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Applications of Deformation Analysis in Geodesy and Geodynamics

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The role of deformation analysis is discussed with respect to its existing or possible future applications in geodesy and geodynamics. Expressions for strain tensors are given in the more general case of Riemannian spaces and specialized for Euclidean spaces and the case of infinitesimal deformation. Among the various applications, special emphasis is given to the study of crustal deformations of the earth, deformations of the gravity field, and gravity field related deformations. Other applications are also considered.

INTRODUCTION

Deformation, loosely speaking, is the alteration of form and shape. Deformation is historically linked to the study of deformable material bodies, originally within the theory of elasticity and later on within continuum mechanics [*Love*, 1927; *Malvern*, 1969; *Truesdell*, 1977; *Eringen*, 1980]. The abstraction of relevant concepts and mathematical tools developed in mechanics leads to their broader use in a variety of problems which are not related to material bodies.

Since form and shape are connected to the metric characteristics of bodies, deformation, in a broad sense, is also concerned with the alteration of such characteristics of general physical or even abstract entities. It is only necessary that the elements of such entities can be brought into a one-to-one correspondence, in the same way that through the identification of material points a correspondence is established between two different states of a deformable body.

Applying deformation analysis in the geosciences, for example, in geodesy, the previous discussion meets a clear example: From its definition, geodesy studies among other topics the material shape and the gravity field of the earth as well as their alteration in space and time. The earth body deformation corresponds to the classical treatment in mechanics of deformable bodies, while the deformation of the gravity field of the earth calls for an abstract application of the same tools. Another, even more obvious, example of abstract deformation is illustrated by mapping the geographic grid of parallels and meridians from the globe onto a plane grid, a standard technique in cartography; the spherical and plane grids under comparison are obviously abstract entities of a nonmaterial nature.

In mechanics, deformation (strain) is not studied by itself but mainly in connection with the underlying forces (stress-strain relation). Although the forces acting to cause earth deformation are of interest in both geodesy and geophysics, geodesists are primarily concerned with the development of methods and techniques for the determination of such deformation.

The object of geodesy is the study of the size, shape, and gravity field of the earth, position determination, time variation of all the above, and their representation. Consequently, deformation methods find manifold applications in geodetic work.

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MATHEMATICAL TOOLS

Since form and shape are described mathematically by the concept of metric, deformation can be represented by mathematical objects which depend on metric alterations.

Usually, in classical mechanics the relevant metric spaces are Euclidean. This is due to the model accepted for the description of the physical three-dimensional space in the neighborhood of the earth. The most convenient way of describing Euclidean space is by means of Cartesian orthogonal coordinate systems. However, curvilinear coordinates within Euclidean spaces are also useful in some applications.

In order to apply the above mathematical techniques outside the realm of mechanics, a generalization to Riemmanian metric spaces is sometimes necessary, for example, in cartographic applications.

Therefore the development of our formulation will start from curvilinear Riemmanian spaces. Euclidean descriptions with curvilinear or Cartesian coordinate systems will then be derived as special cases.

The Riemmanian Case

Let there be two Riemmanian spaces with corresponding metrics dS^2 and ds^2 which describe their metrical properties in the infinitesimal neighborhood of points of the two spaces. Curvilinear coordinate systems $(U^1\ U^2\ U^3)$ and $(u^1\ u^2\ u^3)$ are assigned to the two spaces. The metrics with respect to the chosen coordinate system are given by

$$dS^2 = \sum_{ij} G_{ij}\, dU^i\, dU^j = d\mathbf{U}^T\mathbf{G}\, d\mathbf{U} \tag{1}$$

$$ds^2 = \sum_{ij} g_{ij}\, du^i\, du^j = d\mathbf{u}^T\mathbf{g}\, d\mathbf{u} \tag{2}$$

where G_{ij}, g_{ij} are the corresponding metric tensors and $\mathbf{G}$, $\mathbf{g}$ their matrix representations.

It is furthermore assumed that a one-to-one correspondence exists between the two spaces described by

$$\mathbf{u} = \mathbf{u}(\mathbf{U}) \tag{3}$$

This mapping function must also be continuous and have a continuous inverse, that is, a homeomorphism in mathematical terms. In classical mechanics the sets $\mathbf{U}$ and $\mathbf{u}(\mathbf{U})$ correspond to the same point in two different states of a material body.

For the description of deformations we look at the differences $(ds^2 - dS^2)$ between corresponding elements of the two spaces under comparison. These differences can be expressed either in terms of the first coordinate system $\mathbf{U}$ (Lagrangian approach) or in terms of the second coordinate system $\mathbf{u}$ (Eulerian approach).

When the Lagrangian approach is followed, the Lagrangian strain tensor (strain matrix) $\mathbf{E}$ is introduced by means of

$$ds^2 - dS^2 \stackrel{\text{def}}{=} 2d\mathbf{U}^T\mathbf{E}\, d\mathbf{U} \tag{4}$$

It follows that

$$\mathbf{E} = \tfrac{1}{2}[(\partial\mathbf{u}/\partial\mathbf{U})^T\mathbf{g}(\partial\mathbf{u}/\partial\mathbf{U}) - \mathbf{G}] \tag{5}$$

When the Eulerian approach is followed, the Eulerian strain tensor (strain matrix) $\mathbf{E}^*$ is introduced by means of

$$ds^2 - dS^2 \stackrel{\text{def}}{=} 2d\mathbf{u}^T\mathbf{E}^*\, d\mathbf{u} \tag{6}$$

so that

$$\mathbf{E}^* = \tfrac{1}{2}[\mathbf{g} - (\partial\mathbf{U}/\partial\mathbf{u})^T\mathbf{G}(\partial\mathbf{U}/\partial\mathbf{u})] \tag{7}$$

We have called $\mathbf{E}$ and $\mathbf{E}^*$ in (5) and (7) tensors. However, some clarifications are needed. Tensors obey well-known transformation rules under changes of coordinate systems. These rules apply to $\mathbf{E}$ and $\mathbf{E}^*$ when the first coordinate system $\mathbf{U}$ undergoes a transformation, while the second coordinate system $\mathbf{u}$ undergoes the corresponding transformation induced by (3). On the one hand, function (3) depends on an underlying correspondence between the elements of the two compared Riemmanian spaces; on the other hand, its specific mathematical representation is a consequence of the more or less arbitrary selection of the coordinate systems $\mathbf{U}$ and $\mathbf{u}$. A general discussion of this aspect is given by *Boucher* [1980]. The point will be clarified later when specific applications are analyzed.

Note that all components of the strain tensor must have the same physical dimensions. The reason for this restriction is that scalar invariants of the strain tensor are sometimes used as deformation parameters. Since these invariants (i.e., the trace and determinant) are combinations of the tensor components, they are physically meaningful when the particular components have the same physical dimensions. Recalling (4), the above restriction necessitates the use of curvilinear coordinates of the same physical units in contrast to some usual choices (e.g., spherical coordinates with both angular and linear units). In order to avoid this inconvenience, all coordinates may be transformed to coordinates of the same units through multiplication with suitable scalar factors. For example, when angular coordinates are used in the same set with linear coordinates, they can be transformed to linear ones through multiplication with the radii of curvature of the corresponding coordinate lines.

Equation (4) can be written as

$$ds^2/dS^2 \stackrel{\text{def}}{=} 1 + 2(d\mathbf{U}^T/dS)\mathbf{E}(d\mathbf{U}/dS) \tag{8}$$

or

$$m_L{}^2 \stackrel{\text{def}}{=} 1 + 2\mathbf{L}^T\mathbf{E}\mathbf{L} \tag{9}$$

where m_L is the linear modulus of deformation along the direction of the unit vector $\mathbf{L} = d\mathbf{U}/dS$. Replacing $\mathbf{L}^T\mathbf{L} = 1$ in (9), we obtain

$$m_L{}^2 = \mathbf{L}^T(\mathbf{I} + 2\mathbf{E})\mathbf{L} \tag{10}$$

which represents a quadratic form. The associated quadric surface, being the geometric locus of all points with $m_L{}^2 = 1$, is the well-known Cauchy strain ellipsoid in three dimensions or the Tissot strain ellipse in two dimensions used in cartography.

In the usual description of this quadratic form, the restriction that all curvilinear coordinates in $\mathbf{U}$ be of the same physical dimension is relaxed, and (9) is now written as

$$m_L{}^2 = 1 + 2\mathbf{L}^T\mathbf{KEKL} \tag{11}$$

where $\mathbf{K}$ is the diagonal matrix with elements the corresponding curvatures of the coordinate lines. Similarly, (10) becomes

$$m_L{}^2 = \mathbf{L}^T(\mathbf{K}^2 + 2\mathbf{KEK})\mathbf{L} \tag{12}$$

The Euclidean Case

In the special but common case where the spaces under comparison are Euclidean, it is always possible to assign global orthogonal Cartesian coordinate systems, e.g., $(X^1\ X^2\ X^3)$ and $(x^1\ x^2\ x^3)$. Then the corresponding metrics become

$$dS^2 = d\mathbf{X}^T\, d\mathbf{X} \tag{13}$$

$$ds^2 = d\mathbf{x}^T\, d\mathbf{x} \tag{14}$$

that is, the metric tensors become identity ($\mathbf{G} = \mathbf{g} = \mathbf{I}$).

For a given one-to-one correspondence

$$\mathbf{x} = \mathbf{x}(\mathbf{X}) \tag{15}$$

the Langrangian and Eulerian strain tensors become

$$\mathbf{E} = \tfrac{1}{2}[(\partial\mathbf{x}/\partial\mathbf{X})^T(\partial\mathbf{x}/\partial\mathbf{X}) - \mathbf{I}] \tag{16}$$

$$\mathbf{E}^* = \tfrac{1}{2}[\mathbf{I} - (\partial\mathbf{X}/\partial\mathbf{x})^T(\partial\mathbf{X}/\partial\mathbf{x})] \tag{17}$$

It is also possible to describe Euclidean spaces by curvilinear orthogonal coordinate systems, e.g., $(Q^1\ Q^2\ Q^3)$ and $(q^1\ q^2\ q^3)$. In this case the metrics are written as

$$dS^2 = d\mathbf{Q}^T\mathbf{C}\, d\mathbf{Q} \tag{18}$$

$$ds^2 = d\mathbf{q}^T\mathbf{c}\, d\mathbf{q} \tag{19}$$

Comparison with (13) and (14) gives

$$\mathbf{C} = (\partial\mathbf{X}/\partial\mathbf{Q})^T(\partial\mathbf{X}/\partial\mathbf{Q}) \tag{20}$$

$$\mathbf{c} = (\partial\mathbf{x}/\partial\mathbf{q})^T(\partial\mathbf{x}/\partial\mathbf{q}) \tag{21}$$

Similar expressions in tensor notation are given by *Hotine* [1969].

For a given one-to-one correspondence

$$\mathbf{q} = \mathbf{q}(\mathbf{Q}) \tag{22}$$

the Lagrangian and Eulerian strain tensors are

$$\mathbf{E} = \tfrac{1}{2}[\mathbf{C} - (\partial\mathbf{q}/\partial\mathbf{Q})^T\mathbf{c}(\partial\mathbf{q}/\partial\mathbf{Q})] \tag{23}$$

$$\mathbf{E}^* = \tfrac{1}{2}[(\partial\mathbf{Q}/\partial\mathbf{q})^T\mathbf{C}(\partial\mathbf{Q}/\partial\mathbf{q}) - \mathbf{c}] \tag{24}$$

All the above derivations hold equally well for any dimensions, but our applications are limited to two and three dimensions.

Strain tensors are powerful tools in studying deformations, for they allow a pointwise illustration of alteration of the metric properties. In particular, the tensor field $\mathbf{E}(\mathbf{U})$ describes deformation in the infinitesimal vicinity of each element of the first space with coordinates $(U^1\ U^2\ U^3)$. $\mathbf{E}^*(\mathbf{u})$ does the same for elements of the second space.

Infinitesimal Deformation

When deformations are small, the transformation $\mathbf{u} = \mathbf{u}(\mathbf{U})$ is a transformation close to the identity. In this case the displacements $(\xi^1\ \xi^2\ \xi^3)$

$$\xi = \mathbf{u} - \mathbf{U} = \mathbf{u}(\mathbf{U}) - \mathbf{U} = \xi(\mathbf{U}) \tag{25}$$

are quantities of the first order. Differentiating (25), we obtain

$$\frac{\partial \mathbf{u}}{\partial \mathbf{U}} = \frac{\partial \xi}{\partial \mathbf{U}} + \mathbf{I} \tag{26}$$

and replacing into (5), the Lagrangian strain tensor becomes

$$\mathbf{E} = \frac{1}{2}\left[\left(\mathbf{g}\frac{\partial \xi}{\partial \mathbf{U}}\right)^T + \left(\mathbf{g}\frac{\partial \xi}{\partial \mathbf{U}}\right) + \mathbf{g} - \mathbf{G} + \left(\frac{\partial \xi}{\partial \mathbf{U}}\right)^T \mathbf{g}\left(\frac{\partial \xi}{\partial \mathbf{U}}\right)\right] \tag{27}$$

The previous equation is decomposed into a first-order part, linear in $\partial\xi/\partial\mathbf{U}$,

$$\varepsilon \overset{\text{def}}{=} \frac{1}{2}\left[\left(\mathbf{g}\frac{\partial \xi}{\partial \mathbf{U}}\right)^T + \left(\mathbf{g}\frac{\partial \xi}{\partial \mathbf{U}}\right) + \mathbf{g} - \mathbf{G}\right] \tag{28}$$

and into a second-order part

$$\delta\varepsilon = \frac{1}{2}\left[\left(\frac{\partial \xi}{\partial \mathbf{U}}\right)^T \mathbf{g}\left(\frac{\partial \xi}{\partial \mathbf{U}}\right)\right] \tag{29}$$

so that

$$\mathbf{E} = \varepsilon + \delta\varepsilon \tag{30}$$

Similarly,

$$\frac{\partial \mathbf{U}}{\partial \mathbf{u}} = \mathbf{I} - \frac{\partial \xi}{\partial \mathbf{u}} \tag{31}$$

and replacing into (7), the Eulerian strain tensor becomes

$$\mathbf{E}^* = \frac{1}{2}\left[\mathbf{g} - \mathbf{G} + \left(\mathbf{G}\frac{\partial \xi}{\partial \mathbf{u}}\right)^T + \left(\mathbf{G}\frac{\partial \xi}{\partial \mathbf{u}}\right) - \left(\frac{\partial \xi}{\partial \mathbf{u}}\right)^T \mathbf{G}\left(\frac{\partial \xi}{\partial \mathbf{u}}\right)\right] \tag{32}$$

with first-order part

$$\varepsilon^* \overset{\text{def}}{=} \frac{1}{2}\left[\mathbf{g} - \mathbf{G} + \left(\mathbf{G}\frac{\partial \xi}{\partial \mathbf{u}}\right)^T + \left(\mathbf{G}\frac{\partial \xi}{\partial \mathbf{u}}\right)\right] \tag{33}$$

and second-order part

$$\delta\varepsilon^* = -\frac{1}{2}\left[\left(\frac{\partial \xi}{\partial \mathbf{u}}\right)^T \mathbf{G}\left(\frac{\partial \xi}{\partial \mathbf{u}}\right)\right] \tag{34}$$

being

$$\mathbf{E}^* = \varepsilon^* + \delta\varepsilon^* \tag{35}$$

Here ε and ε^* are called the infinitesimal strain tensors. Not only are they first-order approximations of $\mathbf{E}$ and $\mathbf{E}^*$, but they also stand by themselves as independently defined entities.

In the particular case of Euclidean spaces with associated orthogonal Cartesian coordinate systems $\mathbf{X}$ and $\mathbf{x}$ the strain tensors referred to the displacement $\xi = \mathbf{x} - \mathbf{X}$ become

$$\mathbf{E} = \frac{1}{2}\left[\left(\frac{\partial \xi}{\partial \mathbf{X}}\right)^T + \left(\frac{\partial \xi}{\partial \mathbf{X}}\right) + \left(\frac{\partial \xi}{\partial \mathbf{X}}\right)^T\left(\frac{\partial \xi}{\partial \mathbf{X}}\right)\right] \tag{36}$$

$$\mathbf{E}^* = \frac{1}{2}\left[\left(\frac{\partial \xi}{\partial \mathbf{x}}\right)^T + \left(\frac{\partial \xi}{\partial \mathbf{x}}\right) - \left(\frac{\partial \xi}{\partial \mathbf{x}}\right)^T\left(\frac{\partial \xi}{\partial \mathbf{x}}\right)\right] \tag{37}$$

The infinitesimal strain tensors are

$$\varepsilon = \frac{1}{2}\left[\left(\frac{\partial \xi}{\partial \mathbf{X}}\right) + \left(\frac{\partial \xi}{\partial \mathbf{X}}\right)^T\right] \tag{38}$$

$$\varepsilon^* = \frac{1}{2}\left[\left(\frac{\partial \xi}{\partial \mathbf{x}}\right) + \left(\frac{\partial \xi}{\partial \mathbf{x}}\right)^T\right] \tag{39}$$

From the above developments one can observe that the infinitesimal strain tensors are the symmetric part of the Jacobians of the displacements. The antisymmetric parts are

$$\Omega \overset{\text{def}}{=} \frac{1}{2}\left[\left(\frac{\partial \xi}{\partial \mathbf{X}}\right) - \left(\frac{\partial \xi}{\partial \mathbf{X}}\right)^T\right] \tag{40}$$

$$\Omega^* \overset{\text{def}}{=} \frac{1}{2}\left[\left(\frac{\partial \xi}{\partial \mathbf{x}}\right) - \left(\frac{\partial \xi}{\partial \mathbf{x}}\right)^T\right] \tag{41}$$

Thus the corresponding Jacobians, also called deformation gradients, are decomposed as usual into symmetric and antisymmetric parts

$$\partial\xi/\partial\mathbf{X} = \varepsilon + \Omega \tag{42}$$

$$\partial\xi/\partial\mathbf{x} = \varepsilon^* + \Omega^* \tag{43}$$

Note that since the transformation $\mathbf{x} = \mathbf{x}(\mathbf{X})$ is close to the identity, in the first-order approximation

$$\partial\mathbf{x}/\partial\mathbf{X} \simeq \mathbf{I} \tag{44}$$

and

$$\partial\mathbf{X}/\partial\mathbf{x} = (\partial\mathbf{x}/\partial\mathbf{X})^{-1} \simeq \mathbf{I} \tag{45}$$

hold. Consequently,

$$\partial\xi/\partial\mathbf{X} \simeq \partial\xi/\partial\mathbf{x} \tag{46}$$

and

$$\varepsilon \simeq \varepsilon^* \tag{47}$$

$$\Omega \simeq \Omega^* \tag{48}$$

which means that the distinction between the Langrangian and Eulerian approaches practically disappears. For more details on the infinitesimal case, see, for example *Eringen* [1980].

Strain Rates

When metric alterations are associated with a scalar parameter, t being the independent variable, deformation may be studied by comparing the metrics at two infinitesimally closed values t and $t + dt$. In this case the difference of metrics $ds^2 - dS^2$ is replaced by the square of the derivative, $(dS/dt)^2$, in which case

$$(dS/dt)^2 \overset{\text{def}}{=} 2d\mathbf{U}^T(d/dt)\dot{\mathbf{E}}\, d\mathbf{U} \tag{49}$$

Usually, the parameter t represents time, and

$$\dot{\mathbf{E}} \overset{\text{def}}{=} d/dt\ \mathbf{E} \tag{50}$$

is called the strain rate tensor, in terms of the used curvilinear coordinates $\mathbf{U}$.

Stretch-Rotation Decomposition

An advantage of the analysis concerning infinitesimal deformation is the determination of the antisymmetric part of the Jacobian of the displacements $\partial\xi/\partial\mathbf{U}$ (see (40) and (41)), which represents the infinitesimal rotation of a small neighborhood of the point in question with respect to its original orientation.

In the case of noninfinitesimal deformation a similar approach is possible through the polar decomposition theorem of Cauchy [*Truesdell*, 1977]. Restricting ourselves to the Euclidean case, the Jacobian of the displacements can be uniquely decomposed in two ways,

$$\partial \xi / \partial \mathbf{X} = \mathbf{R V}_R = \mathbf{V}_L \mathbf{R} \tag{51}$$

where $\mathbf{R}$ is an orthogonal matrix called the rotation matrix and $\mathbf{V}_R$, $\mathbf{V}_L$ are positive symmetric matrices called the right and the left stretch matrices, respectively. The rotation matrix represents the change of orientation of a small neighborhood of the point in question.

The independent elements of $\mathbf{R}$ can be computed from the displacement Jacobian matrix by solving one of the following rather complicated matrix equations:

$$\mathbf{R} = \frac{\partial \xi}{\partial \mathbf{X}} \left[\left(\frac{\partial \xi}{\partial \mathbf{X}} \right)^T \frac{\partial \xi}{\partial \mathbf{X}} \right]^{-1/2} = \left[\frac{\partial \xi}{\partial \mathbf{X}} \left(\frac{\partial \xi}{\partial \mathbf{X}} \right)^T \right]^{-1/2} \frac{\partial \xi}{\partial \mathbf{X}} \tag{52}$$

DEFORMATION OF THE EARTH

General Discussion

As is well known, geodetic observations are of a geometric and physical nature. They are also limited to points on and/or outside the surface of the earth. It is beyond the scope of this paper to discuss deformation problems related to the interior of the earth [*Bullen*, 1975], since such deformations do not significantly affect geodetic observations. On the contrary, deformations of the crust are directly connected with geodetic observables. Geometric observations are related to the alteration of the two-dimensional surface of the crust as embedded into three dimensions.

Deformations of the crust, beyond the surface, cause changes in the gravity field of the earth and can be sensed through gravity-dependent geodetic observations.

It must be noted that changes in the gravity vector and gravity potential at discrete points of the surface can be ascribed either to the surface motion or to changes in the gravity field per se. These two effects can be separated when the whole surface is covered by observations, in which case geodynamic boundary value problems are formulated for the determination of variation in the geometry of the boundary surface and variations of the gravity field outside the surface [see *Dermanis and Sansò*, 1982; *Heck*, 1981].

The study of global deformations of the crust surface can be primarily done by space geodetic techniques (laser ranging, very long baseline interferometry, etc.), while traditional geodetic techniques (triangulation, trilateration) are valuable for the study of deformations on a local scale. Global crust deformation and plate tectonic motion determination are the subject of extensive research both theoretical and applied, but there are still problems to be solved (for a discussion see, for example, *Baarda* [1975] and *Livieratos* [1979]).

Local crustal deformation studies are referred to two dimensions, following the classical separation of geodetic practice, that is, the duality planimetry-elevation. In this case, leveling is tied through gravity with internal changes. Since height is anholonomic, that is, path-dependent, geopotential numbers must rather be used through the combination of leveling with gravity. Geopotential differences may be caused by changes of the gravity field due to density redistributions within the crust without any actual change of the geometric shape of the surface. A discussion of the discrepancies between actual and apparent height differences as related to crustal movement is given by *Biró* [1975].

In order to compute strains in a three-dimensional medium (e.g., crust embedded in a Euclidean space) it is necessary to know the continuous field of displacements at every point of the medium or, at least, in the neighborhood of specific points where strain is to be calculated (see (36) and (37)).

Geodetic observations are usually discrete and limited to points on the two-dimensional surface. Even if the three components of the displacement vectors in a network of surface points could be evaluated by analyzing geodetic observations, strains could not be directly computed. In order to evaluate the space derivatives of displacements appearing in (36) and (37) a continuous field of displacements is needed. When only discrete values of displacements are available, a continuous field $\xi(\mathbf{X})$ can be obtained by interpolation. The interpolation concerns the determination of displacements not only on the surface but also within and outside the crust. For points within the crust the strains are computed by analytically differentiating the interpolated function $\xi(\mathbf{X})$, and consequently, the strains depend not only on the discrete geodetic observations but also on the chosen interpolation technique. The more the interpolation technique reflects physical reality, the closer the computed strains are to the real ones.

The same holds for strain computation at points on the crust surface, but an additional problem exists: In physical reality, displacements outside the medium are not defined, and stress-strain relation problems can be faced, introducing boundary conditions on the surface.

Mathematically speaking, interpolation techniques predict displacements even outside the medium, and computed surface strains depend on such predictions and consequently do not reflect physical reality. They have only a descriptive value and should not be used, for example, for conclusions on stress-strain relations.

In practice, the problems connected with deformations in three dimensions are bypassed by computing separately two-dimensional plane strains and vertical motions. Since it is natural to be interested in three-dimensional deformations, one may think of combining two-dimensional plane displacements with vertical motions in order to derive displacements in three dimensions and the relevant strains. However, as already mentioned, vertical motions are severely dependent on changes of the gravity field even when these changes are not accompanied by motion of the terrestrial surface.

As is well known to geodesists, the really estimable quantities are differences of the geopotential, which can be transformed into orthometric height differences and into their time variations only when the gravity field and its time variation are independently known.

On the other hand, the effect of the gravity field variations on the plane horizontal coordinates of triangulation points is negligible. When analyzing a local triangulation network, a common horizontal plane of reference is assumed for the perpendicular projection of network points. In this case, though observations (angular horizontal measurements) at each point depend on the local direction of the plumb line, they are not sensitive to the expected small changes in this direction. This allows the determination of horizontal displacements which essentially reflect actual horizontal motions with negligible effects from gravity field variations.

Works and discussions related to the above can be found in several collective volumes [e.g., *Pavoni and Green*, 1975; *Rikitake*, 1976, 1981; *Mueller*, 1978; *Hieber and Guyenne*, 1978; *Vogel*, 1979; *NASA*, 1979, 1980, 1981; *Isikara and Vogel*, 1981; *U.S. National Research Council*, 1981].

For vertical movements and associated gravity variations see, for example, *Faytel'son and Yurkina* [1974], *Yurkina* [1977], *Hein* [1978], *Drewes* [1980], and *Vanicek et al.* [1980].

Further elaborations with emphasis on analysis are given by *Brunner* [1979], *Kumar* [1979], *Wassef* [1979], *Brunner et al.* [1981], *Dermanis* [1981], *Hein and Kistemann* [1981], *Kelm* [1981], *Koch and Fritsch* [1981], *Livieratos and Vlachos* [1981], and *Niemeier* [1981].

Two-Dimensional Plane Strain

For the study of crustal deformations in two dimensions, usually followed in practice, intrinsic two-dimensional strains are defined for the description of the geometric alterations in the positions of the projections of the surface points onto the local reference plane.

Again, as in the three-dimensional case, strains, being local differential quantities, require continuous displacement information. Geodetic techniques provide only discrete displacements, and strains can be computed only after an explicit or implicit interpolation of displacements.

The definitions of the Lagrangian strain tensor $\mathbf{E}$, the infinitesimal strain tensor $\boldsymbol{\varepsilon}$, the infinitestimal rotation matrix $\boldsymbol{\Omega}$, and their Eulerian counterparts apply equally well in the two-dimensional case. Since the analysis is carried out on the plane (Euclidean space), the equations (36)–(41) are appropriate.

The numerical values of strain tensor elements depend on the reference frame used. Thus one should look for scalar functions of the strain tensor elements which are invariant with respect to the chosen reference frame and moreover have an evident physical interpretation. It is well known that there exists a particular reference frame with respect to which the strain tensor, being symmetric, takes a diagonal form:

$$\Lambda_E = \begin{bmatrix} E_{max} & 0 \\ 0 & E_{min} \end{bmatrix} \tag{53}$$

The quantities E_{max}, E_{min} are called the (Lagrangian) principal strains. In order to compute the principal strains from the components of the strain tensor

$$\mathbf{E} = \begin{bmatrix} E_{xx} & E_{xy} \\ E_{xy} & E_{yy} \end{bmatrix} \tag{54}$$

in any two-dimensional Cartesian coordinate system, well-known spectral techniques must be applied. Indeed, the principal strains E_{min}, E_{max} are the eigenvalues of $\mathbf{E}$ and can be computed as the roots of the characteristic equation

$$\det(\mathbf{E} - \lambda \mathbf{I}) = \lambda^2 - \mathrm{tr}\,(\mathbf{E})\lambda + \det(\mathbf{E}) = 0 \tag{55}$$

which are

$$\begin{aligned} \lambda_1 = E_{max} &= \tfrac{1}{2}\{\mathrm{tr}\,(\mathbf{E}) + [\mathrm{tr}\,(\mathbf{E})^2 - 4\det(\mathbf{E})]^{1/2}\} \\ \lambda_2 = E_{min} &= \tfrac{1}{2}\{\mathrm{tr}\,(\mathbf{E}) - [\mathrm{tr}\,(\mathbf{E})^2 - 4\det(\mathbf{E})]^{1/2}\} \end{aligned} \tag{56}$$

The scalar quantities

$$\begin{aligned} \Delta &= E_{max} + E_{min} \\ \gamma &= E_{max} - E_{min} \end{aligned} \tag{57}$$

are called the dilatation and the maximum shear strain, respectively. Δ represents the isotropic part and γ the anisotropic part of deformation in the infinitesimal vicinity of the point of interest. They are both point functions with the following physical interpretation: Δ is the areal change per unit area (positive for an increase in area), and γ is the shear across the direction of its maximum value (always positive).

The scalar invariants Δ and γ can be expressed, according to *Truesdell and Noll* [1965], in terms of the so-called principal invariants $I_1(\mathbf{E})$ and $I_2(\mathbf{E})$, which in the two-dimensional case are

$$I_1(\mathbf{E}) = \mathrm{tr}\,(\mathbf{E}) \qquad I_2(\mathbf{E}) = \det(\mathbf{E}) \tag{58}$$

Combination of (56) with (57) gives

$$\begin{aligned} \Delta &= I_1(\mathbf{E}) = \mathrm{tr}\,(\mathbf{E}) = E_{xx} + E_{yy} \\ \gamma^2 &= I_1{}^2(\mathbf{E}) - 4I_2(\mathbf{E}) = [\mathrm{tr}\,(\mathbf{E})]^2 - 4\det(\mathbf{E}) \\ &= (E_{xx} - E_{yy})^2 + (2E_{xy})^2 \end{aligned} \tag{59}$$

Introducing the shear components

$$\gamma_1 = E_{xx} - E_{yy} \tag{60}$$

$$\gamma_2 = 2E_{xy} \tag{61}$$

the maximum shear strain becomes

$$\gamma = (\gamma_1{}^2 + \gamma_2{}^2)^{1/2} \tag{62}$$

The above shear components have the following interpretation [*Prescott et al.*, 1979]: γ_1 is the shear across any line parallel to the y direction (north), and γ_2 is the shear across any line parallel to the x direction (east).

The azimuth of the maximum extension E_{max} is computed from the eigenvector problem

$$\mathbf{En} = \lambda \mathbf{n} \tag{63}$$

where

$$\mathbf{n} = \begin{bmatrix} \sin\varphi \\ \cos\varphi \end{bmatrix} \tag{64}$$

Eliminating λ from (63), it follows that

$$\varphi = \tfrac{1}{2}\arctan(-\gamma_2/\gamma_1) \tag{65}$$

When the infinitesimal strain $\boldsymbol{\varepsilon}$ only is considered, corresponding infinitesimal quantities Δ, γ, etc. are defined.

From the Jacobian of the displacements (46), the infinitesimal antisymmetric rotation matrix

$$\boldsymbol{\Omega} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \tag{66}$$

is also defined, where ω is the angle (in radians) of infinitesimal rotation. Similar quantities are analogously defined for the Eulerian mode of description.

There is an extensive literature concerning applications of the above two-dimensional plane strain analysis from geodetic results. Two approaches are basically followed. In the first approach, angular or distance observations at two different epochs are used (observation methods). In the second approach, the coordinates from independently adjusted triangulations at two epochs are used (coordinate method). This is essentially a finite element method where strains are considered constant within the triangular elements.

Concerning horizontal deformations see, for example, *Frank* [1966], *Sato* [1973], *Bibby* [1975, 1981], *Savage* [1976], *Prescott* [1976], *Harada* [1977], *Gerasimenko et al.* [1977], *Livieratos* [1978, 1980], *Baldi et al.* [1979], *Cohen and Cook* [1979], *Harada and Shimura* [1979], *Nyland et al.* [1979], *Prescott et al.* [1979], *Reilly* [1979], *Savage et al.* [1979], *Snay* *and Cline* [1980], *Vlachos* [1980], *Baldi and Marson* [1981], *Cacon* [1981], *Dermanis et al.* [1981], *Lisowski and Thatcher* [1981], and *Bencini et al.* [1982].

DEFORMATION OF THE GRAVITY FIELD OF THE EARTH AND GRAVITY FIELD RELATED DEFORMATIONS

As already discussed in the introduction, while the analysis in the previous chapter concerned the classical treatment of material deformation, the same tools can be abstracted for the study of deformations related to the gravity field of the earth.

In this context, three alternative cases are studied:

1. The most straightforward way of defining deformation of the gravity field is to compare two different gravity vector fields in the vicinity of a fixed point in geometry space.
2. Since traditionally in geodesy we deal with positioning by means of observations related to the gravity field, an alternative approach can be formulated by studying the change of geometric positions identified by the same value of the gravity vector, or any three other appropriate gravity field related parameters, with respect to two different gravity fields under comparison.
3. The third alternative is to study the alterations of the geometric and gravity characteristics, as well as their interrelation, in the vicinity of material points of the earth's surface.

For the study of these cases the following Jacobians are used:

$$\begin{aligned} \partial\mathbf{\Gamma}/\partial\mathbf{X} &= \mathbf{B} \qquad \partial\mathbf{\Gamma}/\partial\mathbf{x} = \mathscr{B} \\ \partial\boldsymbol{\gamma}/\partial\mathbf{X} &= \mathbf{b} \qquad \partial\boldsymbol{\gamma}/\partial\mathbf{x} = \boldsymbol{\beta} \end{aligned} \tag{67}$$

where $\mathbf{x}$ and $\mathbf{X}$ are the 'old' and 'new' position vectors, respectively, and $\boldsymbol{\gamma}$ and $\mathbf{\Gamma}$ the corresponding gravity vectors.

Gravity Deformation at Fixed Space Position

Let us consider a point $\mathbf{X}$ fixed in geometry space and two different potential fields W and w assigning corresponding gravity vectors $\mathbf{\Gamma} = \text{grad } W(\mathbf{X})$ and $\boldsymbol{\gamma} = \text{grad } w(\mathbf{X})$. Since $\mathbf{\Gamma}$ and $\boldsymbol{\gamma}$ are Cartesian coordinates in gravity space, they can be used for the definition of gravity strain in the same way that the space Cartesian coordinates are used in (16) and (17) for the definition of geometry strains. The Lagrangian gravity strain is

$$\mathbf{E}(\mathbf{\Gamma}) = \tfrac{1}{2}[(\partial\boldsymbol{\gamma}/\partial\mathbf{\Gamma})^T(\partial\boldsymbol{\gamma}/\partial\mathbf{\Gamma}) - \mathbf{I}] \tag{68}$$

and the Eulerian gravity strain is

$$\mathbf{E}^*(\boldsymbol{\gamma}) = \tfrac{1}{2}[\mathbf{I} - (\partial\mathbf{\Gamma}/\partial\boldsymbol{\gamma})^T(\partial\mathbf{\Gamma}/\partial\boldsymbol{\gamma})] \tag{69}$$

In the more general case when curvilinear coordinates $\mathbf{\Phi}(\mathbf{\Gamma})$ and $\boldsymbol{\varphi}(\boldsymbol{\gamma})$ are used for the description of gravity space, the above relations are written as in (5) and (7),

$$\mathbf{X} \equiv \mathbf{x} \xrightarrow{\;W\;} \mathbf{\Gamma}$$

$$\mathbf{X} \equiv \mathbf{x} \xrightarrow{\;w\;} \boldsymbol{\gamma}$$

$$\mathbf{E}(\mathbf{\Phi}) = \tfrac{1}{2}[(\partial\boldsymbol{\varphi}/\partial\mathbf{\Phi})^T\mathbf{m}(\partial\boldsymbol{\varphi}/\partial\mathbf{\Phi}) - \mathbf{M}] \tag{70}$$

$$\mathbf{E}(\boldsymbol{\varphi}) = \tfrac{1}{2}[\mathbf{m} - (\partial\mathbf{\Phi}/\partial\boldsymbol{\varphi})^T\mathbf{M}(\partial\mathbf{\Phi}/\partial\boldsymbol{\varphi})] \tag{71}$$

$\mathbf{m}$, $\mathbf{M}$ being the metric tensors associated with the curvilinear frames used. Since gravity space is Euclidean,

$$\mathbf{m} = \mathbf{d}^T\mathbf{d} \qquad \mathbf{M} = \mathbf{D}^T\mathbf{D} \tag{72}$$

where

$$\mathbf{d} = \partial\boldsymbol{\gamma}/\partial\boldsymbol{\varphi} \qquad \mathbf{D} = \partial\mathbf{\Gamma}/\partial\mathbf{\Phi} \tag{73}$$

Introducing the matrices of second-order space derivatives of the potential (gravity gradients),

$$\frac{\partial}{\partial\mathbf{X}}\left(\frac{\partial W}{\partial\mathbf{X}}\right)^T = \frac{\partial\mathbf{\Gamma}}{\partial\mathbf{X}} \overset{\text{def}}{=} \mathbf{B} = \frac{\partial\mathbf{\Gamma}}{\partial\mathbf{x}} \overset{\text{def}}{=} \mathscr{B} \tag{74}$$

$$\frac{\partial}{\partial\mathbf{X}}\left(\frac{\partial w}{\partial\mathbf{X}}\right)^T = \frac{\partial\boldsymbol{\gamma}}{\partial\mathbf{X}} \overset{\text{def}}{=} \mathbf{b} = \frac{\partial\boldsymbol{\gamma}}{\partial\mathbf{x}} \overset{\text{def}}{=} \boldsymbol{\beta} \tag{75}$$

the gravity strain tensors referred to Cartesian coordinates become

$$\mathbf{E}(\mathbf{\Gamma}) = \tfrac{1}{2}(\mathbf{B}^{-1}\mathbf{b}^2\mathbf{B}^{-1} - \mathbf{I}) \tag{76}$$

$$\mathbf{E}^*(\boldsymbol{\gamma}) = \tfrac{1}{2}(\mathbf{I} - \mathbf{b}^{-1}\mathbf{B}^2\mathbf{b}^{-1}) \tag{77}$$

and the gravity strain tensors referred to curvilinear coordinates are

$$\begin{aligned} \mathbf{E}(\mathbf{\Phi}) &= \tfrac{1}{2}[\mathbf{D}^T\mathbf{B}^{-1}\mathbf{b}(\mathbf{d}^{-1})^T\mathbf{m}\mathbf{d}^{-1}\mathbf{b}\mathbf{B}^{-1}\mathbf{D} - \mathbf{M}] \\ &= \tfrac{1}{2}[\mathbf{D}^T\mathbf{B}^{-1}\mathbf{b}^2\mathbf{B}^{-1}\mathbf{D} - \mathbf{D}^T\mathbf{D}] \end{aligned} \tag{78}$$

and

$$\mathbf{E}^*(\boldsymbol{\varphi}) = \tfrac{1}{2}[\mathbf{d}^T\mathbf{d} - \mathbf{d}^T\mathbf{b}^{-1}\mathbf{B}^2\mathbf{b}^{-1}\mathbf{d}] \tag{79}$$

Comparing with (76) and (77),

$$\mathbf{E}(\mathbf{\Phi}) = \mathbf{D}^T\mathbf{E}(\mathbf{\Gamma})\mathbf{D} \tag{80}$$

$$\mathbf{E}^*(\boldsymbol{\varphi}) = \mathbf{d}^T\mathbf{E}(\boldsymbol{\gamma})\mathbf{d} \tag{81}$$

in agreement with the well-known tensor transformation rules.

The two potential fields under comparison could be either the actual instantaneous fields of the earth at two distinct epochs or the actual field and a model field (normal potential).

Deformation of Geometry Space at Fixed Gravity

This case is dual to the previous one, since it follows by simply interchanging geometry with gravity space:

$$\mathbf{X} \xrightarrow{\;W\;} \mathbf{\Gamma} \equiv \boldsymbol{\gamma}$$

$$\mathbf{x} \xrightarrow{\;w\;} \mathbf{\Gamma} \equiv \boldsymbol{\gamma}$$

The situation now is as follows: Consider a position $\mathbf{X}$ and a potential field W assigning a gravity vector $\mathbf{\Gamma} = \text{grad } W(\mathbf{X})$. For a second potential field w there exists a position $\mathbf{x}$ such that the same gravity vector $\mathbf{\Gamma} = \text{grad } w(\mathbf{x})$ is assigned. In this abstract way, positions $\mathbf{X}$ and $\mathbf{x}$ are brought to one-to-one correspondence, in the same way that in continuum mechanics, positions are paired as positions of the same material point at different epochs.

The strain tensors are thus defined as in (16) and (17):

$$\mathbf{E}(\mathbf{X}) = \tfrac{1}{2}[(\partial\mathbf{x}/\partial\mathbf{X})^T(\partial\mathbf{x}/\partial\mathbf{X}) - \mathbf{I}] \tag{82}$$

$$\mathbf{E}^*(\mathbf{x}) = \tfrac{1}{2}[\mathbf{I} - (\partial\mathbf{X}/\partial\mathbf{x})^T(\partial\mathbf{X}/\partial\mathbf{x})] \tag{83}$$

Introducing the matrices of second-order derivatives of the potential,

$$\begin{aligned} \frac{\partial}{\partial\mathbf{X}}\left(\frac{\partial W}{\partial\mathbf{X}}\right)^T &= \left(\frac{\partial\boldsymbol{\gamma}}{\partial\mathbf{X}}\right) \overset{\text{def}}{=} \mathbf{b} = \frac{\partial\mathbf{\Gamma}}{\partial\mathbf{X}} \overset{\text{def}}{=} \mathbf{B} \\ \frac{\partial}{\partial\mathbf{x}}\left(\frac{\partial w}{\partial\mathbf{x}}\right)^T &= \left(\frac{\partial\boldsymbol{\gamma}}{\partial\mathbf{x}}\right) \overset{\text{def}}{=} \boldsymbol{\beta} = \frac{\partial\mathbf{\Gamma}}{\partial\mathbf{x}} \overset{\text{def}}{=} \mathscr{B} \end{aligned} \tag{84}$$

the strain tensors become

$$\mathbf{E}(\mathbf{X}) = \tfrac{1}{2}(\mathbf{B}\boldsymbol{\beta}^{-2}\mathbf{B} - \mathbf{I}) \tag{85}$$

$$\mathbf{E}^*(\mathbf{x}) = \tfrac{1}{2}(\mathbf{I} - \boldsymbol{\beta}\mathbf{B}^{-2}\boldsymbol{\beta}) \tag{86}$$

Generalizations can be easily derived for the case when curvilinear coordinates are used in geometry space.

An application of the above treatment is found in the case where W is the actual potential of the earth and w the normal potential. In this case, $\mathbf{x}$ is the image of $\mathbf{X}$ under the telluroid mapping $\mathbf{X} \mapsto \mathbf{x}$ defined by

$$\text{grad } w(\mathbf{x}) = \text{grad } W(\mathbf{X}) \tag{87}$$

The relevant strain tensors (85) and (86) describe deformation induced by, for example, mapping the surface of the earth into the telluroid [*Bocchio*, 1976, 1979*a*, *b*; *Osada*, 1978, 1980; *Marussi*, 1973, 1974; *Grafarend*, 1978*a*, 1978*b*; *Teisseyre*, 1969].

A similar analysis can be carried out by identifying positions through their natural coordinates $\boldsymbol{\Phi} = (\Phi\ \Lambda\ W/w_0)^T$ or normal coordinates $\boldsymbol{\varphi} = (\varphi\ \lambda\ w/w_0)^T$ instead of the gravity vectors $\boldsymbol{\Gamma}$ or $\boldsymbol{\gamma}$, respectively, where w_0 is a reference constant with potential dimension, that is, a reference normal potential value. The choice of pure numbers W/w_0 and w/w_0 instead of the corresponding potentials W and w is suggested in order to obtain strain tensor components of the same units.

$$\mathbf{X} \xrightarrow{W} \boldsymbol{\Phi} \equiv \boldsymbol{\varphi}$$

$$\mathbf{x} \xrightarrow{w} \boldsymbol{\Phi} \equiv \boldsymbol{\varphi}$$

The strain tensors are defined exactly as in (16) and (17) or (82) and (83). Introducing the matrices

$$\mathbf{F} = \partial\boldsymbol{\Phi}/\partial\mathbf{X} \qquad \mathbf{f} = \partial\boldsymbol{\Phi}/\partial\mathbf{x} \tag{88}$$

the strain tensors become

$$\mathbf{E}(\mathbf{X}) = \tfrac{1}{2}[\mathbf{F}^T(\mathbf{f}\mathbf{f}^T)^{-1}\mathbf{F} - \mathbf{I}] \tag{89}$$

$$\mathbf{E}^*(\mathbf{x}) = \tfrac{1}{2}[\mathbf{I} - \mathbf{f}^T(\mathbf{F}\mathbf{F}^T)^{-1}\mathbf{f}] \tag{90}$$

This case finds application in studying deformations induced by the telluroid mapping $\mathbf{X} \mapsto \mathbf{x}$ defined by

$$w(\mathbf{x}) = W(\mathbf{X}) \qquad \frac{\text{grad } w(\mathbf{x})}{|\text{grad } w(\mathbf{x})|} = \frac{\text{grad } W(\mathbf{X})}{|\text{grad } W(\mathbf{X})|} \tag{91}$$

The Apparent Gravity Deformations

The case most closely connected to observational reality concerns the change of the gravity field at material points which is due to both change of the gravity field itself and change of position due to motion of the surface of the earth. The situation can be shown as

$$P \rightarrow \mathbf{X} \xrightarrow{W} \boldsymbol{\Gamma}$$

$$P \rightarrow \mathbf{x} \xrightarrow{w} \boldsymbol{\gamma}$$

where $\mathbf{X}$ and $\mathbf{x}$ are two different positions of the same material point P and $\boldsymbol{\Gamma}$ and $\boldsymbol{\gamma}$ the corresponding apparent (observable) gravity vectors induced by two different potential fields W and w.

The geometry strain tensors $\mathbf{E}(\mathbf{X})$ and $\mathbf{E}(\mathbf{x})$ are defined as in (16) and (17) or (82) and (83). Introducing the matrix of second-order derivatives of the potential

$$\mathbf{B} = \partial\boldsymbol{\Gamma}/\partial\mathbf{X} \tag{92}$$

and recalling definition (75), we obtain

$$\frac{\partial\mathbf{x}}{\partial\mathbf{X}} = \frac{\partial\mathbf{x}}{\partial\boldsymbol{\gamma}}\frac{\partial\boldsymbol{\gamma}}{\partial\boldsymbol{\Gamma}}\frac{\partial\boldsymbol{\Gamma}}{\partial\mathbf{X}} = \boldsymbol{\beta}\frac{\partial\boldsymbol{\gamma}}{\partial\boldsymbol{\Gamma}}\mathbf{B} \tag{93}$$

Thus the strain tensors become

$$\mathbf{E}(\mathbf{X}) = \frac{1}{2}\left[\mathbf{B}^T\left(\frac{\partial\boldsymbol{\gamma}}{\partial\boldsymbol{\Gamma}}\right)^T\boldsymbol{\beta}^2\frac{\partial\boldsymbol{\gamma}}{\partial\boldsymbol{\Gamma}}\mathbf{B} - \mathbf{I}\right] \tag{94}$$

$$\mathbf{E}^*(\mathbf{x}) = \frac{1}{2}\left[\mathbf{I} - \boldsymbol{\beta}^{-1}\left(\frac{\partial\boldsymbol{\Gamma}}{\partial\boldsymbol{\gamma}}\right)^T\mathbf{B}^{-2}\frac{\partial\boldsymbol{\Gamma}}{\partial\boldsymbol{\gamma}}\boldsymbol{\beta}^{-1}\right] \tag{95}$$

Similar expressions can be derived for the case where instead of $\boldsymbol{\Gamma}$ and $\boldsymbol{\gamma}$ the natural coordinates $\boldsymbol{\Phi}$ and $\boldsymbol{\varphi}$, respectively, are available.

Earth Tidal Deformation

A particular case of the previous formulation concerns the deformation of the earth's crust associated with the lunisolar tides. A material point in a relaxed state has a position $\mathbf{X}$, and the potential W is due only to earth attraction and rotation. An additional potential reflecting lunisolar attraction leads to a new disturbed state where the material point moves to a new position $\mathbf{x}$ and the final potential w depends on the original potential, the potential of the lunisolar attractions, and the variations caused by mass redistribution after elastic yielding. The final position $\mathbf{x}$ of any material point is a function of its original position $\mathbf{X}$, the original and final potential W and w, and a set of parameters (Love and Shida numbers). In consequence, strains depend on the above function. For derivations of tidal strains, see *Melchior* [1978]. For an interesting application see *Marussi* [1979].

OTHER APPLICATIONS

A case completely analogous to the temporal variation of network coordinates caused by crustal deformation is the one where two coordinate sets of the same time-invariant network are available from different sources. Typical examples are the inconsistencies in coordinates as derived from classical adjustment and space techniques (see, for example, *Mueller* [1974]), differences in coordinates as adjusted on a reference ellipsoid and on a projective plane, coordinate differences between the results of compatible adjustments based on data from different observational campaigns, coordinate differences in common parts of networks, alterations in the form of a lower-order network when superfluous, fixed, higher-order control points are used, etc. The physical meaning of deformation in such cases varies with each particular application.

When three-dimensional coordinates from space techniques are compared with classical ellipsoidal geodetic coordinates, deformation parameters may be related to systematic errors caused, for example, when normal parameters of the gravity field are used instead of their actual values in the classical procedures of reduction of the actual observations to the reference surface [*Meissl*, 1973; *Thomson et al.*, 1974].

Another example is the comparison of two forms of a network (or parts of a network) resulting from the utilization of different subsets of the total originally available observations, in which case, deformation parameters may be used for the detection of gross observation inconsistencies (blunders) [*Thapa*, 1980; *Vanicek et al.*, 1981].

A very interesting aspect of the study of deformation appears in engineering surveying [*Pelzer*, 1971, 1974; *Platek*, 1974; *Van Mierlo*, 1977; *Hallerman*, 1981], which concerns the temporal alterations in the shape of man-made structures. Beyond the need for generalization to three dimensions we notice the possibility of relating observed strains with stresses, since the elastic behavior of such structures are more or less known.

A simple two-dimensional Euclidean case is the study of film deformation in photogrammetric applications.

A well-known two-dimensional case is the deformation involved in cartographic projections, when original figures on a Riemmanian space (sphere or ellipsoid) are deformed into corresponding figures on a Euclidean space (plane map) (see, for example, *Marussi* [1951*a*, *b*, 1959], *Bernasconi* [1953, 1956], *Caputo* [1956], *Bocchio* [1969], *Chovitz* [1979], *Hojovec and Jokl* [1981], and *Dermanis and Livieratos* [1982]).

Other geodetic applications involving comparison of figures in two-dimensional Riemmanian spaces are the study of deformations occurring in mapping the geoid onto the reference ellipsoid and similar comparisons between alternative choices of reference ellipsoids (see, for example, *Marussi* [1951*a*, *b*, 1957], *O'Keefe* [1953], *Caputo* [1959], and *Zadro and Carminelli* [1966]).

Since deformation parameters are connected to comparison of metrics, one can study deformation between different metrizations of one and the same physical space, without the need of establishing point-to-point correspondences. An example can be found in the study of atmospheric refraction, where the three-dimensional Euclidean earth space is compared with itself when endowed with a metric related to the index of refraction (see, for example, *Marussi* [1953] and *Grafarend* [1975]).

A related approach is based on the exploitation of certain analogies between stress-strain relations of elasticity theory and error propagation in geodetic networks [*Borre and Krarup*, 1974; *Grafarend*, 1977; *Borre*, 1979].

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