On the interpretation of dilution of precision (DOP) measures in GPS observations

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Abstract

Analytical expressions are derived which express the various Dilution of Precision (DOP) measures of GPS observations in terms of “statistical” characteristics of the spatial distribution of the receiver-to-satellite directions. For the minimal case of four satellites the usual claim that DOP is inversely proportional to the volume of the tetrahedron formed by the tips of the receiver-to-satellite unit vectors is critically examined.

1. Introduction

It is often claimed in the literature related to the Global Positioning System (GPS) that Dilution of Precision (DOP) is proportional to the inverse of the volume \( V \) of the tetrahedron formed by the tips of the unit vectors pointing to the satellites in the minimal 4-satellite configuration (Spilker, 1978, eq. 1-4, Seeber, 1993, p. 293, Hoffman-Wellenhofer et al., 1997, p. 274). Usually reference is made to Milliken and Zoller (1978) who are however very careful in declaring (p. 102) the existence of only a “high correlation” between GDOP (Geometric Dilution of Precision) and tetrahedron volume. The most extensive study of the matter is included in Wundelich (1993) who showed that \( V_f/GDOP = \) (eq. 3.28) where \( f \) depends also on the satellite configuration and “loosely” stated that GDOP is inversely proportional to \( V \). This statement would be correct in the strict sense only if \( f \) happened to be a constant. Wundelich (1993) shows that zero (minimum) volume corresponds to a critical configuration with singular normal equations and no unique solution and thus corresponds to maximum (in fact infinite) GDOP. He also relates minimum GDOP to the maximum of \( V \). Here we shall show that the above claim of proportionality is not strictly true in the sense that GDOP = const.\( \sqrt{V} \sim 1/V \). In addition we will examine the more general case of \( N \) satellites and we shall provide alternative “statistical” interpretations of both GDOP and PDOP (Positional Dilution of Precision).
2. The least squares adjustment solution

Let the receiver \( R \) observe pseudoranges to a series of satellites \( S = 1, 2, \ldots, N \). The model with respect to an arbitrary reference frame has the form (Dermanis, 1999)

\[
p_R^S = \frac{1}{\rho_R^S} \left[ (x^S - x_R)^2 + (y^S - y_R)^2 + (z^S - z_R)^2 \right] + c \Delta \delta = \rho_R^S + \Delta r,
\]

where \( x^S, y^S, z^S \) are the known coordinates of satellite \( S \), \( c \) is the velocity of light, \( \rho_R^S \) is the distance between the receiver and satellite \( S \), \( x_R, y_R, z_R \) are the unknown receiver coordinates, \( \Delta \delta \) is the unknown receiver clock error and \( \Delta r = c \Delta \delta \) its equivalent distance.

Since

\[
\frac{\partial p_R^S}{\partial x_R} = -\frac{x^S - x_R}{\rho_R^S}, \quad \frac{\partial p_R^S}{\partial y_R} = -\frac{y^S - y_R}{\rho_R^S}, \quad \frac{\partial p_R^S}{\partial z_R} = -\frac{z^S - z_R}{\rho_R^S},
\]

the linearized model becomes

\[
p_S = p_R^S - p_{R,0}^S =
\]

\[
= \left[ \begin{array}{c}
    x^S - x_R \\
    y^S - y_R \\
    z^S - z_R
\end{array} \right] - \left[ \begin{array}{c}
    x_R^0 - x_R \\
    y_R^0 - y_R \\
    z_R^0 - z_R
\end{array} \right] + \Delta r + v^S =
\]

\[
= \left[ \begin{array}{c}
    x^S - x_R \\
    y^S - y_R \\
    z^S - z_R
\end{array} \right] - \left[ \begin{array}{c}
    x_R^0 - x_R \\
    y_R^0 - y_R \\
    z_R^0 - z_R
\end{array} \right] + \Delta r + v^S = -n_S^T \Delta x + \Delta r + v_S,
\]

where \( x_R^0, y_R^0, z_R^0 \) are the approximate values of the receiver coordinates,

\[
n_S = \frac{1}{\sqrt{(x^S - x_R^0)^2 + (y^S - y_R^0)^2 + (z^S - z_R^0)^2}}
\]

is the (approximate) unit vector from the receiver to satellite \( S \),

\[
\Delta x = \left[ \begin{array}{c}
    x_R - x_R^0 \\
    y_R - y_R^0 \\
    z_R - z_R^0
\end{array} \right]
\]

is the vector of unknown coordinate corrections and \( v_S \) is the random noise of the \( p_R^S \) observation.
The linearized observation equations for all observations take the form

\[ \mathbf{b} = \begin{bmatrix} p_1 \\ \vdots \\ p_N \end{bmatrix} = \begin{bmatrix} -\mathbf{n}_1^T & \cdots & -\mathbf{n}_N^T \\ 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{x} \\ \Delta \mathbf{r} \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix} = \mathbf{A} \mathbf{x} + \mathbf{v}. \]  

(6)

Assuming that \( \mathbf{v} \sim \sigma^2 \mathbf{I} \), so that the weight matrix is \( \mathbf{P} = \mathbf{I} \), the solution to the least squares adjustment is given by the normal equations \( \hat{\mathbf{x}} = \mathbf{N}^{-1} \mathbf{u} \), where \( \mathbf{N} = \mathbf{A}^T \mathbf{A} \), \( \mathbf{u} = \mathbf{A}^T \mathbf{b} \), while the covariance matrix of the estimated unknowns is \( \mathbf{C}_x = \sigma^2 \mathbf{N}^{-1} \).

Analytically we have

\[ \mathbf{N} = \mathbf{A}^T \mathbf{A} = \begin{bmatrix} -\mathbf{n}_1 & \cdots & -\mathbf{n}_N \\ 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} -\mathbf{n}_1^T & 1 \\ \vdots & \vdots \\ -\mathbf{n}_N^T & 1 \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^N \mathbf{n}_k \mathbf{n}_k^T - \sum_{k=1}^N \mathbf{n}_k^T \\ -\sum_{k=1}^N \mathbf{n}_k^T & N \end{bmatrix} = \mathbf{N} \begin{bmatrix} \frac{1}{N} \sum_{k=1}^N \mathbf{n}_k \mathbf{n}_k^T - \frac{1}{N} \sum_{k=1}^N \mathbf{n}_k^T \\ -\frac{1}{N} \sum_{k=1}^N \mathbf{n}_k^T & 1 \end{bmatrix} = \mathbf{N} \begin{bmatrix} \mathbf{M} - \mathbf{m} \mathbf{m}^T \\ -\mathbf{m}^T \end{bmatrix}, \]  

(7)

where we have set

\[ \mathbf{M} = \frac{1}{N} \sum_{k=1}^N \mathbf{n}_k \mathbf{n}_k^T, \quad \mathbf{m} = \frac{1}{N} \sum_{k=1}^N \mathbf{n}_k. \]  

(8)

Note that \( \mathbf{m} \) is the mean of the satellite directions (not a unit vector!), while their “moment matrix” \( \mathbf{M} \) is related to the “covariance matrix” \( \mathbf{C} \) of the satellite directions through

\[ \mathbf{C} = \frac{1}{N} \sum_{k=1}^N (\mathbf{n}_k - \mathbf{m})(\mathbf{n}_k - \mathbf{m})^T = \frac{1}{N} \sum_{k=1}^N \mathbf{n}_k \mathbf{n}_k^T - \mathbf{m} \mathbf{m}^T = \mathbf{M} - \mathbf{m} \mathbf{m}^T, \]  

(9)

\[ \mathbf{C}^{-1} = (\mathbf{M} - \mathbf{m} \mathbf{m}^T)^{-1} = \mathbf{M}^{-1} + \frac{1}{1 - \mathbf{m}^T \mathbf{M}^{-1} \mathbf{m}} \mathbf{M}^{-1} \mathbf{m} \mathbf{m}^T \mathbf{M}^{-1}, \]  

(10)

while

\[ \mathbf{C}^{-1} \mathbf{m} = \mathbf{M}^{-1} \mathbf{m} + \frac{1}{1 - \mathbf{m}^T \mathbf{M}^{-1} \mathbf{m}} \mathbf{M}^{-1} \mathbf{m} \mathbf{m}^T \mathbf{M}^{-1} \mathbf{m} = \mathbf{M}^{-1} \mathbf{m} + \frac{\mathbf{m}^T \mathbf{M}^{-1} \mathbf{m}}{1 - \mathbf{m}^T \mathbf{M}^{-1} \mathbf{m}} \mathbf{M}^{-1} \mathbf{m} = \frac{1}{1 - \mathbf{m}^T \mathbf{M}^{-1} \mathbf{m}} \mathbf{M}^{-1} \mathbf{m}, \]  

(11)

\[ 1 + \mathbf{m}^T \mathbf{C}^{-1} \mathbf{m} = 1 + \frac{1}{1 - \mathbf{m}^T \mathbf{M}^{-1} \mathbf{m}} \mathbf{m}^T \mathbf{M}^{-1} \mathbf{m} = \frac{1}{1 - \mathbf{m}^T \mathbf{M}^{-1} \mathbf{m}}. \]  

(12)
Analytical inversion gives

\[
N^{-1} = \frac{1}{N} \begin{bmatrix} M & -m \\ -m^T & 1 \end{bmatrix}^{-1} = \frac{1}{N} \begin{bmatrix} M^{-1} + \frac{1}{1 - m^T M^{-1} m} m^T M^{-1} m & \frac{1}{1 - m^T M^{-1} m} M^{-1} m \\ \frac{1}{1 - m^T M^{-1} m} m^T M^{-1} m & \frac{1}{1 - m^T M^{-1} m} \end{bmatrix} = \\
= \frac{1}{N} \begin{bmatrix} C^{-1} & C^{-1} m \\ m^T C^{-1} & 1 + m^T C^{-1} m \end{bmatrix}.
\]

(13)

3. “Statistical” interpretation of the Dilution of Precision

Denoting by \( \text{tr}(B) = \sum_i B_{ii} \) the trace of any matrix \( B \), the geometric dilution of precision is given by

\[
\text{GDOP} = \sqrt{\sigma^2_x + \sigma^2_y + \sigma^2_z + \sigma^2_{\Delta}} = \sqrt{\text{tr}(C)} = \sigma \sqrt{\text{tr}(N^{-1})},
\]

(14)

\[
\frac{(\text{GDOP})^2}{\sigma^2} = \text{tr}(N^{-1}) = \frac{1}{N} [\text{tr}(C^{-1}) + 1 + m^T C^{-1} m],
\]

(15)

\[
\frac{N (\text{GDOP})^2}{\sigma^2} = \text{tr}(C^{-1}) + 1 + m^T C^{-1} m = \\
= \text{tr}(M^{-1}) + \frac{1}{1 - m^T M^{-1} m} \text{tr}(m^T M^{-1} M^{-1} m) + \frac{1}{1 - m^T M^{-1} m} = \\
= \text{tr}(M^{-1}) + \frac{1 + m^T M^{-2} m}{1 - m^T M^{-1} m}.
\]

(16)

Therefore

\[
\text{GDOP} = \frac{\sigma}{\sqrt{N}} \left[ \text{tr}(C^{-1}) + 1 + m^T C^{-1} m \right]^{1/2} = \frac{\sigma}{\sqrt{N}} \left[ \text{tr}(M^{-1}) + \frac{1 + m^T M^{-2} m}{1 - m^T M^{-1} m} \right]^{1/2}.
\]

(17)

The positional dilution of precision is given by

\[
\text{PDOP} = \sqrt{\sigma^2_x + \sigma^2_y + \sigma^2_z} = \sigma \sqrt{\text{tr}(N^{-1} C^{-1})} = \frac{\sigma}{\sqrt{N}} \left[ \text{tr}(C^{-1}) \right]^{1/2}.
\]

(18)
which upon using (10) becomes
\[
\text{PDOP} = \frac{\sigma}{\sqrt{N}} \left[ \text{tr}(\mathbf{C}^{-1}) \right]^{1/2} = \frac{\sigma}{\sqrt{N}} \left[ \text{tr}(\mathbf{M}^{-1}) + \frac{\mathbf{m}^T \mathbf{M}^{-1} \mathbf{m}}{1 - \mathbf{m}^T \mathbf{M}^{-1} \mathbf{m}} \right]^{1/2}.
\] (19)

Equations (17) and (19) provide a direct relation of the dilution of precision measures to the “statistical” characteristics of the spatial configuration of the satellite directions, namely their mean direction and their covariance or moment matrix. Note that the term “statistical” has no probabilistic meaning, but rather relates to the spatial “dispersion” of the satellite directions.

It should be noted that the above relations are independent of the reference frame definition. To see that let a change of reference frame take place described by \( \Delta \mathbf{x} = \mathbf{R} \Delta \mathbf{x} \), where \( \mathbf{R} \) is an orthogonal matrix. The covariance matrix in the new frame is given by
\[
\begin{bmatrix}
\mathbf{C} & \mathbf{0} \\
\mathbf{0} & 1
\end{bmatrix}
\begin{bmatrix}
\mathbf{R} & \mathbf{0} \\
\mathbf{0} & 1
\end{bmatrix}
= \frac{\sigma^2}{N}
\begin{bmatrix}
\mathbf{R} & \mathbf{0} \\
\mathbf{0} & 1
\end{bmatrix}
\begin{bmatrix}
\mathbf{C}^{-1} & \mathbf{m}^T \mathbf{C}^{-1} \\
\mathbf{m} \mathbf{C}^{-1} & 1 + \mathbf{m}^T \mathbf{C}^{-1} \mathbf{m}
\end{bmatrix}
\begin{bmatrix}
\mathbf{R}^T & \mathbf{0} \\
\mathbf{0} & 1
\end{bmatrix}
= \frac{\sigma^2}{N}
\begin{bmatrix}
\mathbf{m}^T \mathbf{C}^{-1} \mathbf{R}^T & \mathbf{m}^T \mathbf{C}^{-1} \mathbf{m} \\
\mathbf{C}^{-1} \mathbf{R}^T & 1 + \mathbf{m}^T \mathbf{C}^{-1} \mathbf{m}
\end{bmatrix},
\]
(20)
so that \( \text{tr} \mathbf{C}_{\Delta \mathbf{x}} = \frac{\sigma^2}{N} \text{tr}(\mathbf{C}^{-1} \mathbf{R}^T) = \frac{\sigma^2}{N} \text{tr}(\mathbf{C}^{-1} \mathbf{R} \mathbf{R}^T) = \frac{\sigma^2}{N} \text{tr}(\mathbf{C}^{-1}) = \text{tr} \mathbf{C}_{\Delta \mathbf{x}} \) and
\[
\text{tr} \mathbf{C}_{\mathbf{x}} = \frac{\sigma^2}{N} \left[ \text{tr}(\mathbf{C}^{-1} \mathbf{R}^T) + 1 + \mathbf{m}^T \mathbf{C}^{-1} \mathbf{m} \right] = \frac{\sigma^2}{N} \left[ \text{tr}(\mathbf{C}^{-1} \mathbf{R} \mathbf{R}^T) + 1 + \mathbf{m}^T \mathbf{C}^{-1} \mathbf{m} \right] = \frac{\sigma^2}{N} \left[ \text{tr}(\mathbf{C}^{-1}) + 1 + \mathbf{m}^T \mathbf{C}^{-1} \mathbf{m} \right] = \text{tr} \mathbf{C}_{\Delta \mathbf{x}} = \text{tr} \mathbf{C}_{\mathbf{x}}.
\] (21)

Therefore both PDOP = \( \sqrt{\text{tr}(\mathbf{C}_{\Delta \mathbf{x}})} \) and GDOP = \( \sqrt{\text{tr}(\mathbf{C}_{\mathbf{x}})} \) are invariant under changes of the reference frame. Note that translation \( \mathbf{d} \) has not been included in the change of frame (i.e., \( \Delta \mathbf{x} = \mathbf{R} \Delta \mathbf{x} + \mathbf{d} \)), since the translation term vanishes in the definition of the unit vectors \( \mathbf{n}_k = \mathbf{x}_k^s - \mathbf{x}_r \), and \( \hat{\mathbf{n}}_s = \mathbf{R} \mathbf{n}_s \). The matrices \( \mathbf{A} \) and \( \mathbf{N} \) depend only on the unit vectors \( \mathbf{n}_s \), \( s = 1, \ldots, N \). As a consequence, the relevant geometric analysis appears “centered” at the receiver point from which the unit vectors appear to emanate.

4. The special minimal case of four satellites

In the case of the minimal required number of 4 satellites, PDOP has been related in the literature to the volume \( V \) contained within the tetrahedron formed by the tips \( P_k \) of the 4 unit vectors pointing to the satellites. This can also be described as the tetrahedron formed by the intersections of the receiver-to-satellites straight lines with a unit sphere centered at the receiver. We shall examine the claim that PDOP \( \sim V^{-1} \), i.e. that PDOP is
proportional to the inverse of the tetrahedron volume, while checking whether GDOP
obeys a similar rule.
The tetrahedron formed by the tips of the unit vectors has sides (see figure 1)
\[ \mathbf{d}_{12} = \mathbf{n}_2 - \mathbf{n}_1, \quad \mathbf{d}_{13} = \mathbf{n}_3 - \mathbf{n}_1, \quad \mathbf{d}_{14} = \mathbf{n}_4 - \mathbf{n}_1. \] (22)

The volume of the parallelepiped formed by the sides \( \mathbf{d}_{12}, \mathbf{d}_{14}, \mathbf{d}_{13} \) (right-hand order) is
well known to be equal to the triple vector product \( [\mathbf{d}_{12}, \mathbf{d}_{14}, \mathbf{d}_{13}] = \mathbf{d}_{12}^T (\mathbf{d}_{14} \times \mathbf{d}_{13}) \) and the
volume of their tetrahedron is one sixth of that, i.e.,
\[ V = \frac{1}{6} \mathbf{d}_{12}^T [\mathbf{d}_{14} \times \mathbf{d}_{13}] = \frac{1}{6} (\mathbf{n}_2 - \mathbf{n}_1) \mathbf{n}_4 \mathbf{n}_3 \mathbf{n}_1^T \times \mathbf{n}_3 - \mathbf{n}_1. \] (23)

The determinant of the coefficient matrix of the normal equations is given by
\[ |\mathbf{N}| = |\mathbf{A}^T \mathbf{A}| = |\mathbf{A}^T| \ ||\mathbf{A}|| = |\mathbf{A}|^2. \] (24)

Taking into account that the determinant remains unchanged if the first column is sub-
tracted from the rest we obtain (Wunderlich, 1993)
\[ |\mathbf{A}| = (-1)^4 |(-\mathbf{A})^T| = \begin{vmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 & \mathbf{n}_4 \\ -1 & -1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} \mathbf{n}_2 - \mathbf{n}_1 & \mathbf{n}_3 - \mathbf{n}_1 & \mathbf{n}_4 - \mathbf{n}_1 \end{vmatrix} = \begin{vmatrix} \mathbf{n}_2 - \mathbf{n}_1 & \mathbf{n}_3 - \mathbf{n}_1 & \mathbf{n}_4 - \mathbf{n}_1 \end{vmatrix} =\]
\[ = \begin{vmatrix} \mathbf{n}_2 - \mathbf{n}_1 & \mathbf{n}_3 - \mathbf{n}_1 & \mathbf{n}_4 - \mathbf{n}_1 \end{vmatrix} = [\mathbf{n}_2 - \mathbf{n}_1, \mathbf{n}_4 - \mathbf{n}_1, \mathbf{n}_3 - \mathbf{n}_1] =
\[ = (\mathbf{n}_2 - \mathbf{n}_1)^T [\mathbf{n}_4 - \mathbf{n}_1] \mathbf{n}_3 - \mathbf{n}_1 = 6V, \] (25)
where according to (25) \( V \) is the volume of the tetrahedron.

Using the property\[ A \begin{bmatrix} B \\ C \end{bmatrix} = |A| \| D - CA^{-1}B \| \] and \( N = 4 \) we obtain from (7) and (13)

\[
|N| = 4^4 \begin{vmatrix} M & -m \\ m^T & 1 \end{vmatrix} = 256 |M| (1 - m^T M^{-1} m),
\]

(26)

\[
|N^{-1}| = \frac{1}{4^4} \begin{vmatrix} C^{-1} & C^{-1}m \\ m^T C^{-1} & 1 + m^T C^{-1} m \end{vmatrix} = \frac{1}{256} |C^{-1} (1 + m^T C^{-1} m - m^T C^{-1} C C^{-1} m)| = \frac{1}{256} |C|,
\]

while

\[
|N^{-1}| = \frac{1}{|N|} = \frac{1}{|A|^2} = \frac{1}{36 V^2}.
\]

(27)

Therefore

\[
|C| = \frac{9}{64} V^2, \quad |M| = \frac{9}{64 (1 - m^T M^{-1} m)} V^2.
\]

(29)

Using the well-known relation \( B^* = |B|^{-1} (B^T)^T \), where \( B^* \) is the matrix of the conjugate determinants of any square non-singular matrix \( B \), we arrive at the representations

\[
\frac{\text{GDOP}^2}{\sigma^2} = \text{tr}(N^{-1}) = \text{tr}(A^{-1} A^{-T}) = \frac{\text{tr}[A^* (A^*)^T]}{|A|^2},
\]

(30)

\[
\frac{4 \text{PDOP}^2}{\sigma^2} = \frac{\text{tr}(C^*)}{|C|}.
\]

(31)

Using the determinant values from (25) and (29) we can relate the two DOPs to the tetrahedron volume \( V \) as follows

\[
\text{GDOP} = \frac{\sigma \sqrt[6]{\text{tr}[A^* (A^*)^T]}}{V},
\]

(32)

\[
\text{PDOP} = \frac{4 \sigma \sqrt[3]{\text{tr}(C^*)}}{V}.
\]

(33)

Although the volume appears as the denominator of the above expressions the nominators are not constant; they depend on the same configuration parameters as the volume.
Therefore neither GDOP or PDOP are proportional to the inverse volume $V^{-1}$ of the tetrahedron in the strict sense.

However there is a chance that the following proposition holds, which can be interpreted as “inverse proportionality” in the loose sense:

“GDOP (or PDOP) is an increasing (or at least non-decreasing) function of the inverse volume of the tetrahedron formed by the tips of the receiver-to-satellite direction (unit) vectors”.

This means that the larger the volume the smaller the GDOP (or PDOP) would become and vice-versa. Such an expectation is enhanced by the fact that zero volume corresponds to the critical configuration where the 4 satellite directions are coplanar, in which case the determinant of the normal equations matrix also vanishes, $\det N = 0$, and the normal equations have no unique solution (infinite parameter variances). Thus when the tetrahedron volume approaches zero both GDOP and PDOP tend to infinity.

To get more inside into the situation we introduce the eigenvalues $\gamma_1 \geq \gamma_2 \geq \gamma_3 > 0$ of the positive-definite covariance matrix $C$, as well as the eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 > 0$ of the also positive-definite normal equations matrix $N$. Thus $C$ and $N$ accept the diagonalizations

$$
C = U_C D_C U_C^T, \quad N = U_N D_N U_N^T,
$$

where $(D_C)_k = \delta_{k \gamma_k}$, $(D_N)_k = \delta_{k \lambda_k}$ are the diagonal matrices of the eigenvalues and $U_C$, $U_N$ the orthogonal matrices having as columns the corresponding eigenvectors.

Since

$$
|C| = |U_C| \cdot |D_C| \cdot |U_C^T| = |D_C| \cdot |U_C^T| \cdot |U_C| = \gamma_1 \gamma_2 \gamma_3 = \frac{9}{64} V^2, \quad (35)
$$

$$
\text{tr}(C^{-1}) = \text{tr}(U_C D_C^{-1} U_C^T) = \text{tr}(D_C^{-1} U_C^T U_C) = \text{tr}(D_C^{-1}) = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_3}, \quad (36)
$$

and similarly

$$
|N| = \lambda_1 \lambda_2 \lambda_3 \lambda_4 = 36 V^2, \quad \text{tr}(N^{-1}) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4}, \quad (37)
$$

the two DOPS admit the representations

$$
PDOP = \sigma \left[ \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_3} \right]^{1/2}, \quad V = \frac{8}{3} \sqrt[3]{\gamma_1 \gamma_2 \gamma_3}, \quad (38)
$$

$$
GDOP = \sigma \sqrt{\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4}}, \quad V = \frac{1}{6} \sqrt[4]{\lambda_1 \lambda_2 \lambda_3 \lambda_4}, \quad (39)
$$

or
From the above relations it follows that if two configurations have the same volume, it does not necessarily follow that they have the same GDOP or PDOP. For example if for two configurations $V'=V$ and thus $\gamma_1'\gamma_2'\gamma_3' = \gamma_1\gamma_2\gamma_3$, it does not follow that
\[
\frac{1}{\gamma_1'} + \frac{1}{\gamma_2'} + \frac{1}{\gamma_3'} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_3}
\]
and hence $\text{PDOP}' = \text{PDOP}$.

On the other hand we are not in a position to assert the existence of indeed different configurations with $V'=V$ but $\text{PDOP}' \neq \text{PDOP}$, because neither the eigenvalues $\gamma_1, \gamma_2, \gamma_3$ nor the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ provide a set of independent parameters for the configuration of the four unit vectors. A deeper and hopefully more conclusive study requires the description of the configuration by a set of 5 independent parameters and the expression of $V$, GDOP and PDOP as functions of those. The number of independent parameters is $5 = 2 \cdot 4 - 3$, since each of the 4 unit vectors requires 2 angles for its determination, while 3 parameters must be subtracted to account for the invariance of the configuration (and hence $V$, GDOP and PDOP) under 3-parametric rotations around the receiver point.

References


