\mathbf{R}_K \times$$

$$\times\left(-[(\mathbf{x}_i - \mathbf{m}_K)\times]\boldsymbol{\omega}_K + \frac{d}{dt}(\mathbf{x}_i - \mathbf{m}_K)\right) =$$

$$= \mathbf{R}_K\left(-\sum_{i\in D_K}[(\mathbf{x}_i - \mathbf{m}_K)\times][(\mathbf{x}_i - \mathbf{m}_K)\times]\right)\boldsymbol{\omega}_K +$$

$$+ \mathbf{R}_K\sum_{i\in D_K}[(\mathbf{x}_i - \mathbf{m}_K)\times]\frac{d}{dt}(\mathbf{x}_i - \mathbf{m}_K) =$$

$$= \mathbf{R}_K\mathbf{C}_{D_K}\boldsymbol{\omega}_K + \mathbf{R}_K\mathbf{h}_{D_K} = \mathbf{0} \,. \tag{15}$$

The solution to the posed problem is given by

$$\mathbf{b}_K = -\mathbf{R}_K\mathbf{m}_K \,, \qquad \boldsymbol{\omega}_K = -\mathbf{C}_{D_K}^{-1}\mathbf{h}_{D_K} \tag{16}$$

where the discrete relative angular momentum with respect to the global network frame is given by eq. (4), while

$$\mathbf{C}_{D_K} = -\sum_{i\in D_K}[(\mathbf{x}_i - \mathbf{m}_K)\times][(\mathbf{x}_i - \mathbf{m}_K)\times] \tag{17}$$

is the discrete matrix of inertia of the plate subnetwork with respect to the same frame. The parameters $\boldsymbol{\theta}_K$ of the rotation matrix $\mathbf{R}_K = \mathbf{R}_K(\boldsymbol{\theta}_K)$ are obtained by numerically solving the system of three first order differential equations (13), which are the well-known geometric Euler equations in a generalized (parameter-free) form.

4 Variation of relative angular momentum due to a change of the global reference frame

We assume that every point of a plate P_K, with coordinates $\mathbf{x}$ in the global geodetic frame, follows the frame of the subnetwork D_K in its motion, so that its coordinates

$$\mathbf{x}' = \mathbf{R}_K(\mathbf{x} - \mathbf{m}_K) \tag{18}$$

with respect to it remain constant

$$\frac{d\mathbf{x}'}{dt} = \mathbf{0} \,. \tag{19}$$

Thus the velocity of any plate point is given by

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}(\mathbf{R}_K^T\mathbf{x}' + \mathbf{m}_K) =$$

$$= \frac{d\mathbf{R}_K^T}{dt}\mathbf{x}' + \mathbf{R}_K^T\frac{d\mathbf{x}'}{dt} + \frac{d\mathbf{m}_K}{dt} =$$

$$= \left(\mathbf{R}_K[\boldsymbol{\omega}_K\times]\right)^T\mathbf{x}' + \frac{d\mathbf{m}_K}{dt} =$$

$$= -[\boldsymbol{\omega}_K\times]\mathbf{R}_K^T\mathbf{x}' + \frac{d\mathbf{m}_K}{dt} =$$

$$= -[\boldsymbol{\omega}_K\times]\mathbf{R}_K^T\mathbf{R}_K(\mathbf{x} - \mathbf{m}_K) + \frac{d\mathbf{m}_K}{dt} =$$

$$= [(\mathbf{x} - \mathbf{m}_K)\times]\boldsymbol{\omega}_K + \frac{d\mathbf{m}_K}{dt} \,. \tag{20}$$

We next study the variation of the relative angular momentum of any tectonic plate from its initial value $\mathbf{h}_{P_K}$ with respect to the original global frame "$\mathbf{x}$" to its new value $\tilde{\mathbf{h}}_{P_K}$ in the modified frame "$\tilde{\mathbf{x}}$" where $\tilde{\mathbf{x}} = \mathbf{Rx}$. The frame "$\tilde{\mathbf{x}}$" is related to the plate frame "$\mathbf{x}'$" by

$$\tilde{\mathbf{x}} = \mathbf{Rx} = \mathbf{R}(\mathbf{R}_K^T \mathbf{x}' + \mathbf{m}_K) . \tag{21}$$

The rotation vector $\boldsymbol{\omega}$ of the frame "$\tilde{\mathbf{x}}$" with respect to the frame "$\mathbf{x}$" is defined by

$$[\boldsymbol{\omega}\times] \equiv \mathbf{R}^T \frac{d\mathbf{R}}{dt} . \tag{22}$$

The velocity of any plate point with respect to the new frame "$\tilde{\mathbf{x}}$" becomes

$$\frac{d\tilde{\mathbf{x}}}{dt} = \frac{d\mathbf{R}}{dt}(\mathbf{R}_K^T \mathbf{x}' + \mathbf{m}_K) + \mathbf{R}\frac{d\mathbf{R}_K^T}{dt}\mathbf{x}' + \mathbf{R}\frac{d\mathbf{m}_K}{dt} =$$

$$= \mathbf{R}[\boldsymbol{\omega}\times](\mathbf{R}_K^T \mathbf{x}' + \mathbf{m}_K) - \mathbf{R}[\boldsymbol{\omega}_K\times]\mathbf{R}_K^T\mathbf{x}' + \mathbf{R}\frac{d\mathbf{m}_K}{dt} =$$

$$= \mathbf{R}[\boldsymbol{\omega}\times]\mathbf{x} - \mathbf{R}[\boldsymbol{\omega}_K\times]\mathbf{R}_K^T\mathbf{R}_K(\mathbf{x} - \mathbf{m}_K) + \mathbf{R}\frac{d\mathbf{m}_K}{dt} =$$

$$= -\mathbf{R}[\mathbf{x}\times]\boldsymbol{\omega} + \mathbf{R}[(\mathbf{x} - \mathbf{m}_K)\times]\boldsymbol{\omega}_K + \mathbf{R}\frac{d\mathbf{m}_K}{dt} =$$

$$= -\mathbf{R}[\mathbf{x}\times]\boldsymbol{\omega} + \mathbf{R}\frac{d\mathbf{x}}{dt} . \tag{23}$$

The contribution of the plate P_K to the relative angular momentum with respect to the original "$\mathbf{x}$" frame is

$$\mathbf{h}_{P_K} = \int_{P_K} [\mathbf{x}\times]\frac{d\mathbf{x}}{dt} dm \tag{24}$$

while in the new "$\tilde{\mathbf{x}}$" frame it becomes

$$\tilde{\mathbf{h}}_{P_K} = \int_{P_K} [\tilde{\mathbf{x}}\times]\frac{d\tilde{\mathbf{x}}}{dt} dm =$$

$$= \int_{P_K} \mathbf{R}[\mathbf{x}\times]\mathbf{R}^T\left(-\mathbf{R}[\mathbf{x}\times]\boldsymbol{\omega} + \mathbf{R}\frac{d\mathbf{x}}{dt} \right) dm =$$

$$= \mathbf{R}\int_{P_K} [\mathbf{x}\times]\left(-[\mathbf{x}\times]\boldsymbol{\omega} + \frac{d\mathbf{x}}{dt} \right) dm =$$

$$= \mathbf{R}\left(-\int_{P_K} [\mathbf{x}\times][\mathbf{x}\times]\, dm \right)\boldsymbol{\omega} + \mathbf{R}\int_{P_K} [\mathbf{x}\times]\frac{d\mathbf{x}}{dt} dm =$$

$$= \mathbf{RC}_{P_K}\boldsymbol{\omega} + \mathbf{Rh}_{P_K} \tag{25}$$

where

$$\mathbf{C}_{P_K} \equiv -\int_{P_K} [\mathbf{x}\times][\mathbf{x}\times]\, dm \tag{26}$$

is the inertia matrix of the plate P_K in the "$\mathbf{x}$" frame.

5 Derivation of the transformation to an estimate of a Tisserand frame

In order to make the frame "$\tilde{\mathbf{x}}$" an estimate of a an earth Tisserand frame we impose the condition that the total angular momentum of all the plates is vanishing in the new frame

$$\tilde{\mathbf{h}} = \sum_K \tilde{\mathbf{h}}_{P_K} = \mathbf{R}\left(\sum_K \mathbf{C}_{P_K} \right)\boldsymbol{\omega} + \mathbf{R}\left(\sum_K \mathbf{h}_{P_K} \right) \equiv$$

$$\equiv \mathbf{RC}_T\boldsymbol{\omega} + \mathbf{Rh}_T = \mathbf{0} . \tag{27}$$

It follows that the solution is given by

$$\boldsymbol{\omega} = -\mathbf{C}_T^{-1}\mathbf{h}_T . \tag{28}$$

After choosing a particular parameterization of the rotation matrix $\mathbf{R} = \mathbf{R}(\boldsymbol{\theta})$, the parameters $\boldsymbol{\theta}$ are determined by solving the system of three first order differential equations (22) with the help of initial values $\boldsymbol{\theta}(t_0)$. It is the initial values, which pick up a particular Tisserand frame, out of all possible ones.
The solution requires the explicit determination of

$$\mathbf{h}_{P_K} = \int_{P_K} [\mathbf{x}\times]\frac{d\mathbf{x}}{dt} dm =$$

$$= \int_{P_K} [\mathbf{x}\times]\left([(\mathbf{x} - \mathbf{m}_K)\times]\boldsymbol{\omega}_K + \frac{d\mathbf{m}_K}{dt} \right) dm =$$

$$= \left(\int_{P_K} [\mathbf{x}\times][\mathbf{x}\times] dm \right) \boldsymbol{\omega}_K - \left(\int_{P_K} [\mathbf{x}\times] dm \right) [\mathbf{m}_K \times] \boldsymbol{\omega}_K + \\ + \left(\int_{P_K} [\mathbf{x}\times] dm \right) \frac{d\mathbf{m}_K}{dt} =$$

$$= -\mathbf{C}_{P_K} \boldsymbol{\omega}_K - [\mathbf{c}_{P_K} \times][\mathbf{m}_K \times] \boldsymbol{\omega}_K + [\mathbf{c}_{P_K} \times] \frac{d\mathbf{m}_K}{dt} \quad (29)$$

where $\mathbf{c}_{P_K}$ is the center of mass of the plate with respect to the original global frame.
The critical point is the evaluation of the integrals

$$\mathbf{C}_{P_K} = -\int_{P_K} [\mathbf{x}\times][\mathbf{x}\times] dm = -\int_{P_K} [\mathbf{x}\times][\mathbf{x}\times] \rho(\mathbf{x}) d\mathbf{x} \quad (30)$$

and

$$\mathbf{c}_{P_K} = \int_{P_K} \mathbf{x}\, dm = \int_{P_K} \mathbf{x}\, \rho(\mathbf{x}) d\mathbf{x} , \quad (31)$$

which requires knowledge of the boundaries of each plate as well as the density distribution. The horizontal boundaries are established from the discontinuities in the directions of the velocities of geodetic stations. The vertical ones require digital terrain models as well as bathymetry and lithospheric depth models. In a first approximation the density may be taken as constant.

6 The rotating plates model

A simplified model does not allow full rigid plate motion but only rotations around the geocenter (origin of the "$\mathbf{x}$" frame), so that plates are only allowed to move floating on the lithosphere. The transformation to the frame attached to the plate is given in this case by

$$\mathbf{x}' = \mathbf{R}_K \mathbf{x} , \quad (32)$$

while the inertia matrix and the relative angular momentum of the plate subnetwork are referred to the origin

$$\mathbf{C}_{D_K} = -\sum_{i \in D_K} [\mathbf{x}_i \times][\mathbf{x}_i \times] \quad (33)$$

$$\mathbf{h}_{D_K} = \sum_{i \in D_K} [\mathbf{x}_i \times] \frac{d\mathbf{x}_i}{dt} \quad (34)$$

The derivation of the rotation parameters is similar to the rigid motion case and yields

$$\boldsymbol{\omega}_K = -\mathbf{C}_{D_K}^{-1} \mathbf{h}_{D_K} \quad (35)$$

The velocity vector of any plate point becomes for this model

$$\frac{d\mathbf{x}}{dt} = [\mathbf{x}\times] \boldsymbol{\omega}_K , \quad (36)$$

the contribution of the plate P_K to the relative angular momentum becomes

$$\mathbf{h}_{P_K} = \int_{P_K} [\mathbf{x}\times] \frac{d\mathbf{x}}{dt} dm = -\mathbf{C}_{P_K} \boldsymbol{\omega}_K \quad (37)$$

while the contribution $\mathbf{C}_{P_K}$ to the tensor of inertia is still given by eq. (26). The derivation of the parameters of the transformation $\tilde{\mathbf{x}} = \mathbf{R}\mathbf{x}$ in a similar manner leads to a solution having the same form

$$\boldsymbol{\omega} = -\mathbf{C}_T^{-1} \mathbf{h}_T \quad (38)$$

where

$$\mathbf{C}_T = \sum_K \mathbf{C}_{P_K} \quad (39)$$

maintains the same value as in the rigid case, while

$$\mathbf{h}_T = \sum_K \mathbf{h}_{P_K} = -\sum_K \mathbf{C}_{P_K} \boldsymbol{\omega}_K , \quad (40)$$

is different in this case. In fact we may combine (38), (39), (40) to write the solution in the form

$$\boldsymbol{\omega} = \left(\sum_K \mathbf{C}_{P_K} \right)^{-1} \sum_K \mathbf{C}_{P_K} \boldsymbol{\omega}_K . \quad (41)$$

This means that the rotation vector $\boldsymbol{\omega}$ of the transformation from the original global geodetic frame "$\mathbf{x}$" to an estimate "$\tilde{\mathbf{x}}$" of the Tisserand frame is in fact a "weighted mean" of the rotation vectors $\boldsymbol{\omega}_K$ of the transformations from the frame "$\mathbf{x}$" to the frames $\mathbf{x}'(P_K)$ best-fitted to each plate, where the inertia tensors $\mathbf{C}_{P_K}$ play the role of weight matrices.

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