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ORIGINAL PAPER

Fundamentals of surface deformation and application to construction monitoring

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Abstract The rigorous approach to plane deformation developed by the author is extended to the case of curved surfaces and is applied to the monitoring of surface-like constructions by repeated surveys. From coordinates of discrete points at two survey epochs, interpolation of the displacements produces displacement functions expressed in terms of surface curvilinear coordinates, which can be used for the computation of coordinate-invariant deformation parameters which are meaningful from the strength-of-materials point of view. In addition, it is shown how to incorporate information from additional in situ measurements by strainmeters, extensiometers, tiltmeters, etc. in the interpolation process. The computed invariant parameters at any surface point are the dilatation, the maximum shear strain, and the maximum bending expressed in terms of the maximum change of radius curvature among all surface directions. In addition to the theoretical tools, a complete algorithm is presented for direct practical implementation.

Keywords Plane deformation · Surface deformation · Material strength · Dilatation · Shear strain · Extensiometer · Tiltmeter · Collocation

Introduction

Deformation of bodies appearing in nature, such as the earth or man-made materials and constructions, is a three-dimensional entity. On the other hand, only the body surface is offered for direct observation. Two-dimensional deformation of surfaces is always related to the underlying three-dimensional in a non-unique way. However, two-dimensional deformation has been studied not in relation to physical bodies but rather with respect to abstract entities. One example is the deformation involved in map projections where the surface of an ellipsoid or a sphere is deformed into the cartographic plane (Grafarend and Krumm 2006). A second example is the routine study of crustal deformation as a plane deformation where the two planes involved, the original plane and the deformed one, are derived from the projection of the observed earth surface on the horizontal plane. Plane deformation has been thoroughly studied, although typically by approximate methods (Love 1927), and provides tools that are also useful for the study of curved surface deformation, as will be shown here. Direct study of the curved earth surface has been undertaken by Voosoghi (2000), though the results are not convenient for direct geophysical interpretation due to the different geophysical processes giving rise to the horizontal component of earth deformation (e.g., plate motion) and to the vertical one (e.g., postglacial uplift).

Here, we will study the deformation of man-made constructions which have one of their dimensions so small with respect to the other two that they can be virtually considered as two-dimensional. Modern architecture often uses such surface-like thin elements in order to achieve its aesthetic goals. It is the surveyor's task to monitor the deformation of such constructions through repeated surveys which allow deriving the displacements of a well-distributed set of points covering the target surface and hence of the surface itself through interpolation. Since the ultimate purpose of such monitoring is the safety of the construction, we must notice that point displacements are of little value to the strength-of-materials specialist who is rather interested for invariant deformation parameters

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which, under the collective name of "strain," enter into the equations relating deformation to the acting forces or "stresses." These are the constitutive equations of the specific material studied in the theory of elasticity (Marsden and Hughes 1983) or the "stress–strain relations" as commonly called in engineering. For this reason, we will present here the underlying theory and the specific algorithms for deriving from the observed displacements appropriate deformation parameters, namely, dilatation Δ, maximum shear strain γ, and the maximum difference in radii of curvature ΔR, which expresses the bending of the surface. For the first two (Δ and γ), it is sufficient to study the surface as locally approximated by the best fitting plane, utilizing new results based on the standard plane deformation theory, though not in its usual approximate form but rather through its rigorous version presented by Biagi and Dermanis (2006) and Dermanis and Kotsakis (2006). For bending (ΔR), we will resort to the local approximation of the surface by a best-fitting ellipsoid.

We shall first shortly present the rigorous plane deformation theory and we will modify it from the usual case of orthonormal reference systems to arbitrary ones in order to make it directly applicable to the curved surface case. Then, we will apply some elementary tools from differential geometry in order to determine the normal section of the surface (curve resulting from the intersection of the surface with a plane passing through the direction perpendicular to it) which demonstrates the maximum change in its radius of curvature and thus the maximum bending.

Fundamentals of planar deformation

We consider the material points P of a plane which is deformed without losing its flat character. Such points change position as a result of deformation, and they are typically identified through their coordinates $\mathbf{x} = \mathbf{x}(t) = [x_1(t)\ x_2(t)]^T$ at a reference epoch t, with respect to an orthonormal reference system with origin O and basis vectors $\vec{\mathbf{e}} = [\vec{e}_1 \vec{e}_2]$. At a later epoch t' as a result of deformation, the same material point has coordinates $\mathbf{x}' = \mathbf{x}'(t') = [x_1'(t')x_2'(t')]^T$ with respect to a different orthonormal reference system with origin O' and basis vectors $\vec{\mathbf{e}}' = \left[\vec{e}_1'\vec{e}_2'\right]$. In general, we need to introduce different reference systems since the whole plane is deforming and thus there exists not a rigid sub-region on which to realize and maintain the same system. This is already a "geodetic" departure from typical deformation theory where the same time-independent reference system is assumed, easily established, e.g., in the rigid environment of a laboratory where the deformation of a material body is studied. Deformation is globally expressed by the "deformation mapping" $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ connecting corresponding coordinates of all plane material points. Local deformation is concerned with what happens in a small neighborhood of any material point, and it is expressed by the "deformation gradient" $\mathbf{F} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \frac{\partial \mathbf{x}'}{\partial \mathbf{x}}$. Each point P with original coordinates $\mathbf{x}$ has its own deformation gradient $\mathbf{F}(\mathbf{x})$. From the elements of the matrix $\mathbf{F}$, we need to extract parameters which are more physically meaningful in describing deformation and, furthermore, unlike $\mathbf{F}$, are independent of the particular arbitrarily introduced reference systems $(O, \vec{e}_1, \vec{e}_2)$ and $\left(O', \vec{e}_1', \vec{e}_2'\right)$. For this purpose, we use the so-called singular value decomposition (SVD)

$$\mathbf{F} = \mathbf{Q}^T\mathbf{L}\mathbf{P} = \mathbf{R}(-\theta_Q)\mathbf{L}\mathbf{R}(\theta_P) = \begin{bmatrix} \cos\theta_Q & -\sin\theta_Q \\ \sin\theta_Q & \cos\theta_Q \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \times \begin{bmatrix} \cos\theta_P & \sin\theta_P \\ -\sin\theta_P & \cos\theta_P \end{bmatrix}. \tag{1}$$

where λ_1^2 and λ_2^2 are the common eigenvalues of the matrices $\mathbf{C} \equiv \mathbf{F}^T\mathbf{F}$ and $\mathbf{B} \equiv \mathbf{F}\mathbf{F}^T$, ordered so that $\lambda_1 > \lambda_2$. The corresponding eigenvectors $\mathbf{p}_1$, $\mathbf{p}_2$ of $\mathbf{F}^T\mathbf{F}=\mathbf{P}^T\mathbf{L}^2\mathbf{P}$ are the columns of the orthogonal matrix $\mathbf{P}^T=[\mathbf{p}_1\mathbf{p}_2]$, while the eigenvectors $\mathbf{q}_1$, $\mathbf{q}_2$ of $\mathbf{F}\mathbf{F}^T=\mathbf{Q}^T\mathbf{L}^2\mathbf{Q}$ are the columns of the orthogonal matrix $\mathbf{Q}^T=[\mathbf{q}_1\mathbf{q}_2]$.

If $\Delta\vec{x} = \vec{\mathbf{e}}\Delta\mathbf{x}$ is the vector from a material point P to an infinitesimally close other point at epoch t and $\Delta\vec{x}' = \vec{\mathbf{e}}'\Delta\mathbf{x}'$, the same vector at epoch t', it holds that $\Delta\mathbf{x}' \approx \mathbf{F}\Delta\mathbf{x} = \mathbf{R}(-\theta_Q)\mathbf{L}\mathbf{R}(\theta_P)\Delta\mathbf{x}$ and $\mathbf{R}(\theta_Q)\Delta\mathbf{x}' \approx \mathbf{L}\mathbf{R}(\theta_P)\Delta\mathbf{x}$. The first rotation $\mathbf{R}(\theta_P)$ corresponds to a rotation from the $\vec{\mathbf{e}} = [\vec{e}_1\ \vec{e}_2]$ basis to another orthonormal basis $[\vec{p}_1\ \vec{p}_2]$, while $\Delta\tilde{\mathbf{x}} = \mathbf{R}(\theta_P)\Delta\mathbf{x}$ are the components of $\Delta\vec{x}$ in this new system (Fig. 1). The rotation $\mathbf{R}(\theta_Q)$ corresponds to a rotation from the $\vec{\mathbf{e}}' = \left[\vec{e}_1'\ \vec{e}_2'\right]$ basis to the $[\vec{p}_1\ \vec{p}_2]$ basis, while $\Delta\tilde{\mathbf{x}}' = \mathbf{R}(\theta_Q)\Delta\mathbf{x}'$ are the components of $\Delta\vec{x}'$ in the $[\vec{p}_1\ \vec{p}_2]$ system. Thus, the original relation $\Delta\mathbf{x}' \approx \mathbf{F}\Delta\mathbf{x}$ with respect to the original two systems takes the form $\Delta\tilde{\mathbf{x}}' \approx \mathbf{L}\Delta\tilde{\mathbf{x}}$ in the common to both epochs system $[\vec{p}_1\ \vec{p}_2]$, or explicitly since $\mathbf{L}$ is diagonal

$$\begin{bmatrix} \Delta\tilde{x}_1' \\ \Delta\tilde{x}_2' \end{bmatrix} = \Delta\tilde{\mathbf{x}}' \approx \mathbf{L}\Delta\tilde{\mathbf{x}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \Delta\tilde{x}_1 \\ \Delta\tilde{x}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1\Delta\tilde{x}_1 \\ \lambda_2\Delta\tilde{x}_2 \end{bmatrix} \tag{2}$$

Points lying on the $\vec{p}_1$ or the $\vec{p}_2$ axes remain on the same axis, but they are displaced (elongated) by factors λ_1 or λ_2, respectively, since $\Delta\tilde{x}_1' = \lambda_1\Delta\tilde{x}_1$ and $\Delta\tilde{x}_2' = \lambda_2\Delta\tilde{x}_2$. Thus, λ_1 and λ_2 are the principal elongations, or maximum and minimum elongations, respectively, since $\lambda_1 > \lambda_2$. The perpendicular directions $\vec{p}_1$ and $\vec{p}_2$ are the directions of maximum and minimum elongation, respectively.

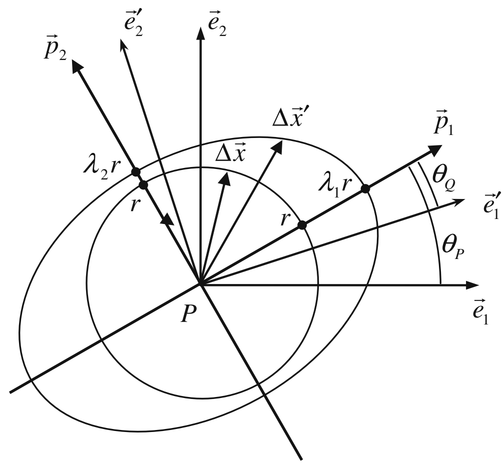

Fig. 1 Deformation through singular value decomposition: The vector $\Delta\vec{x}$ between the reference point P and an infinitesimally nearby point $P+\Delta P$ at the original epoch t becomes $\Delta\vec{x}'$ at a later epoch t'. A circle of radius r is deformed into an ellipse with semi-axes $\lambda_1 r$ and $\lambda_2 r$ along the mutually perpendicular principal directions $\vec{p}_1$ and $\vec{p}_2$, respectively. The direction of maximum elongation $\vec{p}_1$ is located at an angle θ_P from the $\vec{e}_1$ axis at epoch t and at an angle θ_Q from the $\vec{e}'_1$ axis at epoch t'

The unit vector $\vec{p}_1 = \vec{\mathbf{e}}\mathbf{p}_1$ in the direction of maximum elongation λ_1 has components $\mathbf{p}_1 = [\cos\theta_P \ \sin\theta_P]^T$ in the epoch t coordinate system. The minimal elongation λ_2 occurs at the perpendicular direction $\vec{p}_2 = \vec{\mathbf{e}}\mathbf{p}_2$, with $\vec{p}_1 \perp \vec{p}_2$ and $\mathbf{p}_2 = [-\sin\theta_P \ \cos\theta_P]^T$. Thus, θ_P is the (measured counterclockwise) angle from the $\vec{e}_1$ axis to the direction of maximum elongation $\vec{p}_1$ (Fig. 1). The same direction of maximum elongation $\vec{p}_1 = \vec{\mathbf{e}}'\mathbf{q}_1$ has components $\mathbf{q}_1 = [\cos\theta_Q \ \sin\theta_Q]^T$ in the coordinate system of the epoch t' and, thus θ_Q is the (counterclockwise) angle from the $\vec{e}'_1$ axis to the direction of maximum elongation $\vec{p}_1$.

Dilatation Δ is a strain parameter related to forces with radial direction (Fig. 2) either from the point P outwards (expansion, $\Delta>0$) or inwards (shrinking, $\Delta<0$). Shear strain $\gamma = \tan\omega$ relates to tearing forces that tend to deform an orthogonal parallelogram into an inclined one with inclination angle ω (Fig. 3).

An infinitesimal circle with radius Δr, with area $E = \pi\Delta r^2$, is deformed into an ellipse with axes $\lambda_1\Delta r$, $\lambda_2\Delta r$ having an area $E' = \pi\lambda_1\lambda_2\Delta r^2$, and thus $\lambda_1\lambda_2 = E'/E$

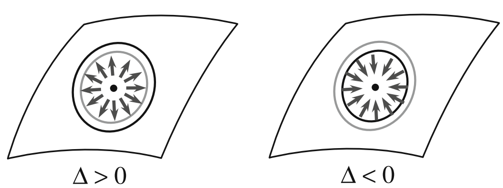

Fig. 2 Dilatation as relative change of an infinitesimal area around the reference point P as a result of outward ($\Delta>0$) or inward ($\Delta<0$) stresses

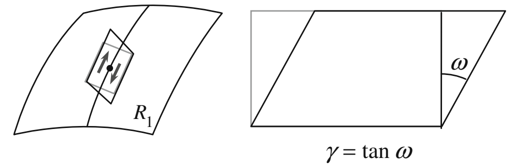

Fig. 3 Shear strain $\gamma = \tan\omega$ as a result of tearing stresses deforming an orthogonal parallelogram into one inclined by an angle ω

is the factor of area increase (or decrease if <1). The dilatation

$$\Delta = \lambda_1\lambda_2 - 1 = \frac{E' - E}{E} \tag{3}$$

expresses the ratio of area variation $E'-E$ with respect to the original area E.

For the maximum shear strain γ, we use the decomposition $\mathbf{F} = s\mathbf{R}(\psi)\Gamma_\phi$, where $\Gamma = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}$ is a matrix of shear along the first axis, $\Gamma_\phi = \mathbf{R}(-\phi)\boldsymbol{\Gamma}\mathbf{R}(\phi)$ represents shear in the direction angle ϕ, and s is a scale parameter related to dilatation. $\mathbf{R}(\psi)$ is an additional rotation not contributing to deformation, which is introduced so that the four parameters γ, ϕ, s, and ψ are as many as elements of $\mathbf{F}$ and can be expressed as functions of them. Comparison with the SVD representation $\mathbf{F} = s\mathbf{R}(\psi)\boldsymbol{\Gamma}_\phi = \mathbf{R}(-\theta_Q)\mathbf{L}\mathbf{R}(\theta_P)$ allows expressing γ, ϕ, s, and ψ as functions of λ_1, λ_2, θ_P, and θ_Q. We shall need only the relations for maximum shear γ and its direction angle ϕ, which are

$$\gamma = \frac{\lambda_1 - \lambda_2}{\sqrt{\lambda_1\lambda_2}}, \quad \sin\omega = \sqrt{\frac{1}{2} + \frac{\gamma}{2\sqrt{\gamma^2+4}}}, \\ \cos\omega = \sqrt{\frac{1}{2} - \frac{\gamma}{2\sqrt{\gamma^2+4}}}, \quad \phi = \theta_P - \omega. \tag{4}$$

In order to prepare for application to surface deformation, we study the case where instead of the Cartesian coordinates $\mathbf{x}$, $\mathbf{x}'$ we have used coordinates $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$ in two arbitrary reference systems with the same origins but oblique no-unit basis elements $\vec{\varepsilon} = [\vec{\varepsilon}_1 \ \vec{\varepsilon}_2]$ and $\vec{\varepsilon}' = [\vec{\varepsilon}'_1 \ \vec{\varepsilon}'_2]$ related to the previous ones through (Fig. 4)

$$\vec{\boldsymbol{\varepsilon}} = [\vec{\varepsilon}_1 \ \vec{\varepsilon}_2] = [\vec{e}_1 \ \vec{e}_2]\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \vec{\mathbf{e}}\mathbf{S}, \tag{5}$$
$$\vec{\boldsymbol{\varepsilon}}' = [\vec{\varepsilon}'_1 \ \vec{\varepsilon}'_2] = [\vec{e}'_1 \ \vec{e}'_2]\begin{bmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{bmatrix} = \vec{\mathbf{e}}'\mathbf{S}'.$$

Since $\vec{x} = \overrightarrow{OP} = \vec{\mathbf{e}}\mathbf{x} = \vec{\varepsilon}\xi = \vec{\mathbf{e}}\mathbf{S}\xi$ and $\vec{x}' = \overrightarrow{O'P} = \vec{\mathbf{e}}'\mathbf{x}' = \vec{\varepsilon}'\xi' = \vec{\mathbf{e}}'\mathbf{S}'\xi'$ for the position vectors of the same material points, the corresponding coordinates are related by $\mathbf{x} = \mathbf{S}\xi$, $\mathbf{x}' = \mathbf{S}'\xi'$, $\xi = \mathbf{S}^{-1}\mathbf{x}$, and $\xi' = \mathbf{S}'^{-1}\mathbf{x}'$. According to the chain rule of differentiation the Cartesian deformation

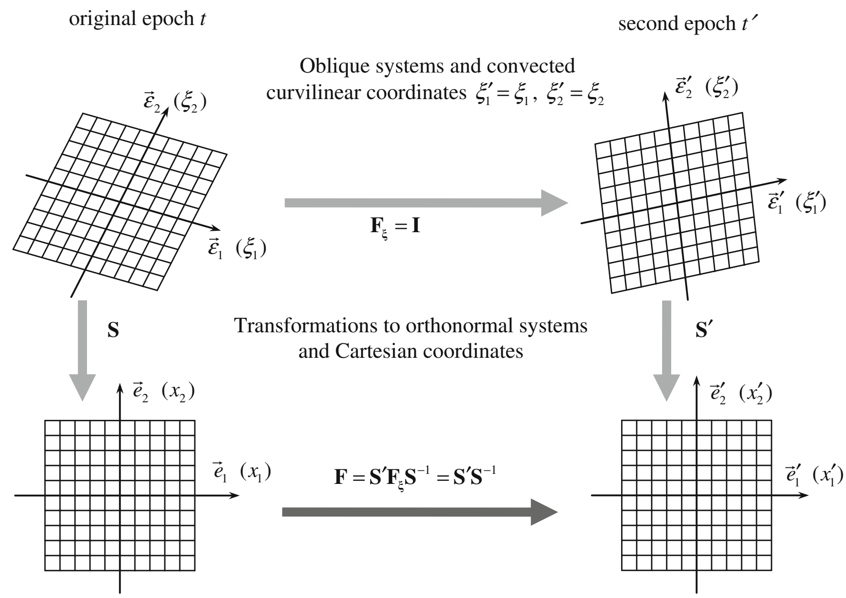

Fig. 4 Computation of the gradient matrix $\mathbf{F}$ for local Cartesian coordinates from the original one $\mathbf{F}_\xi = \mathbf{I}$ for curvilinear convected coordinates ($\xi_1' = \xi_1, \xi_2' = \xi_2$) through the transformations $\mathbf{S}$ and $\mathbf{S}'$ from the oblique to orthonormal reference systems for both epochs t and t'

gradient, $\mathbf{F} = \frac{\partial \mathbf{x}'}{\partial \mathbf{x}}$ is related to the "oblique" one $\mathbf{F}_\xi = \frac{\partial \xi'}{\partial \xi}$ through

$$\mathbf{F} = \frac{\partial \mathbf{x}'}{\partial \mathbf{x}} = \frac{\partial \mathbf{x}'}{\partial \xi'} \frac{\partial \xi'}{\partial \xi} \frac{\partial \xi}{\partial \mathbf{x}} = \mathbf{S}'\mathbf{F}_\xi\mathbf{S}^{-1}. \tag{6}$$

The metric matrices of the original orthonormal systems are the identity matrices $\vec{\mathbf{e}}^T \cdot \vec{\mathbf{e}} = \mathbf{I}$ and $\vec{\mathbf{e}}'^T \cdot \vec{\mathbf{e}}' = \mathbf{I}$. However, the oblique systems have metric matrices

$$\mathbf{G} \equiv \begin{bmatrix} \vec{\varepsilon}_1 \cdot \vec{\varepsilon}_1 & \vec{\varepsilon}_1 \cdot \vec{\varepsilon}_2 \\ \vec{\varepsilon}_2 \cdot \vec{\varepsilon}_1 & \vec{\varepsilon}_2 \cdot \vec{\varepsilon}_2 \end{bmatrix} = \begin{bmatrix} \vec{\varepsilon}_1 \\ \vec{\varepsilon}_2 \end{bmatrix} \cdot [\,\vec{\varepsilon}_1 \quad \vec{\varepsilon}_2\,] = \vec{\boldsymbol{\varepsilon}}^T \cdot \vec{\boldsymbol{\varepsilon}}$$

$$= (\vec{\mathbf{e}}\mathbf{S})^T \cdot (\vec{\mathbf{e}}\mathbf{S}) = \mathbf{S}^T\vec{\mathbf{e}}^T \cdot \vec{\mathbf{e}}\mathbf{S} = \mathbf{S}^T\mathbf{S} \tag{7}$$

and similarly $\mathbf{G}' = \vec{\boldsymbol{\varepsilon}}'^T \cdot \vec{\boldsymbol{\varepsilon}}' = \mathbf{S}'^T\mathbf{S}'$.

Intrinsic surface deformation

On a curved surface, it is not possible to have Cartesian coordinates; instead, curvilinear coordinates $\mathbf{u} = [u_1\ u_2]^T$ may be used. If $\vec{x} = \vec{\mathbf{E}}\mathbf{X}$ is the position vector of any point P on the surface with respect to a three-dimensional orthonormal system $\vec{\mathbf{E}} = [\vec{E}_1\,\vec{E}_2\,\vec{E}_3]$, the surface itself is described by the parametric equations $\mathbf{X} = \mathbf{X}(u_1, u_2) = \mathbf{X}(\mathbf{u})$ or in vector form $\vec{X} = \vec{X}(u_1, u_2) = \vec{X}(\mathbf{u})$. The local basis at point P associated with the curvilinear coordinates is simply $\vec{\varepsilon}_1 = \frac{\partial \vec{X}}{\partial u_1} = \vec{\mathbf{E}}\frac{\partial \mathbf{X}}{\partial u_1} \equiv \vec{\mathbf{E}}\mathbf{X}_{u_1}$ and $\vec{\varepsilon}_2 = \frac{\partial \vec{X}}{\partial u_2} = \vec{\mathbf{E}}\frac{\partial \mathbf{X}}{\partial u_2} \equiv \vec{\mathbf{E}}\mathbf{X}_{u_2}$, or in compact form $\vec{\boldsymbol{\varepsilon}} = \frac{\partial \vec{X}}{\partial \mathbf{u}} = \vec{\mathbf{E}}\frac{\partial \mathbf{X}}{\partial \mathbf{u}}$. A small neighborhood of P on the surface can be sufficiently approximated by the "tangent" plane spanned by the vectors $\vec{\varepsilon}_1$, $\vec{\varepsilon}_2$, while the curvilinear coordinate differences $u_1 - u_1(P)$ and $u_2 - u_2(P)$ can be sufficiently approximated by the planar oblique rectilinear coordinates with respect to the local basis ξ_1, and ξ_2, respectively.

The metric matrix is simply

$$\mathbf{G} = \vec{\boldsymbol{\varepsilon}}^T \cdot \vec{\boldsymbol{\varepsilon}} = \left(\vec{\mathbf{E}}\frac{\partial \mathbf{X}}{\partial \mathbf{u}}\right)^T \cdot \left(\vec{\mathbf{E}}\frac{\partial \mathbf{X}}{\partial \mathbf{u}}\right)$$

$$= \left(\frac{\partial \mathbf{X}}{\partial \mathbf{u}}\right)^T (\vec{\mathbf{E}}^T \cdot \vec{\mathbf{E}}) \frac{\partial \mathbf{X}}{\partial \mathbf{u}} = \left(\frac{\partial \mathbf{X}}{\partial \mathbf{u}}\right)^T \frac{\partial \mathbf{X}}{\partial \mathbf{u}}. \tag{8}$$

If we consider the above surface-related parameters to correspond to a reference epoch t, then for the same material point P at a later epoch t', we have on the deformed surface a curvilinear system $\mathbf{u}' = [u_1' u_2']^T$ and a local basis $\vec{\boldsymbol{\varepsilon}}' = \frac{\partial \vec{X}}{\partial \mathbf{u}'} = \vec{\mathbf{E}}'\frac{\partial \mathbf{X}'}{\partial \mathbf{u}'}$ with metric matrix $\mathbf{G}' = \left(\frac{\partial \mathbf{X}'}{\partial \mathbf{u}'}\right)^T \frac{\partial \mathbf{X}'}{\partial \mathbf{u}'}$. Note that we have assumed different three-dimensional reference systems $\vec{\mathbf{E}}$, $\vec{\mathbf{E}}'$ for the two epochs t, t', respectively, so that there is no need to maintain the same external reference system during repeated surveys dedicated to the surface deformation monitoring. This facilitates the surveying procedure since no maintenance of fixed points around the construction is required, while results are the same for the proposed system-invariant parameters as explained in detail by Dermanis (2010).

Due to the local approximation of the surface by its tangent plane, the deformation gradient $\mathbf{F}_\xi = \frac{\partial \xi'}{\partial \xi}$ is equal to the curvilinear coordinate gradient

$$\mathbf{F}_\xi = \frac{\partial \mathbf{u}}{\partial \mathbf{u}'} = \mathbf{F}_\mathbf{u}. \tag{9}$$

Now, dilatation Δ and maximum shear strain γ can be computed from the deformation gradient

$$\mathbf{F} = \mathbf{S}'\mathbf{F}_\mathbf{u}\mathbf{S}^{-1} \tag{10}$$

provided that we can find the appropriate matrices which relate the oblique bases $\vec{\boldsymbol{\varepsilon}}$, $\vec{\boldsymbol{\varepsilon}}'$ to the orthonormal ones $\vec{\mathbf{e}}$, $\vec{\mathbf{e}}'$ through the relations $\vec{\boldsymbol{\varepsilon}} = \vec{\mathbf{e}}\mathbf{S}$, $\vec{\boldsymbol{\varepsilon}}' = \vec{\mathbf{e}}'\mathbf{S}'$. Since the metric matrices are related to the basis transformation matrices through the relations $\mathbf{G}=\mathbf{S}^T\mathbf{S}$, $\mathbf{G}'=\mathbf{S}'^T\mathbf{S}'$, it suffices to diagonalize the metric matrices by $\mathbf{G}=\mathbf{R}(-\Theta)\mathbf{M}\mathbf{R}(\Theta)$, $\mathbf{G}'=\mathbf{R}(-\Theta')\mathbf{M}'\mathbf{R}(\Theta')$, in which case

$$\mathbf{S}=\mathbf{M}^{1/2}\mathbf{R}(\Theta), \quad \mathbf{S}'=\mathbf{M}'^{1/2}\mathbf{R}(\Theta'). \tag{11}$$

Equation 10 can be further simplified if we use the so-called convected coordinates (see, e.g., Marsden and Hughes 1983, p. 41), $\mathbf{u}'=\mathbf{u}$, i.e., if we identify each material point P at epoch t' by the coordinates $u_1'=u_1$, $u_2'=u_2$, which it had at the reference epoch t. With this particular choice, the deformation gradient becomes

$$\begin{aligned}\mathbf{F}=\mathbf{S}'\mathbf{S}^{-1}&=\mathbf{M}'^{1/2}\mathbf{R}(\Theta')\left[\mathbf{M}^{1/2}\mathbf{R}(\Theta)\right]^{-1}\\&=\mathbf{M}'^{1/2}\mathbf{R}(\Theta')\mathbf{R}(-\Theta)\mathbf{M}^{-1/2}\\&=\mathbf{M}'^{1/2}\mathbf{R}(\Theta'-\Theta)\mathbf{M}^{-1/2}.\end{aligned} \tag{12}$$

Once $\mathbf{F}$ is calculated, dilatation and maximum shear strain can be derived from it through its singular value decomposition, as already explained.

Surface bending

Dilatation and maximum shear strain derived trough the approximation of the surface by its tangent plane do not manage to say the whole story about surface deformation. To realize this, consider the case of developable surfaces, like that of a cylinder or cone, which can be formed by bending a plane surface piece without introducing any deformation in its tangent plane at any point. In such a case, the surface remains undeformed from the intrinsic point of view (no dilatation or shear strain) despite the fact that that it is obviously deformed when viewed from outside. Thus, bending of a surface does not contribute to the intrinsic deformation of a two-dimensional surface, but is important in the case of material "surfaces" (thin shells) having the third dimension very small but finite. For this reason, we need an appropriate parameter which describes locally the bending of a surface. Bending of plane curves can be parameterized through the change of their curvature k, which is the inverse $k=1/R$ of the radius of the locally best-fitting circle (radius of curvature). Plane curves lying on a surface can be derived by intersecting the surface with a plane. Of particular importance among such intersections are the "normal sections" with planes containing the line perpendicular to the tangent space at the point of interest. The surface is locally approximated by a best-fitting tri-axial ellipsoid (second-order approximation instead of the first-order one of the tangent plane), with one axis perpendicular to the tangent plane and the other two parallel to it (Fig. 5). The curvatures $k_1>k_2$ at the mutually perpendicular directions of the parallel axes are called principal curvatures. They can be calculated with the help of the first and second fundamental forms of the surface (Stoker 1969). The first fundamental form is related to the distance element ds on the surface by

$$\begin{aligned}I=ds^2=d\mathbf{X}^Td\mathbf{X}&=\left(\frac{\partial\mathbf{X}}{\partial\mathbf{u}}d\mathbf{u}\right)^T\frac{\partial\mathbf{X}}{\partial\mathbf{u}}d\mathbf{u}\\&=d\mathbf{u}^T\left(\frac{\partial\mathbf{X}}{\partial\mathbf{u}}\right)^T\frac{\partial\mathbf{X}}{\partial\mathbf{u}}d\mathbf{u}=d\mathbf{u}^T\mathbf{G}d\mathbf{u},\end{aligned} \tag{13}$$

where $\mathbf{G}$ is the already introduced metric matrix. The second fundamental form is

$$\begin{aligned}II=-d\mathbf{X}^Td\mathbf{N}&=-\left(\frac{\partial\mathbf{X}}{\partial\mathbf{u}}d\mathbf{u}\right)^T\frac{\partial\mathbf{N}}{\partial\mathbf{u}}d\mathbf{u}\\&=d\mathbf{u}^T\left(-\frac{\partial\mathbf{X}}{\partial\mathbf{u}}\right)^T\frac{\partial\mathbf{N}}{\partial\mathbf{u}}d\mathbf{u}=d\mathbf{u}^T\mathbf{L}d\mathbf{u},\end{aligned} \tag{14}$$

where $\vec{N}=\vec{\mathbf{E}}\mathbf{N}$ is the unit vector normal to the surface determined from $\vec{N}=\frac{\vec{\varepsilon}_1\times\vec{\varepsilon}_2}{|\vec{\varepsilon}_1\times\vec{\varepsilon}_2|}$, and

$$\mathbf{L}=-\left(\frac{\partial\mathbf{X}}{\partial\mathbf{u}}\right)^T\frac{\partial\mathbf{N}}{\partial\mathbf{u}}=\left[\frac{\partial}{\partial\mathbf{u}}\left(\frac{\partial\mathbf{X}}{\partial\mathbf{u}}\right)^T\right]\mathbf{N} \tag{15}$$

is the matrix with elements

$$L_{ik}=-\left(\frac{\partial\mathbf{X}}{\partial u_i}\right)^T\frac{\partial\mathbf{N}}{\partial u_k}=\left(\frac{\partial^2\mathbf{X}}{\partial u_i\partial u_k}\right)^T\mathbf{N}. \tag{16}$$

From the matrices $\mathbf{G}$ and $\mathbf{L}$, we can compute the local mean curvature $H=\frac{1}{2}(k_1+k_2)$ and the local Gaussian curvature $K=k_1k_2$ (Stoker 1969, Eqs. 4.25 and 4.26)

$$H=\frac{g_{22}L_{11}-2g_{12}L_{12}+g_{11}L_{22}}{2\det\mathbf{G}}, \quad K=\frac{\det\mathbf{L}}{\det\mathbf{G}}, \tag{17}$$

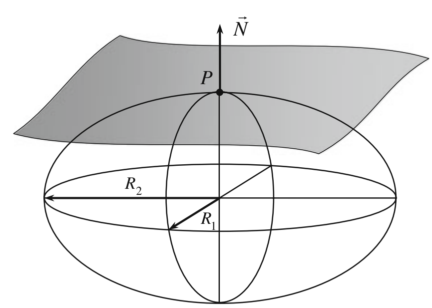

Fig. 5 Local approximation of a surface by a triaxial ellipsoid with one of its axes along the normal vector $\vec{N}$ at the reference point P. The semi-axes R_1 and R_2 of the other two axes correspond to the principal curvatures (maximum and minimum for all normal sections with planes passing through $\vec{N}$) $k_1=1/R_1$ and $k_2=1/R_2$, respectively

from which the principal curvatures are computed

$$k_1 = H + \sqrt{H^2 - K}, \quad k_2 = H - \sqrt{H^2 - K}. \tag{18}$$

Any plane passing through $\vec{N}$ intersects the surface into a "normal section" curve $\vec{X}(s) = \vec{\mathbf{E}}\mathbf{X}(s)$ having a tangent vector $\vec{t} = \vec{\mathbf{E}}\mathbf{T}$ with components

$$\mathbf{T} = \frac{\partial \mathbf{X}}{\partial s} = \frac{\partial \mathbf{X}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial s} \equiv \frac{\partial \mathbf{X}}{\partial \mathbf{u}}\dot{\mathbf{u}}. \tag{19}$$

Recalling that the local tangent plane basis is $\vec{\varepsilon} = \vec{\mathbf{E}}\frac{\partial \mathbf{X}}{\partial \mathbf{u}}$, it follows that $\vec{t} = \vec{\mathbf{E}}\mathbf{T} = \vec{\mathbf{E}}\frac{\partial \mathbf{X}}{\partial \mathbf{u}}\dot{\mathbf{u}} = \vec{\varepsilon}\dot{\mathbf{u}} = \vec{\varepsilon}\mathbf{t}$ and thus $\mathbf{t} = \dot{\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial s} = [\dot{u}_1 \ \dot{u}_2]^T$ are the components of the tangent vector $\vec{t}$ in the local basis $\vec{\varepsilon}$. The curvature of a normal section with tangent vector $\vec{t} = \vec{\varepsilon}\dot{\mathbf{u}}$ is given by (Stoker 1969, Eq. 4.22)

$$k(\dot{\mathbf{u}}) = \frac{II}{I} = \frac{\dot{\mathbf{u}}^T \mathbf{L}\,\dot{\mathbf{u}}}{\dot{\mathbf{u}}^T \mathbf{G}\,\dot{\mathbf{u}}}. \tag{20}$$

In order to determine the principal directions $\vec{t}_1 = \vec{\varepsilon}\mathbf{t}_1 = \vec{\varepsilon}\,\dot{\mathbf{u}}_{(1)}$, $\vec{t}_2 = \vec{\varepsilon}\mathbf{t}_2 = \vec{\varepsilon}\,\dot{\mathbf{u}}_{(2)}$ to which the principal curvatures correspond, we must set $\frac{\partial k}{\partial \dot{\mathbf{u}}} = \frac{2\dot{\mathbf{u}}^T\mathbf{L} - 2k\dot{\mathbf{u}}^T\mathbf{G}}{I} = \mathbf{0}$, which leads to the homogeneous system $(\mathbf{L} - k\mathbf{G})\,\dot{\mathbf{u}} = \mathbf{0}$, having a non-trivial solution $(\dot{\mathbf{u}} \neq \mathbf{0})$ only when $\det(\mathbf{L} - k\mathbf{G}) = k^2 - 2Hk + K = 0$, from which Eq. 18 follows. In this case, however, the matrix $\mathbf{L} - k_i\mathbf{G}$ $(i = 1, 2)$ is singular for the principal curvatures k_1, k_2 and the homogeneous equations $(\mathbf{L} - k_i\mathbf{G})\dot{\mathbf{u}}_{(i)} = \mathbf{0}$ cannot be uniquely solved for the components $\dot{\mathbf{u}}_{(i)}$ of the principal directions. However, we can determine the ratios $\rho_1 = \left(\frac{\dot{u}_1}{\dot{u}_2}\right)_{(1)}$ and $\rho_2 = \left(\frac{\dot{u}_1}{\dot{u}_2}\right)_{(2)}$ using

$$\rho_i = -\frac{L_{12} - k_i g_{12}}{L_{11} - k_i g_{11}} = -\frac{L_{22} - k_i g_{22}}{L_{12} - k_i g_{12}}, \quad i = 1, 2. \tag{21}$$

Since, however, $|\vec{t}_1| = |\vec{t}_2| = 1$, the components of the principal directions in the tangent plane basis $\vec{\varepsilon}$ can be computed from

$$\mathbf{t}_i = \dot{\mathbf{u}}_{(i)} = \frac{1}{\sqrt{g_{11}\rho_i^2 + 2g_{12}\rho_i + g_{22}}}\begin{bmatrix}\rho_i \\ 1\end{bmatrix}, \quad i = 1, 2. \tag{22}$$

The normal curvature $k(\dot{\mathbf{u}})$ at any direction with unit tangent vector $\vec{t} = \vec{\varepsilon}\,\mathbf{t} = \vec{\varepsilon}\dot{\mathbf{u}}$ is much easier to compute by means of the angle θ between $\vec{t}_1$ and $\vec{t}$ ($\vec{t} \cdot \vec{t}_1 = \mathbf{t}^T\mathbf{t}_1 = \cos\theta$) using "Euler's theorem" (Lipschultz 1969, p. 196)

$$k(\theta) = k_1\cos^2\theta + k_2\sin^2\theta = (k_1 - k_2)\cos^2\theta + k_2. \tag{23}$$

Our remaining problem is to find the direction θ of the normal section in which the maximum change in the radius of curvature $R = \frac{1}{k}$ occurs. Repeating the above procedure at the new epoch t' (Fig. 6), we compute the principal curvatures k_1', k_2' of the deformed surface and the curvature $k'(\theta')$ at any direction using 23. We assume that we have in hand the function $\mathbf{X}'(\mathbf{u}')$ of the embedding of the surface and the function $\mathbf{u}'(\mathbf{u})$ of curvilinear coordinate correspondence for the same material point. For the question which is the appropriate curve of the deformed surface with which the normal section at a given direction θ of the original surface should be compared with, there are two possible answers: The first is to compare the normal section $\vec{X}(s) = \vec{\mathbf{E}}\mathbf{X}(\mathbf{s})$ with its image $\vec{X}'(s) = \vec{\mathbf{E}}\mathbf{X}'(\mathbf{s})$, where $\mathbf{X}'(s) = \mathbf{X}'(\mathbf{X}(s))$, which can also be described by $\mathbf{X}'(s) = \mathbf{X}'(\mathbf{u}'(s)) = \mathbf{X}'(\mathbf{u}'(\mathbf{u}(s)))$. The second is to compare the normal section $\vec{X}(s)$ with the normal section which has the same tangent vector $\vec{t}'$ as the deformed curve $\vec{X}'(s)$. Unfortunately, the first choice results into equations which are very complicated to be treated analytically, and for this reason, we will follow the second approach of comparison between corresponding normal sections.

The vector $\vec{t}'$ tangent to $\vec{X}'(s)$ has three-dimensional components $\mathbf{T}' = \frac{\partial \mathbf{X}'}{\partial s} = \frac{\partial \mathbf{X}'}{\partial \mathbf{u}'}\frac{\partial \mathbf{u}'}{\partial \mathbf{u}}\frac{d\mathbf{u}}{ds} \equiv \frac{\partial \mathbf{X}'}{\partial \mathbf{u}'}\mathbf{t}'$, and hence, its components in the local basis $\vec{\varepsilon}' = \vec{\mathbf{E}}\frac{\partial \mathbf{X}'}{\partial \mathbf{u}'}$ are

$$\mathbf{t}' = \frac{\partial \mathbf{u}'}{\partial \mathbf{u}}\frac{d\mathbf{u}}{ds} = \mathbf{F}_{\mathbf{u}}\dot{\mathbf{u}} = \mathbf{F}_{\mathbf{u}}\mathbf{t}. \tag{24}$$

Since $\mathbf{t} = \mathbf{t}(\theta) = \mathbf{t}_1\cos\theta + \mathbf{t}_2\sin\theta$ in the original surface, it follows that

$$\mathbf{t}'(\theta) = \mathbf{F}_{\mathbf{u}}\mathbf{t}(\theta) = \mathbf{F}_{\mathbf{u}}(\mathbf{t}_1\cos\theta + \mathbf{t}_2\sin\theta). \tag{25}$$

The direction $\theta' = \theta'(\theta)$ of $\vec{t}'$ in the tangent plane of the deformed surface is determined from

$$\cos\theta' = \vec{t}' \cdot \vec{t}_1' = \mathbf{t}_1'^T\mathbf{t}' = \mathbf{t}_1'^T\mathbf{F}_{\mathbf{u}}(\mathbf{t}_1\cos\theta + \mathbf{t}_2\sin\theta) = \beta_1\cos\theta + \beta_2\sin\theta \tag{26}$$

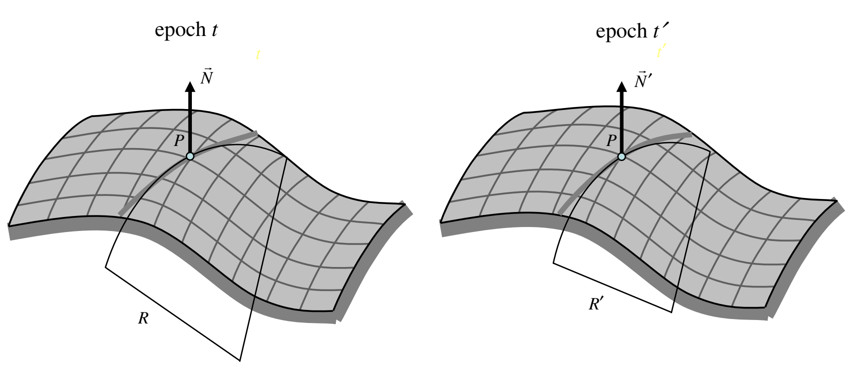

Fig. 6 Conjugate normal sections at the two epochs t and t' and their corresponding radii of curvature R and R'. For any normal section at t, the conjugate normal section at t' is the one that has the same direction (tangent vector) as the deformed image at t' of the original normal section at t

where $\mathbf{t}_1^{'}$, $\mathbf{t}_2^{'}$ are computed from an equation similar to 22 using the corresponding quantities $g_{ik}^{'}$, $\rho_i^{'}$, $i = 1, 2$, computed for the second epoch t' from equations similar to 8, 15–18, 21, and 22, while the new auxiliary parameters have been introduced, namely,

$$\beta_1 = \mathbf{t}_1^{'T}\mathbf{F}_\mathbf{u}\mathbf{t}_1, \quad \beta_2 = \mathbf{t}_1^{'T}\mathbf{F}_\mathbf{u}\mathbf{t}_2. \tag{27}$$

We shall also need the derivative

$$\frac{d\theta'}{d\theta} = \frac{\mathbf{t}_1^{'T}\mathbf{F}_\mathbf{u}(\mathbf{t}_1 \sin\theta - \mathbf{t}_2 \cos\theta)}{\sin\theta'} = \frac{\mathbf{t}_1^{'T}\mathbf{F}_\mathbf{u}\mathbf{t}_1 \sin\theta - \mathbf{t}_1^{'T}\mathbf{F}_\mathbf{u}\mathbf{t}_2 \cos\theta}{\sin\theta'} = \frac{\beta_1 \sin\theta - \beta_2 \cos\theta}{\sin\theta'}. \tag{28}$$

The curvature $k'(\theta) = k'(\theta'(\theta))$ of the corresponding normal section on the deformed surface is according to Euler's equation

$$k' = k_1^{'}\cos^2\theta' + k_2^{'}\sin^2\theta' = (k_1^{'} - k_2^{'})\cos^2\theta' + k_2^{'}. \tag{29}$$

As shown in Appendix 1, the maximum of the difference in the radii of curvature

$$\Delta R(\theta) = R' - R = \frac{1}{k'(\theta)} - \frac{1}{k(\theta)} = \max \tag{30}$$

is obtained from the solution of the fundamental equation

$$F(\theta) = \frac{\left(1 - \frac{k_1}{k_2}\right)\sin 2\theta}{\left[1 - \left(1 - \frac{k_1}{k_2}\right)\cos^2\theta\right]^2} - 2\left(\frac{k_2}{k'_2}\right) \times \frac{\left(1 - \frac{k_1^{'}}{k_2^{'}}\right)(\beta_1 \cos\theta + \beta_2 \sin\theta)(\beta_1 \sin\theta - \beta_2 \cos\theta)}{\left[1 - k_2^{'}\left(1 - \frac{k_1^{'}}{k_2^{'}}\right)(\beta_1 \cos\theta + \beta_2 \sin\theta)^2\right]^2} = 0. \tag{31}$$

This is a complicated nonlinear equation that can only be resolved by numerical methods using appropriate routines from one of the available mathematic programming libraries.

Monitoring surface deformation and bending by repeated surveys

We assume that from surveys repeated in two epochs t and t', we have the corresponding spatial coordinates X_i, Y_i, Z_i, and $X_i^{'} = X_i + U_i, Y_i^{'} = Y_i + V_i, Z_i^{'} = Z_i + W_i$, for a number of discrete points i=1,…,N. We shall use the horizontal coordinates X, Y as surface coordinates u_1=X, u_2=Y at epoch t (Fig. 7) and also as surface coordinates $u_1^{'} = X$, $u_2^{'} = Y$ at epoch t' (convected coordinates). From the discrete data $U_i(X_i,Y_i)$, $V_i(X_i,Y_i)$, $W_i(X_i,Y_i)$, $Z_i(X_i,Y_i)$, we can form by means of some interpolation technique the corresponding displacement functions $U(X,Y)$, $V(X,Y)$, $W(X,Y)$, and the height function $Z(X,Y)$ as well as their corresponding derivatives

$$\begin{array}{lllll}
U_X = \frac{\partial U}{\partial X}, & U_Y = \frac{\partial U}{\partial Y}, & U_{XX} = \frac{\partial^2 U}{\partial X^2}, & U_{YY} = \frac{\partial^2 U}{\partial Y^2}, & U_{XY} = \frac{\partial^2 U}{\partial X \partial Y}; \\
V_X = \frac{\partial V}{\partial X}, & V_Y = \frac{\partial V}{\partial Y}, & V_{XX} = \frac{\partial^2 V}{\partial X^2}, & V_{YY} = \frac{\partial^2 V}{\partial Y^2}, & V_{XY} = \frac{\partial^2 V}{\partial X \partial Y}; \\
W_X = \frac{\partial W}{\partial X}, & W_Y = \frac{\partial W}{\partial Y}, & W_{XX} = \frac{\partial^2 W}{\partial X^2}, & W_{YY} = \frac{\partial^2 W}{\partial Y^2}, & W_{XY} = \frac{\partial^2 W}{\partial X \partial Y}; \\
Z_X = \frac{\partial Z}{\partial X}, & Z_Y = \frac{\partial Z}{\partial Y}, & Z_{XX} = \frac{\partial^2 Z}{\partial X^2}, & Z_{YY} = \frac{\partial^2 Z}{\partial Y^2}, & Z_{XY} = \frac{\partial^2 Z}{\partial X \partial Y};
\end{array}$$

which we shall need later on. For the sake of completeness, we give a short presentation of interpolation of functions $f(X,Y)$ using the collocation method in Appendix 2. The next step is the formulation of the matrices

$$\frac{\partial \mathbf{X}}{\partial \mathbf{u}} = \left[\frac{\partial \mathbf{X}}{\partial u_1} \quad \frac{\partial \mathbf{X}}{\partial u_2}\right] \equiv [\mathbf{X}_{u_1} \quad \mathbf{X}_{u_2}] = \begin{bmatrix} \frac{\partial X}{\partial X} & \frac{\partial X}{\partial Y} \\ \frac{\partial Y}{\partial X} & \frac{\partial Y}{\partial Y} \\ \frac{\partial Z}{\partial X} & \frac{\partial Z}{\partial Y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ Z_X & Z_Y \end{bmatrix}, \tag{32}$$

$$\frac{\partial \mathbf{X}'}{\partial \mathbf{u}'} = \left[\frac{\partial \mathbf{X}'}{\partial u_1} \quad \frac{\partial \mathbf{X}'}{\partial u_2}\right] \equiv \left[\mathbf{X}'_{u_1'} \quad \mathbf{X}'_{u_2'}\right] = \begin{bmatrix} \frac{\partial X'}{\partial X} & \frac{\partial X'}{\partial Y} \\ \frac{\partial Y'}{\partial X} & \frac{\partial Y'}{\partial Y} \\ \frac{\partial Z'}{\partial X} & \frac{\partial Z'}{\partial Y} \end{bmatrix} = \begin{bmatrix} 1 + U_X & U_Y \\ V_X & 1 + V_Y \\ Z_X + W_X & Z_Y + W_Y \end{bmatrix} \tag{33}$$

and the computation of the metric matrices

$$\mathbf{G} = \left(\frac{\partial \mathbf{X}}{\partial \mathbf{u}}\right)^T \frac{\partial \mathbf{X}}{\partial \mathbf{u}}, \quad \mathbf{G}' = \left(\frac{\partial \mathbf{X}'}{\partial \mathbf{u}'}\right)^T \frac{\partial \mathbf{X}'}{\partial \mathbf{u}'}. \tag{34}$$

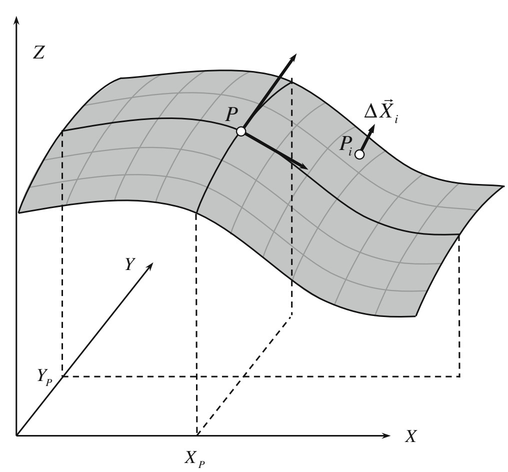

Fig. 7 Monitoring of the surface using convected curvilinear coordinates X,Y for both epochs t and t', with interpolation of the observed displacements $\Delta\vec{X}_i = \Delta\vec{X}(X_i, Y_i)$ with components U_i, V_i, W_i at discrete points P_i, i=1,2,…,n

Diagonalization of the metric matrices gives

$$\mathbf{G}=\begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22}\end{bmatrix}=\begin{bmatrix}\cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta\end{bmatrix}\begin{bmatrix} m_1 & 0 \\ 0 & m_2\end{bmatrix}\begin{bmatrix}\cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta\end{bmatrix}$$
$$=\mathbf{R}(-\Theta)\mathbf{M}\mathbf{R}(\Theta), \tag{35}$$

$$\mathbf{G}'=\begin{bmatrix} g'_{11} & g'_{12} \\ g'_{12} & g'_{22}\end{bmatrix}=\begin{bmatrix}\cos\Theta' & -\sin\Theta' \\ \sin\Theta' & \cos\Theta'\end{bmatrix}\begin{bmatrix} m'_1 & 0 \\ 0 & m'_2\end{bmatrix}\begin{bmatrix}\cos\Theta' & \sin\Theta' \\ -\sin\Theta' & \cos\Theta'\end{bmatrix}$$
$$=\mathbf{R}(-\Theta')\mathbf{M}'\mathbf{R}(\Theta'), \tag{36}$$

where using the auxiliary quantities

$$A=\frac{g_{11}+g_{22}}{2},\quad B=\sqrt{\left(\frac{g_{11}-g_{22}}{2}\right)^2+g_{12}^2},\qquad R=\frac{g_{11}-g_{22}}{2B}, \tag{37}$$

m_1, m_2, and Θ are computed from

$$m_1=A+B,\quad m_2=A-B, \tag{38}$$

$$\sin\Theta=\operatorname{sgn}(g_{12})\sqrt{\frac{1-R}{2}},\quad \cos\Theta=\sqrt{\frac{1+R}{2}} \tag{39}$$

with analogous formulas for m'_1, m'_2 and Θ′.

Application of Eq. 12 gives the deformation gradient matrix

$$\mathbf{F}=\mathbf{M}'^{1/2}\mathbf{R}(\Theta'-\Theta)\mathbf{M}^{-1/2} \tag{40}$$
$$=\begin{bmatrix}\sqrt{\frac{m'_1}{m_1}}\cos(\Theta'-\Theta) & \sqrt{\frac{m'_1}{m_2}}\sin(\Theta'-\Theta) \\ -\sqrt{\frac{m'_2}{m_1}}\sin(\Theta'-\Theta) & \sqrt{\frac{m'_2}{m_2}}\cos(\Theta'-\Theta)\end{bmatrix}.$$

Once $\mathbf{F}$ is determined, we may compute and diagonalize the matrix $\mathbf{C}=\mathbf{F}^T\mathbf{F}=\mathbf{R}(-\theta_P)\mathbf{L}^2\mathbf{R}(\theta_P)$ with

$$\lambda_1^2=\widetilde{A}+\widetilde{B},\ \lambda_2^2=\widetilde{A}-\widetilde{B},$$
$$\sin\theta_P=\operatorname{sgn}(C_{12})\sqrt{\frac{1-\widetilde{R}}{2}},\qquad \cos\theta_P=\sqrt{\frac{1+\widetilde{R}}{2}} \tag{41}$$

where

$$\widetilde{A}=\frac{C_{11}+C_{22}}{2},\quad \widetilde{B}=\sqrt{\left(\frac{C_{11}-C_{22}}{2}\right)^2+C_{12}^2},\quad \widetilde{R}=\frac{C_{11}-C_{22}}{2\widetilde{B}}, \tag{42}$$

and then compute the dilatation $\Delta=\lambda_1\lambda_2-1$, the maximum shear strain $\gamma=\frac{\lambda_1-\lambda_2}{\sqrt{\lambda_1\lambda_2}}$, and its direction angle $\phi=\theta_P-\omega$ using Eq. 4.

The spatial components $\mathbf{N}$, $\mathbf{N}'$ of the unit normal vectors $\vec{N}$, $\vec{N}'$ are given by

$$\mathbf{N}_0=[\mathbf{X}_{u_1}\times]\mathbf{X}_{u_2}=\begin{bmatrix}0 & -Z_X & 0 \\ Z_X & 0 & -1 \\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}0 \\ 1 \\ Z_Y\end{bmatrix}=\begin{bmatrix}-Z_X \\ -Z_Y \\ 1\end{bmatrix},$$
$$\mathbf{N}\equiv\begin{bmatrix}N_X \\ N_Y \\ N_Z\end{bmatrix}=\frac{1}{\sqrt{\mathbf{N}_0^T\mathbf{N}_0}}\mathbf{N}_0 \tag{43}$$

$$\mathbf{N}'_0=\left[\mathbf{X}'_{u'_1}\times\right]\mathbf{X}'_{u'_2}=\begin{bmatrix}0 & -(Z_X+W_X) & V_X \\ Z_X+W_X & 0 & -(1+U_X) \\ -V_X & 1+U_X & 0\end{bmatrix}\begin{bmatrix}U_Y \\ 1+V_Y \\ Z_Y+W_Y\end{bmatrix}.$$
$$=\begin{bmatrix}-(Z_X+W_X)(1+V_Y)+V_X(Z_Y+W_Y) \\ (Z_X+W_X)U_Y-(1+U_X)(Z_Y+W_Y) \\ -V_XU_Y+(1+U_X)(1+V_Y)\end{bmatrix},\ \mathbf{N}'\equiv\begin{bmatrix}N'_X \\ N'_Y \\ N'_Z\end{bmatrix}=\frac{1}{\sqrt{\mathbf{N}'^T_0\mathbf{N}'_0}}\mathbf{N}'_0 \tag{44}$$

Taking into account that

$$\frac{\partial^2\mathbf{X}}{\partial X^2}=\frac{\partial}{\partial X}\begin{bmatrix}1 \\ 0 \\ Z_X\end{bmatrix}=\begin{bmatrix}0 \\ 0 \\ Z_{XX}\end{bmatrix},$$
$$\frac{\partial^2\mathbf{X}}{\partial Y^2}=\frac{\partial}{\partial Y}\begin{bmatrix}0 \\ 1 \\ Z_Y\end{bmatrix}=\begin{bmatrix}0 \\ 0 \\ Z_{YY}\end{bmatrix},$$
$$\frac{\partial^2\mathbf{X}}{\partial X\partial Y}=\frac{\partial}{\partial Y}\begin{bmatrix}1 \\ 0 \\ Z_X\end{bmatrix}=\begin{bmatrix}0 \\ 0 \\ Z_{XY}\end{bmatrix},$$
$$\frac{\partial^2\mathbf{X}'}{\partial X^2}=\frac{\partial}{\partial X}\begin{bmatrix}1+U_X \\ V_X \\ Z_X+W_X\end{bmatrix}=\begin{bmatrix}U_{XX} \\ V_{XX} \\ Z_{XX}+W_{XX}\end{bmatrix}, \tag{45}$$
$$\frac{\partial^2\mathbf{X}'}{\partial Y^2}=\frac{\partial}{\partial Y}\begin{bmatrix}U_Y \\ 1+V_Y \\ Z_Y+W_Y\end{bmatrix}=\begin{bmatrix}U_{YY} \\ V_{YY} \\ Z_{YY}+W_{YY}\end{bmatrix},$$
$$\frac{\partial^2\mathbf{X}'}{\partial X\partial Y}=\frac{\partial}{\partial Y}\begin{bmatrix}1+U_X \\ V_X \\ Z_X+W_X\end{bmatrix}=\begin{bmatrix}U_{XY} \\ V_{XY} \\ Z_{XY}+W_{XY}\end{bmatrix}$$

the second fundamental form matrices are computed from

$$\mathbf{L}=\begin{bmatrix}\left(\frac{\partial^2\mathbf{X}}{\partial X^2}\right)^T\mathbf{N} & \left(\frac{\partial^2\mathbf{X}}{\partial X\partial Y}\right)^T\mathbf{N} \\ \left(\frac{\partial^2\mathbf{X}}{\partial X\partial Y}\right)^T\mathbf{N} & \left(\frac{\partial^2\mathbf{X}}{\partial Y^2}\right)^T\mathbf{N}\end{bmatrix}=N_Z\begin{bmatrix}Z_{XX} & Z_{XY} \\ Z_{XY} & Z_{YY}\end{bmatrix} \tag{46}$$

$$\mathbf{L}' = \begin{bmatrix} \left(\frac{\partial^2 \mathbf{X}'}{\partial X^2}\right)^T \mathbf{N}' & \left(\frac{\partial^2 \mathbf{X}'}{\partial X \partial Y}\right)^T \mathbf{N}' \\ \left(\frac{\partial^2 \mathbf{X}'}{\partial X \partial Y}\right)^T \mathbf{N}' & \left(\frac{\partial^2 \mathbf{X}'}{\partial Y^2}\right)^T \mathbf{N}' \end{bmatrix}$$
$$= \begin{bmatrix} U_{XX}N_X' + V_{XX}N_Y' + (Z_{XX} + W_{XX})N_Z' & U_{XY}N_X' + V_{XY}N_Y' + (Z_{XY} + W_{XY})N_Z' \\ U_{XY}N_X' + V_{XY}N_Y' + (Z_{XY} + W_{XY})N_Z' & U_{YY}N_X' + V_{YY}N_Y' + (Z_{YY} + W_{YY})N_Z' \end{bmatrix} \tag{47}$$

At this point, we have all the necessary data for computing

$$H = \frac{g_{22}L_{11} - 2g_{12}L_{12} + g_{11}L_{22}}{2(g_{11}g_{22} - g_{12}^2)}, \quad K = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2}, \tag{48}$$

$$H' = \frac{g_{22}'L_{11}' - 2g_{12}'L_{12}' + g_{11}'L_{22}'}{2(g_{11}'g_{22}' - g_{12}'^2)}, \quad K' = \frac{L_{11}'L_{22}' - L_{12}'^2}{g_{11}'g_{22}' - g_{12}'^2} \tag{49}$$

$$k_1 = H + \sqrt{H^2 - K}, \quad k_2 = H - \sqrt{H^2 - K}, \tag{50}$$

$$k_1' = H' + \sqrt{H'^2 - K'}, \quad k_2' = H' - \sqrt{H'^2 - K'}, \tag{51}$$

$$\rho_1 \equiv \frac{1}{\sigma_1} = -\frac{L_{12} - k_1 g_{12}}{L_{11} - k_1 g_{11}} = -\frac{L_{22} - k_1 g_{22}}{L_{12} - k_1 g_{12}}, \quad \rho_2 \equiv \frac{1}{\sigma_2} = -\frac{L_{12} - k_2 g_{12}}{L_{11} - k_2 g_{11}} = -\frac{L_{22} - k_2 g_{22}}{L_{12} - k_2 g_{12}}, \tag{52}$$

$$\rho_1' \equiv \frac{1}{\sigma_1'} = -\frac{L_{12}' - k_1' g_{12}'}{L_{11}' - k_1' g_{11}'} = -\frac{L_{22}' - k_1' g_{22}'}{L_{12}' - k_1' g_{12}'}, \quad \rho_2' \equiv \frac{1}{\sigma_2'} = -\frac{L_{12}' - k_2' g_{12}'}{L_{11}' - k_2' g_{11}'} = -\frac{L_{22}' - k_2' g_{22}'}{L_{12}' - k_2' g_{12}'}. \tag{53}$$

$$\beta_1 = \frac{1 + \rho_1 \rho_1'}{\sqrt{(g_{11}'\rho_1'^2 + 2g_{12}'\rho_1' + g_{22}')(g_{11}\rho_1^2 + 2g_{12}\rho_1 + g_{22})}}, \tag{54}$$

$$\beta_2 = \frac{1 + \rho_2 \rho_1'}{\sqrt{(g_{11}'\rho_1'^2 + 2g_{12}'\rho_1' + g_{22}')(g_{11}\rho_2^2 + 2g_{12}\rho_2 + g_{22})}}. \tag{55}$$

If $\rho_i > 1$ or $\rho_1' > 1$, we use instead $\sigma_i = 1/\rho_i < 1$ or $\sigma_1' = 1/\rho_1' < 1$ and modify accordingly the equations for the computation of β_1 and β_2 in order to avoid numerical problems when $\rho_i \to \infty$ or $\rho_i' \to \infty$. With θ determined from the numerical solution of the nonlinear Eq. 31, we finally compute

$$cos\theta' = \beta_1 \cos\theta + \beta_2 \sin\theta, \tag{56}$$

$$k = k_1 \cos^2\theta + k_2 \sin^2\theta, \quad k' = k_2' + (k_1' - k_2')\cos^2\theta', \tag{57}$$

$$\Delta R_{\max} = \frac{1}{k} - \frac{1}{k'}. \tag{58}$$

A more intuitively appealing non-dimensional quantity is the relative change of radius of curvature

$$\frac{\Delta R_{\max}}{R} = \frac{R' - R}{R} = \frac{R'}{R} - 1 = \frac{k}{k'} - 1. \tag{59}$$

The change of sign in 58 and 59 is due to the fact that the choice of the order in the tangent vectors $\partial_X \mathbf{X}$, $\partial_Y \mathbf{X}$ produces an outward normal vector $\mathbf{N}$ and thus negative curvatures (Stoker 1969, p. 89).

Incorporation of additional "in situ" observations

In addition or in place of the displacement observables provided by classical surveying techniques, in situ measurements may be utilized using such instruments as strainmeters, extensometers, tiltmeters, and other sensors. Such observables can always be expressed as linear functionals $L_i(f)$ of the interpolated field $f(X,Y)$, which is either one of the displacement components $U(X,Y)$, $V(X,Y)$, $W(X,Y)$, or the height $Z(X,Y)$. As explained in Appendix 2, these observations are incorporated in the collocation interpolation method by simply applying the covariance propagation law as described in Eq. 71 to derive the necessary matrices $\mathbf{C}$ and $\mathbf{c}_P$ to be used in the final interpolation formula $f(P) = \mathbf{c}_P^T(\mathbf{C} + \Sigma)^{-1}\mathbf{y}$.

Strainmeters, extensometers, tiltmeters, and other related instruments typically produce "observables" referring to a specific point, which depend on the three-dimensional strain tensor $\mathbf{E} \approx \frac{1}{2}(\mathbf{J} + \mathbf{J}^T)$, where $\mathbf{J} = \mathbf{F} - \mathbf{I}$ is the displacement gradient and $\mathbf{F} = \frac{\partial \mathbf{x}'}{\partial \mathbf{x}}$ is the deformation gradient in three dimensions. The relations of the observables on $\mathbf{E}$ as well as on the antisymmetric rotation matrix $\mathbf{\Omega} = \frac{1}{2}(\mathbf{J} - \mathbf{J}^T)$ are based on the assumption of a homogeneous strain field, such that $\mathbf{F}$, $\mathbf{J}$, $\mathbf{E}$, and $\mathbf{\Omega}$ are constant in the area around the instrument. This assumption may be reasonable for crustal deformation, but may be violated in construction monitoring. Furthermore, the observables refer to three-dimensional deformation and cannot easily be adapted to our case of two-dimensional surface deformation. For these reasons, we will go back to the primary measurements which always relate to a two-point instrument baseline, which from its original value $\mathbf{b} = \mathbf{x}_2 - \mathbf{x}_1$ at epoch t obtains the deformed value $\mathbf{b}' = \mathbf{x}_2' - \mathbf{x}_1' = \mathbf{b} + \mathbf{s}$ at a later epoch t', where $\mathbf{s} = (\mathbf{x}'_2 - \mathbf{x}_2) - (\mathbf{x}'_1 - \mathbf{x}_1) = \mathbf{u}_2 - \mathbf{u}_1$ is the relative displacement. The displacement at any point $\mathbf{x}$ is given by $\mathbf{u} = \mathbf{x}' - \mathbf{x} = [U\ V\ W]^T$.

The first primary measurement is the *extension* (Agnew 1986)

$$\varepsilon = \frac{|\mathbf{b}'| - |\mathbf{b}|}{|\mathbf{b}|} \equiv \frac{s' - s}{s} \approx \frac{\mathbf{b}_0^T \mathbf{s}}{s} \tag{60}$$

where $s = |\mathbf{b}|$, $s' = |\mathbf{b}'|$ are the baseline lengths and $\mathbf{b}_0 = \frac{1}{s}\mathbf{b}$ is the unit vector in the original baseline direction. The second primary measurement by tiltmeters relates to the infinitesimal rotation vector

$$\boldsymbol{\omega}_{\text{inf}} \equiv \frac{1}{s^2}[\mathbf{b}'\times]\mathbf{b} = \frac{1}{s}[\mathbf{s}\times]\mathbf{b}_0, \tag{61}$$

from which the vector of *deformational tilt* follows as

$$\boldsymbol{\omega}_t = [\mathbf{n}\times]\boldsymbol{\omega}_{\text{inf}} = [\mathbf{n}\times]\left(\frac{1}{s^2}[\mathbf{s}\times]\mathbf{b}\right)$$

$$= -\frac{1}{s^2}[\mathbf{n}\times][\mathbf{b}\times]\mathbf{s} = \frac{1}{s^2}\left[(\mathbf{n}^T\mathbf{b})\mathbf{I} - \mathbf{b}\mathbf{n}^T\right]\mathbf{s} \equiv \mathbf{T}\mathbf{s}, \tag{62}$$

while the actually measured quantity is the magnitude $\omega_t = \sqrt{\boldsymbol{\omega}_t^T \boldsymbol{\omega}_t}$. The unknown signals U, V, W to be interpolated are entering in $\mathbf{s} = [\,U_2 - U_1 \quad V_2 - V_1 \quad W_2 - W_1\,]^T$ through their values at the baseline end points. The extension ε already has the form of linear(ized) functional on the unknown functions, while ω_t does not share this property. This can be achieved on the basis of linearization using corrections $\delta U = U - U_0$, $\delta V = V - V_0$, $\delta W = W - W_0$ to a reference displacement field U_0, V_0, W_0, obtained by trend removal or by collocation without the in situ measurements, so that $\mathbf{s} = \mathbf{s}_0 + \delta\mathbf{s}$ and the reduced observables become

$$\delta\varepsilon = \varepsilon - \varepsilon_0 = \frac{\mathbf{b}_0^T \delta\mathbf{s}}{s}, \quad \delta\omega_t = \omega_t - \omega_{t0} = \frac{1}{\omega_{t0}}\mathbf{s}_0^T \mathbf{S}\delta\mathbf{s} \tag{63}$$

where with $\mathbf{b} = [b_1\ b_2\ b_3]^T$

$$\varepsilon_0 = \frac{\mathbf{b}_0^T \mathbf{s}_0}{s}, \tag{64}$$

$$\mathbf{S} = \mathbf{T}^T\mathbf{T} = \frac{1}{s^4}\begin{bmatrix} b_3^2 & 0 & -b_1 b_3 \\ 0 & b_3^2 & -b_2 b_3 \\ -b_1 b_3 & -b_2 b_3 & b_1^2 + b_2^2 \end{bmatrix},$$

$$\omega_{t0} = \sqrt{\mathbf{s}_0^T \mathbf{S} \mathbf{s}_0}.$$

Now the reduced observables $\delta\varepsilon$ and $\delta\omega_t$ are linear functions of $\delta\mathbf{s} = \begin{bmatrix} \delta U_2 - \delta U_1 \\ \delta V_2 - \delta V_1 \\ \delta W_2 - \delta W_1 \end{bmatrix}$ and thus linear functionals of the unknown functions δU, δV, δW. They can therefore be directly used in the collocation approach by applying straightforward rules for covariance propagation.

A demonstrative application example

In order to demonstrate the applicability of the proposed methodology, an illustrative example is presented based on simulated data consisting of initial and final epoch coordinates of a set of points distributed in a more or less regular way on a dome having the shape of a spherical cap. The dome has a radius of 40 m and a base diameter of 50 m. The center of the sphere is located at a height of 20 m below the ground, which is used as the horizontal (X,Y) plane of the reference system. The Z-axis is placed at the center of the dome and thus passes through the center of the sphere. The discrete available horizontal displacements $U=\Delta X$, $V=\Delta Y$ are depicted in Fig. 8 and the vertical ones $W=\Delta Z$ in Fig. 9. For the interpolated displacements, only their modulus is depicted in Fig. 10. The deformation parameters are predicted on a dense horizontal grid from which corresponding two-dimensional isoline images are generated referring to the horizontal projection of the spherical surface. Dilatation Δ is depicted in Fig. 11, maximum shear strain γ in Fig. 12, and relative maximum change of the radius of curvature (bending) $\Delta R/R$ in Fig. 13.

Conclusions and suggestions

The applicability of the new methodology for deriving deformation parameters has been demonstrated by means of

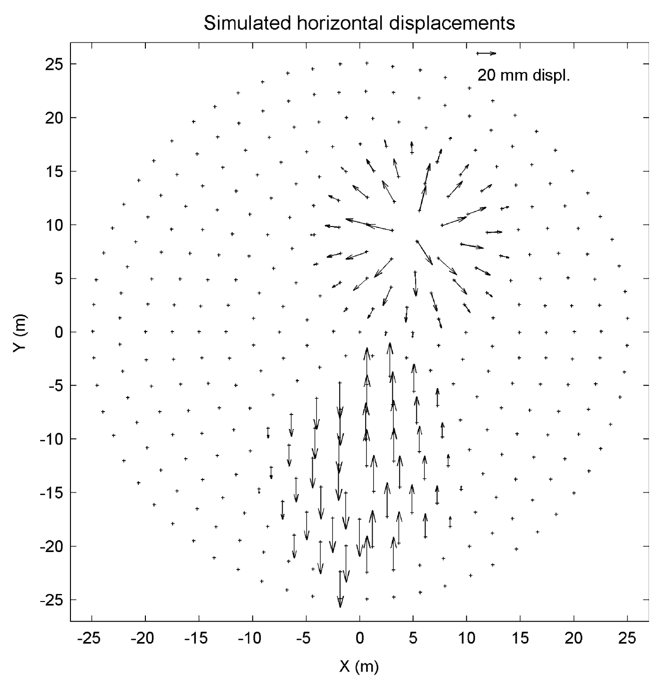

Fig. 8 Horizontal components of the observed (simulated) discrete displacements

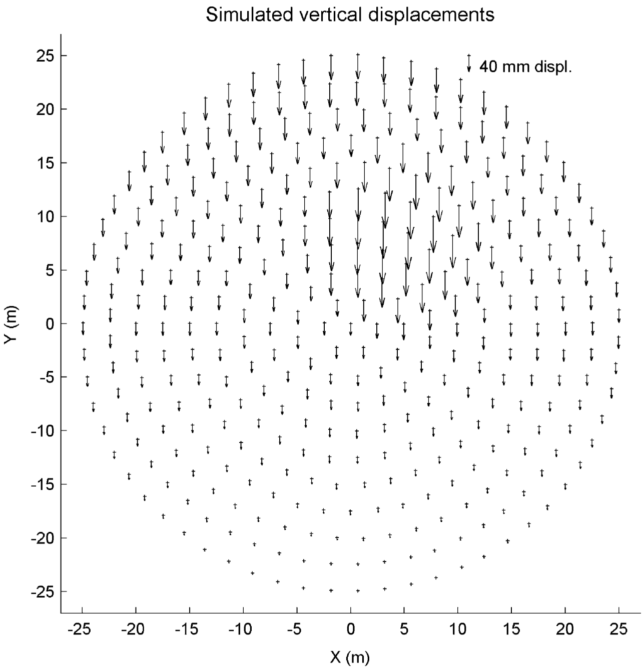

Fig. 9 Vertical components of the observed (simulated) discrete displacements

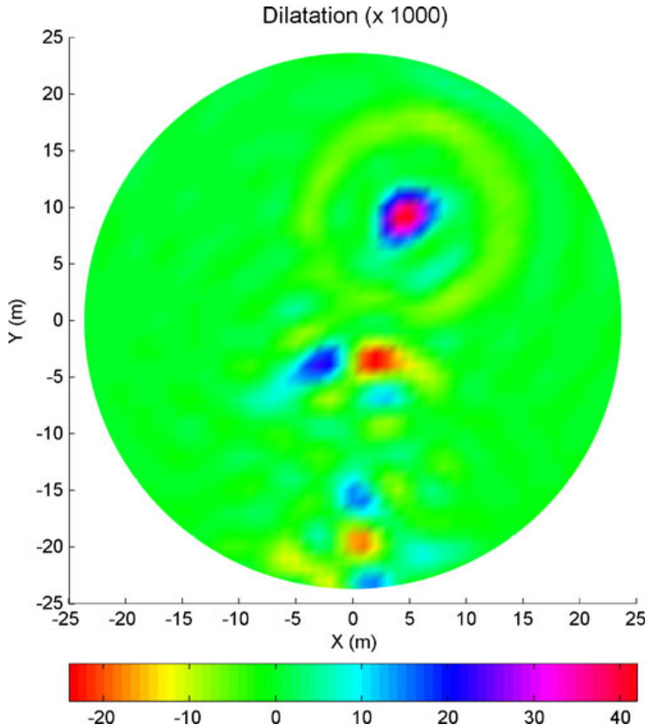

Fig. 11 Computed dilatation Δ

a simple example. The implemented deformation parameters are independent of the choice of the reference system in both epochs. Furthermore, they are selected on the basis of their relevance to construction monitoring, from the point of view of the strength of the materials used for the construction of relatively thin shells. The first step is an interpolation of the observed displacements, considered as functions on the horizontal projection of the curved surface

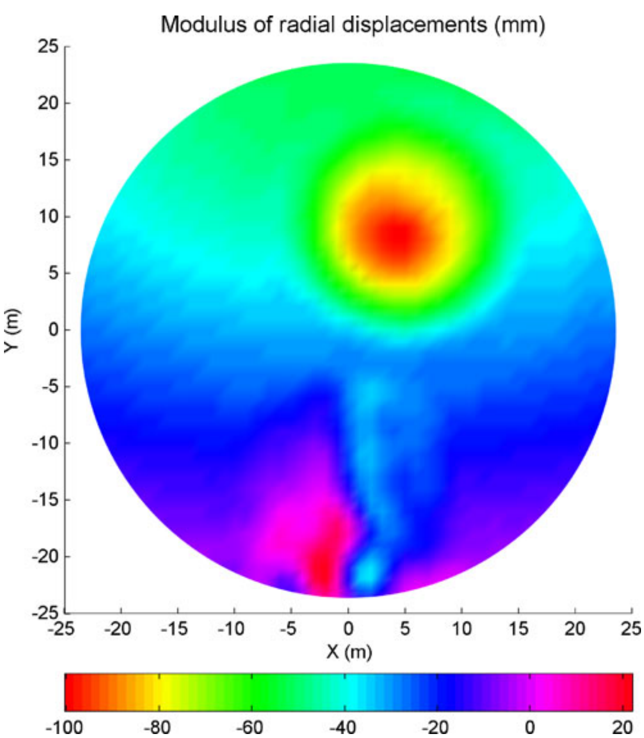

Fig. 10 Modulus of the interpolated displacements

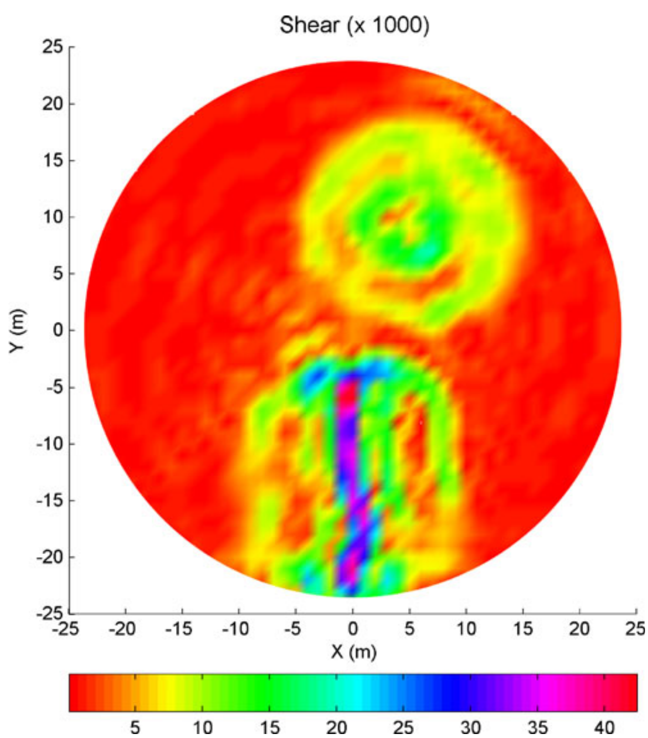

Fig. 12 Computed maximum shear γ

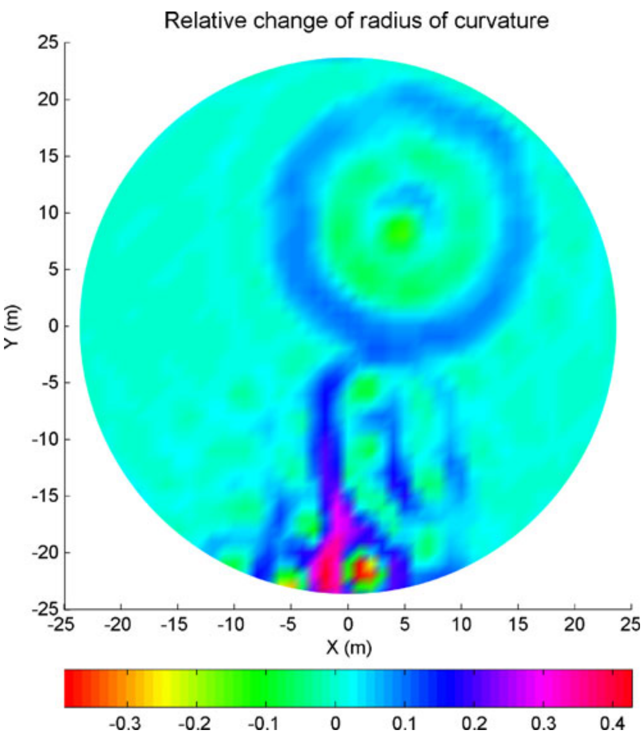

Fig. 13 Computed relative maximum changes in radius of curvature $\Delta R_{\max}/R=(R'-R)/R$

under study. However, the interpolation can be carried in an intrinsic way, on the surface itself, considering the displacements as function of an appropriate set of intrinsic surface coordinates, e.g., spherical coordinates for spherical or near-spherical surface, cylindrical coordinates for near-cylindrical surfaces, and so on. The basic theory is also given here for any set of intrinsic surface coordinates u_1, u_2 so that the modified algorithm can be easily derived by the reader. A remaining open problem that requires further study is the exploitation of virtually continuous data received by laser scanners. The main problem in this case is the identification of pairs of homologous coordinates referring to the same material point. In addition, different numerical methods must be adapted for the determination of first- and second-order derivatives for dense quasi-continuous data, such as the implementation of moving interpolation methods, similar, e.g., to those in template filters used in image analysis for the determination of the Laplacian.

Acknowledgment I wish to express my gratitude to Dr. Ludovico Biagi of the Politecnico di Milano, Polo di Como who developed the necessary software for the implementation of the suggested methodology and performed the calculations for the presented application example.

Appendix 1: Determination of the angle θ of maximum radii of curvature variation for corresponding normal sections

The maximum of the difference in the radii of curvature

$$\Delta R(\theta)=R'-R=\frac{1}{k'(\theta)}-\frac{1}{k(\theta)}=\max \tag{65}$$

is obtained at the angle θ for which $\frac{\partial \Delta R}{\partial\theta}=0$, or explicitly

$$\frac{\partial \Delta R}{\partial\theta}=-\frac{1}{k'^2}\frac{\partial k'}{\partial\theta}+\frac{1}{k^2}\frac{\partial k}{\partial\theta}=0. \tag{66}$$

From the differentiation of Euler's equation, $k=k_1\cos^2\theta+k_2\sin^2\theta$ follows that $\frac{\partial k}{\partial\theta}=(k_2-k_1)\sin 2\theta$ and $\frac{\partial k'}{\partial\theta'}=(k_2'-k_1')\sin 2\theta'$. Since $\frac{\partial k'}{\partial\theta}=\frac{\partial k'}{\partial\theta'}\frac{d\theta'}{d\theta}=(k_2'-k_1')\sin 2\theta'\frac{d\theta'}{d\theta}$ and $\frac{d\theta'}{d\theta}=\frac{\beta_1\sin\theta-\beta_2\cos\theta}{\sin\theta'}$, Eq. 66 becomes

$$\begin{aligned}
&\frac{k_2-k_1}{k^2}\sin 2\theta-\frac{k_2'-k_1'}{k'^2}\sin 2\theta'\frac{d\theta'}{d\theta}\\
&=\frac{(k_2-k_1)\sin 2\theta}{\left[(k_1-k_2)\cos^2\theta+k_2\right]^2}-\frac{(k_2'-k_1')\sin 2\theta'}{\left[(k_1'-k_2')\cos^2\theta'+k_2'\right]^2}\frac{\beta_1\sin\theta-\beta_2\cos\theta}{\sin\theta'}\\
&=\frac{(k_2-k_1)\sin 2\theta}{\left[(k_1-k_2)\cos^2\theta+k_2\right]^2}-\frac{2(k_2'-k_1')\cos\theta'(\beta_1\sin\theta-\beta_2\cos\theta)}{\left[(k_1'-k_2')\cos^2\theta'+k_2'\right]^2}=0.
\end{aligned} \tag{67}$$

Replacing $\cos\theta'=\beta_1\cos\theta+\beta_2\sin\theta$, our fundamental equation for the determination of θ becomes

$$\begin{aligned}
&\frac{(k_2-k_1)\sin 2\theta}{\left[(k_1-k_2)\cos^2\theta+k_2\right]^2}\\
&\quad-\frac{2(k_2'-k_1')(\beta_1\cos\theta+\beta_2\sin\theta)(\beta_1\sin\theta-\beta_2\cos\theta)}{\left[(k_1'-k_2')(\beta_1\cos\theta+\beta_2\sin\theta)^2+k_2'\right]^2}\\
&=0
\end{aligned} \tag{68}$$

Appendix 2: Interpolation of plane functions $f(X,Y)$ using the collocation method

We shall present shortly the method of interpolation known in geodesy as collocation which is based on the stochastic interpretation of the unknown plane function $f(X,Y)$ as a random field so that its values at points other than those where the values $f(X_i,Y_i)$, $i=1,\ldots,n$, have been observed are random variables predicted using the principle of minimum mean square error of prediction among all inhomogeneous linear function of the available data.

In general, a function $f(P)=f(X,Y)$ with discrete data $f_i=f(P_i)=f(X_i,Y_i)$ can be interpolated by modeling it as a linear combination $f(P)=a_1\varphi_1(P)+a_2\varphi_2(P)+\ldots+a_m\varphi_m(P)$ of known basis functions $\varphi_k(P)$, $k=1,\ldots,m$. For all the available data $f_i\equiv f(P_i)$, $i=1,\ldots,n$, the corresponding set of n equations $f_i = a_1\varphi_1(P_i) + a_2\varphi_2(P_i) + ... + a_m\varphi_m(P_i)$ in the m unknowns a_k takes the matrix form **f**=**Fa**, with $F_{ik} = \varphi_k(P_i)$. When $n>m$, there is no exact solution, but instead we may set $\mathbf{f} = \mathbf{Fa} + \mathbf{v}$ and obtain a least-squares smoothing interpolation through $\widehat{\mathbf{a}} = (\mathbf{F}^T\mathbf{PF})^{-1}\mathbf{F}^T\mathbf{Pf}$ for a positive-definite weight matrix **P**. For $n=m$, an exact interpolation is obtained through $\widehat{\mathbf{a}} = \mathbf{F}^{-1}\mathbf{f}$. However, the most flexible case is that of exact interpolation with $n<m$ and an infinite number of solutions, in which case a unique one $\widehat{\mathbf{a}} = \mathbf{W}^{-1}\mathbf{F}^T(\mathbf{FW}^{-1}\mathbf{F}^T)^{-1}\mathbf{f}$ satisfying $\mathbf{a}^T\mathbf{Wa}$ = min can be obtained, for some preferably diagonal weight matrix **W**. Setting $(\mathbf{f}_P)_k = \varphi_k(P)$, the interpolated function takes the form

$$f(P) = \sum\nolimits_{k=1}^{m} \widehat{\mathbf{a}}_k\varphi_k(P) = \mathbf{f}_P^T\widehat{\mathbf{a}} = \mathbf{f}_P^T\mathbf{W}^{-1}\mathbf{F}^T(\mathbf{FW}^{-1}\mathbf{F}^T)^{-1}\mathbf{f}. \tag{69}$$

If the two-point function $k(P,Q) = \sum_{k=1}^{m} W_{kk}^{-1}\varphi_k(P)\varphi_k(Q)$ is introduced, the solution can be also written in the compact form $f(P) = \mathbf{k}_P^T\mathbf{K}^{-1}\mathbf{f}$, where $K_{ij} = k(P_i,P_j)$ and $(\mathbf{k}_P)_i = k(P,P_i)$. Thus, the solution can be obtained even without explicitly defining the weights W_{kk} or the base functions $\{\varphi_k(P)\}$ by introducing rather directly the function $k(P,Q)$. A probabilistic interpretation is possible if we set $W_{kk} = 1/\sigma_k^2$ and interpret σ_k^2 as the variances of the zero mean uncorrelated coefficients a_k. When a_k are considered random variables, $f(P) = \sum_{k=1}^{m} a_k\varphi_k(P)$ becomes a zero mean stochastic process (random function) and $k(P,Q) = E\{f(P)f(Q)\}$ is simply the covariance function $C(P,Q)$ of $f(P)$. The interpolating equations $f(P) = \mathbf{k}_P^T\mathbf{K}^{-1}\mathbf{f}$ become in this case the (minimum mean square error) prediction equations

$$f(P) = \mathbf{c}_P^T\mathbf{C}^{-1}\mathbf{f} \tag{70}$$

with $C_{ij} = C(P_i,P_j)$ and $(\mathbf{c}_P)_i = C(P,P_i)$ determined from the single choice of a positive-definite covariance function $C(P,Q) = C(X_P,Y_P;X_Q,Y_Q)$. Since the prediction point P appears only in the vector $\mathbf{c}_P$, it is possible to obtain the required first- and second-order partial derivatives of $f(P) = f(X,Y)$ by differentiating directly the elements $(\mathbf{c}_P)_i = C(P,P_i) = C(X,Y;X_i,Y_i)$. For example, $\frac{\partial^2 f}{\partial X\partial Y}(X,Y) = \mathbf{c}^T\mathbf{C}^{-1}\mathbf{f}$ with $c_i = \frac{\partial^2}{\partial X\partial Y}(\mathbf{c}_P)_i = \frac{\partial^2}{\partial X\partial Y}C(X,Y;X_i,Y_i)$ and similar relations hold for the other partial derivatives. Replacing $f(X,Y)$ with $Z(X,Y)$, $U(X,Y)$, $V(X,Y)$, $W(X,Y)$, we may interpolate–predict the required derivatives, e.g., $U_X = \frac{\partial U}{\partial X}$, $U_{XY} = \frac{\partial^2 U}{\partial X\partial Y}$, etc. If instead of $f_i = f(P_i)$ we have observations $y_i = f(P_i) + v_i$ contaminated with zero mean noise v_i with covariance matrix $\mathbf{\Sigma}$, Eq. 70 is simply replaced by $f(P) = \mathbf{c}_P^T(\mathbf{C}+\Sigma)^{-1}\mathbf{y}$. In order to achieve independence from the reference system used, we need to assume that the random field $f(P)$ is not only homogeneous but also isotropic, in which case the covariance function depends only on the distance r between the points P and Q, i.e., $C(P,Q) = C(r)$. An example of such a covariance function is $C(r) = \sigma^2 e^{kr^2}$, where σ^2 is the variance of the random field, approximated, e.g., by $s^2 = \frac{1}{n}\sum_i f_i^2$, where it is assumed that a constant mean m, approximated by $\overline{f} = \frac{1}{n}\sum_i f_i$, has already been removed from the data ($f_i \rightarrow f_i - \overline{f}$). The parameter $k = -\ln 2/R^2$ relates to the correlation length R defined by $C(R) = \frac{1}{2}C(0)$, which must be of the same order of magnitude as the mean distance between neighboring points. The parameters σ^2 and k are selected so that $C(r)$ best fits an empirical covariance function estimated from the data.

Before interpolating (in particular when coordinates $\mathbf{X}_i$ and $\mathbf{X}_i'$ do not refer to the same reference system), it is necessary to perform a "trend removal" by a least-squares fitting where the coordinates $\mathbf{X}_i'$ are rotated (**R**) and translated (**d**) into a new set $\widetilde{\mathbf{X}}_i' = \mathbf{RX}_i' + \mathbf{d}$, satisfying $\sum_i |\mathbf{X}_i - \widetilde{\mathbf{X}}_i'|^2 = \min$. While the deformation parameters themselves are invariant under changes of the coordinate systems at both epochs under comparison, the same is not true for the interpolation–prediction process. The suggested trend removal satisfies the theoretical requirement that the interpolated displacement fields are zero mean random fields and at the same time secures that the final results are not affected by the adopted reference systems at the two epochs.

The strongest point of the above interpolation prediction is the fact that it can accommodate as input data not only values $f_i=f(P_i)$ of the interpolated field at particular points but also any type of observable depending on the unknown field, which after linearization can be expressed as a continuous linear functional $L_i(f)$. In this case, the elements of the involved matrices become

$$C_{ij} = L_{i,P}L_{i,Q}C(P,Q), \quad (\mathbf{c}_P)_i = L_{i,Q}C(P,Q), \tag{71}$$

where the additional subscripts P and Q clarify with respect to which of the two variables of $C(P,Q)$ the functionals L_i and L_j are acting, according to a scheme that is called "law of covariance propagation."

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