THE FINITE ELEMENT APPROACH TO THE GEODETIC
COMPUTATION OF TWO- AND THREE-DIMENSIONAL
DEFORMATION PARAMETERS:
A STUDY OF FRAME INVARIANCE AND PARAMETER
ESTIMABILITY

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Summary

Transformation formulas for deformation parameters, under frame changes at two compared
epochs are derived, which allow to draw direct conclusions about their invariance and their
related estimability. The study is carried out in two, as well as in three dimensions where
concepts analogous to the usual plane dilatation and plane maximum shear strain are
introduced. Apart from the rigorous approach for the computation of the strain matrix and its
invariants, the widely used approximate infinitesimal approach is also examined. Emphasis
is given to the case where there exists no undeformed part of the geodetic network, in which
case the network shapes at the two epochs cannot be uniquely connected by a common
choice of reference frame. The results can also be directly applied to connected shapes by
simply setting equal the transformation parameters at the two epochs.
1. INTRODUCTION

Geodetic observations, classical and modern, are an established means for the determination of deformation parameters of the earth crust. The analysis of geodetic observations is typically performed with models involving coordinates, which refer to more-or-less arbitrary reference frames. As a consequence it is possible to use coordinate estimates in order to compute any coordinate function, even those which depend not only on the network shape, but also on its arbitrary position introduced by the adoption of a particular reference frame. For this reason it is essential to distinguish between coordinate functions which are estimable quantities and those which are not. As shown by Grafarend and Schaffrin (1976), estimable quantities are those which remain invariant under changes of the reference frame.

The first study of the estimability-invariance of deformation parameters was presented in Dermanis (1981) for the commonly used finite element method in the plane, where a piecewise linear interpolation scheme is used. Furthermore this approach may also be characterized as infinitesimal, because second order terms of the displacement gradient are ignored in the computation of the strain matrix. These results have been generalized to a wider class of interpolation techniques in Dermanis (1985), without the infinitesimal approximation and in particular for the "collocation" approach to the prediction of coordinate displacements at any point in the network area.

An attempt to examine the three dimensional case was made by Zaiser (1984), while Grafarend (1986) studied three-dimensional deformation by separating global from local effects.

In the present work we study the estimability-invariance characteristics of deformation parameters obtained through the finite element method, by using a "dimension-free" approach with results that can be immediately specialized to three or two dimensions. In the last case previously obtained results are simply verified. The shortcomings of the widely used infinitesimal approach are also pointed out.

An important point, when studying the invariance characteristics of deformation parameters, is to allow for different frame definitions (and changes) at the two epochs under comparison. In fact a common frame definition is based on the possibility to "connect" the two network shapes, through the additional assumption that a subnetwork (of at least three points) maintains an invariant shape. Such an assumption, even if justified by means of statistical hypothesis tests, brings additional information in the model, other than that already contained in the available observations. Furthermore such shape invariance may not be indicated by statistical tests, or it may be indicated for more than one subnetworks. The last case appears typically with fault crossing geodetic observations, where two shape invariant subnetworks can be pinpointed, one at each side of the fault with points remote from the active area. This allows for two different connections of the network and in fact leads to two different definitions of reference frame at the second epoch with respect to that of the first.

Since deformation refers to change of shape, independently of position, it is only reasonable that it is studied through properly defined parameters which are independent of frame definitions. In this case the network adjustments at different epochs can be performed with independent and arbitrary frame definitions, by means of minimal or inner constraints (free networks). The need for a different frame definition at each epoch is well known in classical continuum mechanics, where frame invariance is studied in continuous time, by means of a continuous change of frame which is a function of time, i.e. different at each epoch (Truesdell, 1977).
2. BASIC DEFINITIONS

Let \( x, x' \) be the coordinate vectors of a point \( P \) at epochs \( t, t' \), respectively, while \( u = x' - x \) is the corresponding "displacement" vector. The coordinate Jacobian matrix \( J \) and the displacement Jacobian matrix \( E \) are given by

\[
J = \frac{\partial x'}{\partial x}, \quad E = \frac{\partial u}{\partial x} = J - I
\]

The Lagrangean strain matrix \( S \) is defined in terms of the line elements \( ds, ds' \) at the two epochs \( t, t' \)

\[
\frac{1}{2} (ds'^2 - ds^2) = dx^T S dx
\]

It can be directly shown that in terms of the Jacobians the strain matrix becomes

\[
S = \frac{1}{2} (J^T J - I) = \frac{1}{2} (E + ET + ET E)
\]

The Jacobian matrices \( J, E \) and the strain matrix \( S \) at a certain point \( P \) can be computed only if the coordinates of all points in a neighborhood of \( P \) at the two specific epochs \( t, t' \) are known. On the contrary, geodetic observations provide coordinate estimates at only discrete points. This information has to be interpolated in order to obtain the continuous information required for the computation of \( J, E, S \) and relevant deformation parameters. Needless to say that the results of such an approach depend not only on the available discrete point coordinates but also on the particular interpolation technique being used. Furthermore coordinates refer to arbitrary reference frames at the two epochs \( t, t' \) (which may coincide or not) and only those deformation parameters which are independent of such frame definitions are of any practical value.

The most simple interpolation technique is the piecewise linear interpolation related with finite elements. The region of interest is divided in disjoint elements (triangles in the plane, quadrilaterals in three dimensions) which have the known points as vertices. Assuming a different linear field of coordinate changes

\[
x'(x) = J x + g
\]

within each finite element, the coefficient matrices \( J \) and \( g \) can be computed from the system of equations arising from the application of equation (4) to the vertices of the element. For example in 3 dimensions we have 4 vertices, say \( P_\alpha, P_\beta, P_\gamma, P_\delta \), and the relevant system is

\[
[x'_\alpha, x'_\beta, x'_\gamma, x'_\delta] = J [x_\alpha, x_\beta, x_\gamma, x_\delta] + [g g g g]
\]

or with obvious notation

\[
X' = J X + g 1^T
\]
In order to eliminate $\mathbf{g}$ we introduce the differencing matrix

$$\mathbf{D} = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with the property $\mathbf{1}_3^T \mathbf{D} = \mathbf{0}$. Upon multiplication of equation (6) with $\mathbf{D}$ from the right we obtain

$$(\mathbf{X}' \mathbf{D}) = \mathbf{J} (\mathbf{X} \mathbf{D}) + \mathbf{g} \mathbf{1}^T \mathbf{D} = \mathbf{J} (\mathbf{X} \mathbf{D}) \quad \Rightarrow \quad \mathbf{J} = (\mathbf{X}' \mathbf{D}) (\mathbf{X} \mathbf{D})^{-1}$$

In the 2-dimensional case the same formulation applies with only 3 vertices $\mathbf{X} = [x, x', x_\gamma]$, $\mathbf{X}' = [x, x', x'\gamma]$, and differencing matrix $\mathbf{D} = [-1, 1, 1]^T$. The displacement vector $\mathbf{g}$ can be computed from $\mathbf{g} = \mathbf{X}' \mathbf{1} - \mathbf{J} \mathbf{X} \mathbf{1}$, but it is not needed for the analysis of deformation.

3. STUDY OF INVARIANCE - ESTIMABILITY

In order to study the dependence of the various deformation parameters on the choice of reference frames, two frame changes are introduced (one for each epoch $t$ and $t'$) by means of the corresponding coordinate transformations

$$\begin{align*}
\mathbf{x} &= k \ R \ x + \mathbf{b} \quad (t) \\
\mathbf{x}' &= k' \ R' \ x' + \mathbf{b}' \quad (t')
\end{align*}$$

Using these transformations we easily get from equation (8) the transformation law for the coordinate Jacobian, the displacement Jacobian and the strain matrix:

$$\begin{align*}
\mathbf{J} &= \mu \ R' \ J \ R^T \\
\mathbf{E} &= \mu \ R' \ E \ R^T + \mu \ R' \ R^T - \mathbf{I} \\
\mathbf{S} &= \mu^2 \ R \ S \ R^T + \frac{\mu^2 - 1}{2} \mathbf{I}
\end{align*}$$

If $v_i$ are the unit eigenvectors of $\mathbf{S}$ corresponding to eigenvalues $\lambda_i$, the strain matrix can be diagonalized as

$$\mathbf{S} = \mathbf{V} \Lambda \mathbf{V}^T \quad \Leftrightarrow \quad \mathbf{V}^T \mathbf{S} \mathbf{V} = \Lambda = \text{Diag}(\lambda_1, \lambda_2, \lambda_3)$$
where \( V \) is the orthogonal matrix with columns the eigenvectors \( v_i \) and \( \Lambda \) is the diagonal matrix with diagonal elements the ordered eigenvalues \( \lambda_i \) of \( S \). Replacing in equation (12) it follows that the transformed strain matrix admits the corresponding diagonalization

\[
\begin{align*}
\bar{S} &= \mu^2 R V \Lambda V^T R^T + \frac{\mu^2 - 1}{2} I = \\
&= (R V) \left[ \mu^2 \Lambda + \frac{\mu^2 - 1}{2} I \right] (R V)^T = \bar{V} \bar{\Lambda} \bar{V}^T
\end{align*}
\]

where

\[
\begin{align*}
\bar{V} &= R V \quad \quad (v_i' = R v_i) \\
\bar{\Lambda} &= \mu^2 \Lambda + \frac{\mu^2 - 1}{2} I \quad \quad (\lambda_i' = \lambda_i + \frac{\mu^2 - 1}{2})
\end{align*}
\]

3.1. THE INFINITESIMAL APPROACH

It is a usual practice in planar applications to use instead of the strain matrix \( S \) a "first order" approximation \( S_o \), which is based on the decomposition of the displacement Jacobian matrix \( E \) into a symmetric part \( S_o \) and an antisymmetric one \( \Omega \)

\[
E = S_o + \Omega \quad \left( S_o = \frac{1}{2} (E + E^T), \quad \Omega = \frac{1}{2} (E - E^T) \right)
\]

The approximation \( S_o \) differs from the true \( S \) only in neglecting the second order term \( E E^T \), but the consequence for the transformation under changes of reference frames are quite significant, as it will be shown in the following.

The above approach originates from the study not of deformation itself (which refers to the comparison of shape at two discrete epoches) but rather from the study of the rate of deformation. This means that an interpolation in time is also needed, in order to obtain not displacements but displacement velocities. Division of equation (17) with the time interval \( \Delta t' - \Delta t \) will give results corresponding to a linear interpolation of point position between epoches. In this case the velocities are simply the displacements divided by the time interval and the matrix \( E \) should be interpreted as the Jacobian matrix of velocity with respect to position.

When a common frame has been established for both epoches, and therefore for any intermediate epoch of the time-wise interpolation, the approach becomes rigorous but with a physical interpretation different from the (formally identical) infinitesimal approximation to the discrete epoch deformation problem. However when coordinates of both epoches refer to independent frames, a smooth change of frame is also interpolated, which of course is physically absurd. As a consequence deformation parameters reflect not only the true rate of deformation of the earth but also the rate of pseudo-deformation arising from the change of frame with time.
3.2. THE FINITE ELEMENT APPROACH IN THE PLANE

In the planar case the orthogonal eigenvector matrix $V$ is a plane rotation matrix

$$V = R(\phi) = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} = \frac{1}{1 + q^2} \begin{bmatrix} 1 - q^2 & 2q \\ -2q & 1 - q^2 \end{bmatrix}$$

and the eigenvalue matrix takes the form

$$\Lambda = \begin{bmatrix} e_{\text{max}} & 0 \\ 0 & e_{\text{min}} \end{bmatrix} = \begin{bmatrix} \frac{\Delta + \gamma}{2} & 0 \\ 0 & \frac{\Delta - \gamma}{2} \end{bmatrix}$$

where $e_{\text{max}}$ and $e_{\text{min}}$ are the principal strains,

$$\Delta = \text{tr}(S) = e_{\text{max}} + e_{\text{min}} = S_{11} + S_{22}$$

is the dilatation and

$$\gamma = e_{\text{max}} - e_{\text{min}}$$

is the maximum shear strain. From the well known solution to the planar eigenvalue problem we have

$$2S = R(\phi) (2\Lambda) R(-\phi) = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \Delta + \gamma & 0 \\ 0 & \Delta - \gamma \end{bmatrix} \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} =$$

$$= \begin{bmatrix} \Delta + \gamma \cos 2\phi & -\gamma \sin 2\phi \\ -\gamma \sin 2\phi & \Delta - \gamma \cos 2\phi \end{bmatrix}.$$ 

Introducing the auxiliary parameters (shear components)

$$\gamma_1 = S_{11} - S_{22} = \gamma \cos 2\phi$$

$$\gamma_2 = 2S_{12} = -\gamma \sin 2\phi$$

it is directly seen that

$$\tan 2\phi = \frac{-\gamma_2}{\gamma_1}, \quad \gamma = \sqrt{\gamma_1^2 + \gamma_2^2}$$
where \( \phi \) is the direction of maximum principal strain \( e_{\text{max}} \).

Under the transformations (9) the relevant deformation parameters will undergo corresponding transformations described by equations (15) and (16), which in the planar case become

\[
(26) \quad R(\bar{\phi}) = R(\theta) R(\phi) \quad \Rightarrow \quad \bar{\phi} = \phi + \theta
\]

\[
(27) \quad \bar{e}_{\text{max}} = \mu^2 e_{\text{max}} + \frac{\mu^2 - 1}{2}
\]

\[
(28) \quad \bar{e}_{\text{min}} = \mu^2 e_{\text{min}} + \frac{\mu^2 - 1}{2}
\]

For the dilatation and the maximum shear strain equations (20) and (21) give

\[
(29) \quad 1 + \bar{\Delta} = \mu^2 (1 + \Delta)
\]

\[
(30) \quad \bar{\gamma} = \mu^2 \gamma
\]

From equation (26) it is seen that the directions of the principal strains remain invariant, while the other parameters are influenced only by scale changes of the reference frames at the two epochs \( t, t' \) under comparison. In order to study the transformation of the shear components \( \gamma_1, \gamma_2 \) we note that

\[
(31) \quad \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \Delta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_{11} \\ S_{12} \\ S_{12} \\ S_{22} \end{bmatrix} = \begin{bmatrix} I & \Psi \\ \Psi & I \end{bmatrix} \text{vec}\mathbf{S}
\]

and

\[
(32) \quad \text{vec}\mathbf{S} = \begin{bmatrix} I & \Psi \\ \Psi & I \end{bmatrix}^{-1} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \Delta \end{bmatrix} = \frac{1}{2} \begin{bmatrix} I & -\Psi \\ -\Psi & I \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ 0 \end{bmatrix}
\]

with similar relations holding for the transformed parameters. Using the property \( \text{vec}(\mathbf{A} \mathbf{B} \mathbf{C}) = (\mathbf{B}^T \otimes \mathbf{A}) \text{vec}\mathbf{C} \), the transformation equation becomes in "vec" form

\[
(33) \quad \text{vec}\mathbf{\bar{S}} = \mu^2 (R \otimes R) \text{vec}\mathbf{S} + \frac{\mu^2 - 1}{2} \text{vec}\mathbf{I}
\]
Applying equation (31) to transformed quantities, relating vec\(\tilde{S}\) to vec\(S\) through equation (33) and vec\(S\) to the deformation parameters through equation (32) we obtain after some algebraic computations

\[
\begin{bmatrix}
\tilde{\gamma}_1 \\
\tilde{\gamma}_2 \\
\tilde{\Delta}
\end{bmatrix} = \mu^2 \begin{bmatrix}
R^2 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
\gamma_1 \\
\gamma_2 \\
\Delta
\end{bmatrix} + (\mu^2 - 1) \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

For the shear components we have explicitly

\[
\begin{bmatrix}
\tilde{\gamma}_1 \\
\tilde{\gamma}_2
\end{bmatrix} = \mu^2 \begin{bmatrix}
\cos 2\theta & \sin 2\theta \\
-\sin 2\theta & \cos 2\theta
\end{bmatrix}
\begin{bmatrix}
\gamma_1 \\
\gamma_2
\end{bmatrix}, \quad \tilde{\gamma} = \mu^2 \gamma
\]

while the last of the equations in (34) yields again equation (29).

### 3.3. THE INFINITESIMAL APPROACH IN THE PLANE

The infinitesimal approach is based on the decomposition of the displacement Jacobian matrix into a symmetric and an antisymmetric part

\[
E = \begin{bmatrix}
e_{xx} & e_{xy} \\
e_{xy} & e_{yy}
\end{bmatrix} = \begin{bmatrix}
e_{xx} & \frac{1}{2}(e_{xy}+e_{yx}) \\
\frac{1}{2}(e_{xy}+e_{yx}) & e_{yy}
\end{bmatrix} + \begin{bmatrix}
0 & \frac{1}{2}(e_{xy}-e_{yx}) \\
\frac{1}{2}(e_{xy}-e_{yx}) & 0
\end{bmatrix} = \begin{bmatrix}
S_{11} & S_{12} \\
S_{12} & S_{22}
\end{bmatrix} + \begin{bmatrix}
0 & \omega \\
-\omega & 0
\end{bmatrix} = S_o + \Omega
\]

From the definition of \(\gamma_1, \gamma_2, \Delta\) and \(\omega\) it follows that

\[
\begin{bmatrix}
\gamma_1 \\
\gamma_2 \\
2\omega \\
\Delta
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 1 & 0 \\
0 & -1 & 1 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
e_{xx} \\
e_{xy} \\
e_{yx} \\
e_{yy}
\end{bmatrix} = \begin{bmatrix}
\mathbf{1} & \mathbf{\Psi} \\
\mathbf{\Psi} & \mathbf{1}
\end{bmatrix} \text{vec}E
\]

and

\[
\text{vec}E = \begin{bmatrix}
\mathbf{1} & \mathbf{\Psi} & \mathbf{I}
\end{bmatrix}^{-1}
\begin{bmatrix}
\gamma_1 \\
\gamma_2 \\
2\omega \\
\Delta
\end{bmatrix} = \frac{1}{2}
\begin{bmatrix}
\mathbf{1} & -\mathbf{\Psi} \\
-\mathbf{\Psi} & \mathbf{1}
\end{bmatrix}
\begin{bmatrix}
\gamma_1 \\
\gamma_2 \\
2\omega \\
\Delta
\end{bmatrix}
\]
The transformation equation (11) in "vec" form becomes

$$\text{vec}\tilde{E} = \mu (\mathbf{R} \otimes \mathbf{R'}) \text{vec}E + \mu \text{vec}(\mathbf{R'} \mathbf{R}^\top) - \text{vec}I$$  \hspace{1cm} (39)

Combining equation (38), applied to the transformed quantities, with equations (39) and (38) we obtain, after some algebraic manipulation, the transformation property

$$\begin{bmatrix} \tilde{\gamma}_1 \\ \tilde{\gamma}_2 \\ 2\tilde{\omega} \\ \tilde{\Delta} \end{bmatrix} = \mu \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^\top \mathbf{R'} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ 2\omega \\ \Delta \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 2\mathbf{R'}\mathbf{R}\mathbf{e}_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 2\mathbf{e}_2 \end{bmatrix}$$  \hspace{1cm} (40)

or explicitly

$$\begin{bmatrix} \tilde{\gamma}_1 \\ \tilde{\gamma}_2 \end{bmatrix} = \mu^2 \begin{bmatrix} \cos(\theta + \theta') & \sin(\theta + \theta') \\ -\sin(\theta + \theta') & \cos(\theta + \theta') \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}, \quad \tilde{\gamma} = \mu^2 \gamma$$  \hspace{1cm} (41)

$$\begin{bmatrix} 2\tilde{\omega} \\ \tilde{\Delta} \end{bmatrix} = \mu \begin{bmatrix} \cos(\theta' - \theta) & \sin(\theta' - \theta) \\ -\sin(\theta' - \theta) & \cos(\theta' - \theta) \end{bmatrix} \begin{bmatrix} 2\omega \\ \Delta \end{bmatrix} + \begin{bmatrix} 2\mu \sin(\theta' - \theta) \\ 2\mu \cos(\theta' - \theta) - 2 \end{bmatrix}$$  \hspace{1cm} (42)

From equation (25) applied to transformed parameters the formula for the tangent of the sum of two angles gives

$$\tilde{\phi} = \phi + \frac{\theta + \theta'}{2}$$  \hspace{1cm} (43)

In the special case of connected networks a common frame change for both epochs should only be allowed, in which case \(\theta' = \theta\) gives: \(\tilde{\phi} = \phi + \theta, \quad \tilde{\omega} = \mu \omega, \quad \tilde{\Delta} = \mu \Delta + 2 (\mu - 1)\).

3.4. THE FINITE ELEMENT APPROACH IN THREE DIMENSIONS

In three dimensions the strain matrix can be computed from either the coordinate or from the displacement Jacobian matrix (equation 3) which are computed from

$$\begin{bmatrix} x_{\beta}^1 - x_{\alpha}^1 \\ x_{\gamma}^1 - x_{\alpha}^1 \\ x_{\delta}^1 - x_{\alpha}^1 \end{bmatrix} \begin{bmatrix} x_{\beta}^1 - x_{\alpha}^1 \\ x_{\gamma}^1 - x_{\alpha}^1 \\ x_{\delta}^1 - x_{\alpha}^1 \end{bmatrix}^{-1}$$  \hspace{1cm} (44)

$$\begin{bmatrix} x_{\beta}^2 - x_{\alpha}^2 \\ x_{\gamma}^2 - x_{\alpha}^2 \\ x_{\delta}^2 - x_{\alpha}^2 \end{bmatrix} \begin{bmatrix} x_{\beta}^2 - x_{\alpha}^2 \\ x_{\gamma}^2 - x_{\alpha}^2 \\ x_{\delta}^2 - x_{\alpha}^2 \end{bmatrix}^{-1}$$  \hspace{1cm} (44)

$$\begin{bmatrix} x_{\beta}^3 - x_{\alpha}^3 \\ x_{\gamma}^3 - x_{\alpha}^3 \\ x_{\delta}^3 - x_{\alpha}^3 \end{bmatrix} \begin{bmatrix} x_{\beta}^3 - x_{\alpha}^3 \\ x_{\gamma}^3 - x_{\alpha}^3 \\ x_{\delta}^3 - x_{\alpha}^3 \end{bmatrix}^{-1}$$  \hspace{1cm} (44)

$$\begin{bmatrix} x_{\beta}^4 - x_{\alpha}^4 \\ x_{\gamma}^4 - x_{\alpha}^4 \\ x_{\delta}^4 - x_{\alpha}^4 \end{bmatrix} \begin{bmatrix} x_{\beta}^4 - x_{\alpha}^4 \\ x_{\gamma}^4 - x_{\alpha}^4 \\ x_{\delta}^4 - x_{\alpha}^4 \end{bmatrix}^{-1}$$  \hspace{1cm} (44)
\[
E = (UD)(XD)^{-1} = \begin{bmatrix}
    u^1_b - u^1_a & u^1_y - u^1_a & u^1_z - u^1_a \\
    u^2_b - u^2_a & u^2_y - u^2_a & u^2_z - u^2_a \\
    u^3_b - u^3_a & u^3_y - u^3_a & u^3_z - u^3_a
\end{bmatrix}
\begin{bmatrix}
    x^1_b - x^1_a & x^1_y - x^1_a & x^1_z - x^1_a \\
    x^2_b - x^2_a & x^2_y - x^2_a & x^2_z - x^2_a \\
    x^3_b - x^3_a & x^3_y - x^3_a & x^3_z - x^3_a
\end{bmatrix}^{-1}
\]

Using any computer routine for the solution of the eigenvalue problem, the three unit eigenvectors \( \nu_1, \nu_2, \nu_3 \) and their corresponding eigenvalues \( \varepsilon_1 > \varepsilon_2 > \varepsilon_3 \) can be computed. \( \varepsilon_1 \) is the maximum, \( \varepsilon_2 \) is the intermediate and \( \varepsilon_3 \) is the minimum principal strain. Planar dilatations and maximum shear strains can be defined for each of the three principal planes \((2,3), (1,3), (1,2)\), perpendicular to the principal directions \( \nu_1, \nu_2, \nu_3 \), respectively:

\[
\Delta_{12} = \varepsilon_1 + \varepsilon_2 \quad \gamma_{12} = \varepsilon_1 - \varepsilon_2 \\
\Delta_{23} = \varepsilon_2 + \varepsilon_3 \quad \gamma_{23} = \varepsilon_2 - \varepsilon_3 \\
\Delta_{13} = \varepsilon_1 + \varepsilon_3 \quad \gamma_{13} = \varepsilon_1 - \varepsilon_3
\]

The total dilatation is

\[
\Delta = \frac{1}{3} (\Delta_{12} + \Delta_{23} + \Delta_{13})
\]

The direction of each principal axis is defined by the corresponding vector \( \nu_i \), or by two appropriate direction angles, e.g. "longitude" and "latitude"

\[
\Lambda_i = \arctan \left( \frac{\nu_i^2}{\nu_i^1} \right), \quad \Phi_i = \arctan \left( \frac{\nu_i^1}{\sqrt{(\nu_i^1)^2 + (\nu_i^2)^2}} \right)
\]

From the transformation equation (16) and the above definitions, it follows that under reference frame changes

\[
\left( \frac{1}{2} + \tilde{e}_i \right) = \mu^2 \left( \frac{1}{2} + e_i \right)
\]

\[
(1 + \tilde{\Lambda}_{ik}) = \mu^2 (1 + \Lambda_{ik})
\]

\[
\tilde{\gamma}_{ik} = \mu^2 \gamma_{ik}, \quad ik = 12, 23, 13
\]

\[
(1 + \tilde{\Delta}) = \mu^2 (1 + \Delta)
\]

3.5. THE INFINITESIMAL APPROACH IN THREE DIMENSIONS

The infinitesimal approach in three dimensions should be avoided because of its approximate character and the corresponding awkward transformation properties under changes of reference frames. To show that this also applies to the three-dimensional case, we use the definitions of \( S_o, \Omega \) (eq. 17) and the transformation law for \( E \) (eq. 11) to obtain
\[ (51) \quad \tilde{S} = \frac{\mu}{2} (R'S_o R^T + R S_o R^T) + \frac{\mu}{2} (R' \Omega R^T - R \Omega R^T) + \frac{\mu}{2} (R'R^T + R R^T) - I \]

\[ (52) \quad \tilde{\Omega} = \frac{\mu}{2} (R'S_o R^T - R S_o R^T) + \frac{\mu}{2} (R' \Omega R^T + R \Omega R^T) + \frac{\mu}{2} (R'R^T - R R^T) \]

or in "vec" form

\[ (53) \quad \begin{bmatrix} \text{vec}(I + \tilde{S}_o) \\ \text{vec}\tilde{\Omega} \end{bmatrix} = \mu \begin{bmatrix} R \otimes R' + R' \otimes R & R \otimes R' - R' \otimes R \\ R \otimes R' - R' \otimes R & R \otimes R' + R' \otimes R \end{bmatrix} \begin{bmatrix} \text{vec}(I + S_o) \\ \text{vec}\Omega \end{bmatrix} \]

It is obvious that the strain and rotation are not separated, except for the special case of connected networks, where only \( R' = R \) is allowed and consequently

\[ (54) \quad \tilde{S}_o = \mu R S_o R^T + (\mu - 1) I, \quad \tilde{\Omega} \equiv [\tilde{\omega} \times] = \mu R \Omega R^T \quad \Rightarrow \quad \tilde{\omega} = R \omega \]

4. ESTIMABLE QUANTITIES AND INVARIANTS

Denote by \((x_\alpha, x_\beta, x_\gamma)\), \((x'_\alpha, x'_\beta, x'_\gamma)\) the triplet of coordinate vectors of three points \(P_\alpha\), \(P_\beta\), \(P_\gamma\) at epochs \(t\), \(t'\), respectively. There are four choices of similarity invariant measures of deformations generated by positional angles \(\psi_{\beta\alpha\gamma}\), \(\psi'_{\beta\alpha\gamma}\) and distance ratios \(\frac{||x_\beta - x_\alpha||^2}{||x_\gamma - x_\alpha||^2}\), \(\frac{||x'_\beta - x'_\alpha||^2}{||x'_\gamma - x'_\alpha||^2}\) respectively, following Grafarend and Schaffrin (1976).

**1st choice**

\[ (55) \quad \cos\psi_{\beta\alpha\gamma} = \frac{x_\beta - x_\alpha \cdot x_\gamma - x_\beta}{||x_\beta - x_\alpha||^2 \cdot ||x_\gamma - x_\alpha||^2} \]

\[ (56) \quad \cos\psi'_{\beta\alpha\gamma} = \frac{x'_\beta - x'_\alpha \cdot x'_\gamma - x'_\beta}{||x'_\beta - x'_\alpha||^2 \cdot ||x'_\gamma - x'_\alpha||^2} \]

\[ (57) \quad \operatorname{rate}\psi = \frac{\cos\psi_{\beta\alpha\gamma}}{\cos\psi'_{\beta\alpha\gamma}} \]

**2nd choice**

\[ (58) \quad \operatorname{diff} \psi = \cos\psi_{\beta\alpha\gamma} - \cos\psi'_{\beta\alpha\gamma} \]

or

\[ (59) \quad \ln \operatorname{rate} \psi = \ln \cos\psi_{\beta\alpha\gamma} - \ln \cos\psi'_{\beta\alpha\gamma} \]
3rd choice

\[
\text{rate dist} = \frac{||x_\beta - x_\alpha||^2}{||x_\gamma - x_\alpha||^2} \frac{||x_\gamma' - x_\alpha'||^2}{||x_\beta' - x_\alpha'||^2} = \frac{||x_\beta - x_\alpha||^2}{||x_\gamma - x_\alpha||^2} \frac{||x_\gamma - x_\alpha||^2}{||x_\beta - x_\alpha||^2}
\]

4th choice

\[
\text{diff dist} = \frac{||x_\beta - x_\alpha||^2}{||x_\gamma - x_\alpha||^2} - \frac{||x_\beta' - x_\alpha'||^2}{||x_\gamma' - x_\alpha'||^2}
\]

or

\[
\ln \text{rate dist} = \ln \frac{||x_\beta - x_\alpha||^2}{||x_\gamma - x_\alpha||^2} - \ln \frac{||x_\beta - x_\alpha||^2}{||x_\gamma - x_\alpha||^2} = 2 \left\{ \ln ||x_\beta - x_\alpha|| - \ln ||x_\gamma - x_\alpha|| - \ln ||x_\beta' - x_\alpha'|| + \ln ||x_\gamma' - x_\alpha'|| \right\}
\]

5. AN EXAMPLE

An example of geodetic strain patterns has been given by Chen (1991) and Kakkuri and Chen (1992), who use the infinitesimal approach with the correct interpretation (strains rates). Figure 5.1. summarizes the first order triangulation network of Finnland. Figure 5.2 illustrates the positive and negative dilatation distribution \( \Delta \) in the horizontal plane, while figure 5.3. illustrates the maximum shear strain distribution \( \gamma \) within the horizontal Finnish 1st order network. Principal strains \( e_{\max}, e_{\min} \), as well as the orientation \( \phi \) and \( \phi + \frac{\pi}{2} \) are plotted for this network in figure 5.4. The known tectonic strain patterns are graphically given by figure 5.5. compared to the geodetically determined rates of change in figure 5.6.

6. CONCLUSIONS

The basic results of this work are given by the transformation formulas for deformation parameters, which allow to draw direct conclusions about their invariance and therefore their estimability. There are no estimable parameters when a different scale has been used for the frame definition at each of the two epoches under comparison. This is not a such a rare case as it might seem, since distances are presently measured indirectly by time intervals and the clocks used may have relative drifts. Estimability is therefore guaranteed only in the case of distance measurements at both epoches with instruments which either preserve the "unit of length" definition (stable oscillators) or else they are calibrated by comparison on an undeformed baseline.

For the rigorous approach, estimable up to scale, i.e., only when \( \mu = 1 \), are in the planar case the maximum and minimum principal strains and their directions, the dilatation and the
maximum shear strain. In the three-dimensional case estimable are the three principal strains (maximum, intermediate and minimum) and their directions, the planar dilatations and maximum shear strains of the three principal planes and the total dilatation.

The same results hold in the infinitesimal approach only in the special case of geodetic networks connected through a common undeformed subnetwork. Otherwise only maximum shear strain in the plane appears to be an estimable quantity. For this reason the infinitesimal approximation should be avoided in the case where independently adjusted networks (considered different for the two epoches even if they consist of the same physical points) with different arbitrary frame definitions at the two different epoches are to be compared.

REFERENCES


Figure 5.1: The Finnish first order triangulation network.
(From Chen, 1991)
Figure 5.2: Horizontal positive and negative dilatation distribution Δ in Finlann.
(From Chen, 1991)
Figure 5.3: Horizontal maximum shear strain distribution $\gamma$ in Finmland.
(From Chen, 1991)
Figure 5.4: Horizontal principal strains \( e_{\text{max}} \), \( e_{\text{min}} \), and their orientation \( (\phi, \phi + \frac{\pi}{2}) \) in Finland.

(From Chen, 1991)
Figure 5.5: General structure of strain patterns in Finland (simplified from fig. 5.4)
(From Chen, 1991)
Figure 5.6: Geodetically determined rates of change of lines across the tectonic boundaries (in mm/year).
(From Chen, 1991)
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