Estimability analysis of geodetic, astrometric and geodynamical quantities in very long baseline interferometry

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**Summary.** Within the framework of Newtonian kinematics VLBI observations are analysed with respect to estimability of geodetic and astrometric quantities. An Earth model of either rigid or deformable type is designed; instrumental clock offsets and clock drifts are included. Observational patterns are studied in all detail reviewed in seven tables. Appendix A is an introduction to the set-up of the observational model for a deformable Earth both in terms of coordinate-free and coordinate-related geometry. Appendix B is a study of the invariance characteristics of VLBI observations. Interrelations of three fundamental quantities, length unit, time unit and velocity of light are discussed. An overall result of an Earth model of deformable type is the need of simultaneous observations to more than one source; VLBI time delay observations cannot distinguish between secular changes of network size (expansion or shrinking) and a common secular drift (deceleration or acceleration) of clocks used.

**Introduction**

The scope of the present work is first to introduce a careful set-up of very long baseline interferometry (VLBI) observation equations within the framework of Newtonian mechanics and a generally deformable Earth in both a coordinate-free or synthetic and frame-dependent way using concepts of the modern theory of the kinematics of deformable bodies, e.g. Macvean (1968), Truesdell (1977) and Wang & Truesdell (1973). Comparing the order of magnitude of the terms in the general model with the observational noise level certain terms may be discarded and thus more simple models are obtained. On the basis of such simplified models the problem of estimability of parameters present in VLBI observations is considered. It turns out that with a carefully designed pattern of observations in a geometric mode, for instance simultaneous observations, both geometry of a rigid network and network rotation parameters at the discrete epochs of observations are estimable. As a byproduct the relative configuration of observed radio sources on the celestial sphere (astrometry) can also be estimated as well as clock-related parameters. In addition, the possibility of estimating such time-discrete or instantaneous parameters together with time-discrete geometry of a deformable network is investigated. The estimability of time-discrete
rigid network rotation parameters poses the limit on resolution of Earth rotation by VLBI observations.

Usually the concept of estimability of parameters is introduced in a stochastic set-up, e.g. in relation to the so-called linear Gauss–Markov model for the adjustment of observations distorted by errors. In essence estimability has nothing to do with stochastics and can be judged upon by only examining the deterministic part of the linear (or even non-linear) model connecting observables with parameters. If the true values of observables were known, estimable parameters are those that can be uniquely determined from these known values. In this respect we say that parameters are estimable if they are determinable from observational errors approaching zero. The case where, in addition to independent parameters and observational errors, parameters depending on underlying functions are also present within the model, is more complicated. We shall refer to such function dependent parameters as signals. There are two approaches possible to the treatment of signals following Dermanis (1979). In the first the underlying function (or an unknown part of it) is treated as a zero-mean second-order stochastic process. Signals form then a zero-mean second-order random vector just like the observational errors. In this case an independent parameter is estimable if it is uniquely determinable for both errors and signals becoming zero. In a second approach, the so-called deterministic approach, the relation of signals to an underlying function is completely ignored. Signals are treated as independent parameters. An independent parameter is estimable if it is uniquely determinable for errors becoming zero. In addition we have the previously absent concept of an estimable signal, e.g. a signal which is uniquely determinable for errors becoming zero. This is actually the approach taken here for investigating the estimability of parameters, related to functions describing Earth rotation and deformation. The values of such functions at discrete time epochs are the considered signals.

A third possible approach, the so-called model function approach, is the modelling of underlying unknown functions in terms of finite parameters. These parameters may become estimable if a sufficient number of observations is available. The need on the one hand for a larger number of parameters to make the model function a more sufficient approximation to the real one, and the need on the other hand for a smaller number of parameters to increase the number of degrees of freedom for a more powerful adjustment, pose a problem that cannot be a priori resolved. We do not pursue this approach here since the conclusion about estimability has no universal validity in this case, but strongly depends on the function model being used.

Appendix A is an introduction to VLBI observation equations (functionals) referring to the so-called deterministic approach. There is a close link between the concept of estimability and invariance of observations following Baarda (1973) and Grafarend & Schaffrin (1976). Those quantities which belong to the invariance class of the observations are estimable. For instance, if only angles are observed, distance ratios are estimable since both are invariant under a similarity transformation. Appendix B is an investigation of the transformation leaving VLBI observations invariant within the bounds of Newtonian space-time structure. It is shown that without clock drifts only shape and size parameters of rigid VLBI network or instantaneous shape and size parameters of a deformable Earth are candidates as estimables. If clock drifts are included within the model, then only shape parameters are in the set of estimable quantities, since scale of the network is coupled with unknown scale of time. Whether such parameters are actually estimable depends on the number and the specific pattern of the observation. This problem is solved in Section 1 for a rigid network, in Section 2 for a deformable network and in Section 3 for both rigid and deformable networks in the case of clock drifts present in the model.
In more detail, a familiar concept in the theory of satellite observations for geodetic purposes is the so-called geometric observational mode, e.g. Aardom (1973), Tsimis (1973). As opposed to the so-called dynamical mode where the satellite orbits or the gravitational field driving them are present as unknown functions, in the geometric mode these functions are disregarded utilizing simultaneous observations. What remains present in the relevant model are the satellite positions at the epochs of observation. These positions are exactly the signals of the previously mentioned deterministic approach. In the case of VLBI observations from stations of a rigid network, the 'orbits' of the observed radio sources are the variations with time of their directions with respect to the network. They are governed by the relative configuration of the observed radio sources on the celestial sphere and the rotation of the network with respect to the celestial sphere. The use of simultaneous observations allows us to use only the values of functions describing network rotation at observational epochs. Such time-discrete values constitute signals treated as independent non-stochastic parameters and the question about their estimability from available or conceivable observations becomes relevant. If a deformable network is considered its geometry, shape and size becomes a function of time. For simultaneous observations only the values of functions describing network geometry at the observation epochs are relevant. Such values become additional non-stochastic signals, and their estimability is also a matter of consideration.

The use of simultaneous observations to radio sources has been first considered by Aardom (1972), and more recently by Cannon (1979). We note that estimability is investigated here without reference to the so-called critical configurations of stations and radio source directions leading to additional rank deficiencies in the operator mapping parameters into observables, e.g. Tsimis (1973). The consideration of critical configurations deserves a study of its own.

More specifically the observational model is as follows. We refer to Newtonian space-time structure, mainly after a reduction of observations due to relativity, refraction, parallax, proper motion of radio sources. Special attention is paid to a careful terrestrial model, the rigid and the deformable one. Again we assume a priori reductions due to tides etc. Local rotation is generally modelled by vorticity being separated from the global rotation vector characterized by a homogeneous vorticity vector field. In between the concept of plate tectonics is introduced, namely plate rotations and translations. In detail, time delay and delay rate of VLBI signals travelling between terrestrial stations are represented by 6 and 11 model terms, respectively, including the station-to-station baseline vector, baseline retardation, baseline deformation, and time offset, frequency drift of time. These observational models are finally simplified after a careful analysis of the numerical impact of different terms.

1 Estimable quantities for a rigid network of stations engaged in simultaneous observations

1.1 Geodetic Estimables

In this mode g stations comprising a rigid network co-observe simultaneously a radio source at each epoch. Concentrating on the possibility of extracting only geodetic information from the observations the interrelation between configuration of radio sources, network orientation and instantaneous radio source directions with respect to the network is disregarded. Each instantaneous radio source direction with respect to the network is treated as an independent unknown.
Since observations are invariant with respect to translations and rotations of a reference frame, we can only hope to estimate shape and size of the network. We refer to a simplified model of equations (A25) and (A26) with or without clock offsets when only delays or both delays and delay derivatives are observed. Only the case where both types of observations are available and clock offsets are included as unknown parameters is realistic. The other cases are examined for comparison purposes, especially in comparison to the results of Aardom (1972).

The situation of the unknowns is as follows: shape and size of a rigid network of \( g \) stations is sufficiently described by \( 3g - 6 \) parameters. For a proof consider a three-station network whose shape and size is fully described by three parameters, e.g. the lengths of its three sides. In order to attach any new station to the network, at least three new parameters are needed, e.g. three distances to three 'old' stations. Thus the total number of parameters for \( g \) stations is \( 3 + (g - 3)3 = 3g - 6 \). Alternatively, if a reference frame is used there exist \( 3g \) coordinates, but the six degrees of freedom for frame rotation and translation have been subtracted.

At each epoch of observation the direction of the observed radio source with respect to the network is described by two parameters, e.g. its angles with two network sides. If the stations co-observe for \( i \) epochs, \( 2i \) parameters are totally needed for the instantaneous radio source directions. According to equation (A19), the observations of delay derivatives depend also on the instantaneous rotation vector of the network at each observation epoch. This vector can be referred to the rigid network by three parameters, two for its relative orientation and one for its magnitude and therefore a total of \( 3i \) additional parameters are introduced.

Only differences of clock offsets appear in the observations. If \( b_i \) is the absolute clock offset at station \( i \), then the set \( b_{ij} = b_i - b_j, i = 2, 3, \ldots, g, \) is a complete independent set of relative offsets in the sense that any other relative offset can be expressed in terms of this set, e.g. \( b_{ij} = b_j - b_i = b_j - 1 - (b_i - 1) = b_{ij} - b_{ij} \). Thus clock offsets introduce \( g - 1 \) parameters.

The situation of the observations is as follows: a network of \( g \) stations has \( C_g^g = g!/[2!(g - 2)!] \) baselines and if they all co-observe at a certain epoch the same number of delay observations can be reproduced by cross-correlation of station tapes in all possible pairs. However, these observations are not functionally independent; for each network triangle \( ijk \) a condition of the form \( \tau_{ij} + \tau_{jk} + \tau_{ik} = 0 \) holds. An independent complete set of observations is provided by the set \( \tau_{ij} = t_i - t_j, i = 2, 3, \ldots, g, \) since any other delay can be expressed in terms of this set, e.g. \( \tau_{ij} = t_j - t_i = t_j - t_i - (t_i - t_j) = \tau_{ij} - \tau_{ij} \). The total number of independent delay observations for \( i \) observational epochs is \( i(g - 1) \) and the same number holds for observations of delay derivatives.

The determination of all the above considered estimable parameters, e.g. network shape and size, instantaneous radio source directions and rotation vector values for all observation epochs, is secured once the number of observations \( n \) is not less than the number of unknowns \( m \), i.e. \( n \geq m \). Next the number \( i \) of observation epochs necessary for such an inequality to hold is derived in terms of the number \( g \) of network stations.

Note that two other side inequalities have to be satisfied:

(i) Since only delay observations are sensitive to clock offsets their number must be at least equal to the number of clock offset parameters. This leads to the trivially satisfied inequality \( i(g - 1) \geq g - 1 \).

(ii) Since only observations of delay derivatives are sensitive to the rotation vector, their number must not be less than the number of rotation vector parameters. This leads to the inequality \( i(g - 1) \geq 3i \) yielding \( g \geq 4 \).
1(a) Observations of delays only, no clock offsets:

Unknowns: \( m = 3g - 6 + 2i \)
Observations: \( n = i(g - 1) \)
\[ n \geq m \Rightarrow i(g - 1) \geq (3g - 6) + 2i \Rightarrow \]
\[ i \geq 3(g - 2)(g - 3)^{-1}, \quad \text{always } i \geq 4 \]  

(1.1)

1(b) Observations of delays only with clock offsets:

Unknowns: \( m = 3g - 6 + 2i + g - 1 = 4g - 7 + 2i \)
Observations: \( n = i(g - 1) \)
\[ n \geq m \Rightarrow i(g - 1) \geq (4g - 7) + 2i \Rightarrow \]
\[ i \geq (4g - 7)(g - 3)^{-1}, \quad \text{always } i \geq 5 \]  

(1.2)

1(c) Observations of delays and delay rates, no clock offsets:

Unknowns: \( m = 3g - 6 + 2i + 3i = 3g - 6 + 5i \)
Observations: \( n = 2i(g - 1) \)
\[ n \geq m \Rightarrow 2i(g - 1) \geq (3g - 6 + 5i) \Rightarrow \]
\[ i \geq 3(g - 2)(2g - 7)^{-1}, \quad \text{always } i \geq 2 \]  

(1.3)

1(d) Observations of delays and delay rates with clock offsets:

Unknowns: \( m = 3g - 6 + 2i + 3i + g - 1 = 4g - 7 + 5i \)
Observations: \( n = 2i(g - 1) \)
\[ n \geq m \Rightarrow 2i(g - 1) \geq (4g - 7 + 5i) \Rightarrow \]
\[ i \geq (4g - 7)(2g - 7)^{-1}, \quad \text{always } i \geq 3 \]  

(1.4)

The detailed results of the above inequalities—solutions of the different diophantic inequalities—for various values of \( g \) can be taken from Table 1. For the first two cases

| Table 1. Necessary number of observation epochs \( i \) for the determination of estimable parameters in a rigid network of \( g \) stations co-observing one radio source at each epoch. |
|-----------------|-----------------|-----------------|-----------------|
|                 | No clock offsets| With clock offsets| With clock offsets and clock drifts |
|                 | \( g \) \( i \geq \) | \( g \) \( i \geq \) | \( g \) \( i \geq \) |
| Delay observations only |
| Delay only | 4 | 6 | 4 | 9 | 4 | 12 |
| \( \geq 6 \) | 5 | 5 | 5 | 7 | 5 | 9 |
| Delay and delay rate observations |
| Delay only | 4 | 6 | 4 | 9 | 4 | 12 |
| \( \geq 8 \) | 7 | 6 | 7 | 7 |
| \( \geq 10 \) | 9 | 7 |
| With clock offsets and clock drifts |
| Delay only | 4 | 6 | 4 | 9 | 4 | 12 |
| \( \geq 8 \) | 2 | \( \geq 13 \) | 3 |
results are identical with those of Aardom (1972) obtained by a different type of reasoning. The case where clock drifts are included, but is discussed in Section 3.

1.2 ESTIMABILITY OF ASTROMETRIC AND NETWORK ROTATION PARAMETERS

The estimability of astrometric and network rotation parameters will be examined here utilizing the results of the previous section.

Astrometric parameters are the parameters necessary for the determination of the relative configuration of the directions of radio sources. This configuration is assumed to be time independent, e.g. we consider a 'directionally rigid' network of sources without proper motion, $\dot{e}(\gamma) = 0$. The relative configuration of $h$ sources is fully described by $2h - 3$ parameters. For the first two sources only one parameter is needed, e.g. the angle between them, while for each new source two new parameters are necessary, e.g. the angles with two old sources. This leads to a total of $1 + (h - 1)2 = 2h - 3$ parameters.

Network rotation parameters appear in VLBI observations in two ways. First, delay observations are sensitive as already seen to the instantaneous rotation vector at each observation epoch. Secondly, due to network rotation with respect to the inertial source configuration, instantaneous directions to sources with respect to the network depend on the source configuration and also on parameters describing the instantaneous relative orientation of the network with respect to the source configuration as a whole. These last parameters, three for each epoch, can be visualized in a coordinate-free approach, e.g. as two angles of a source direction with two network sides plus the angle of the plane of two source directions with the plane of two network sides. However, the use of two arbitrary frames, one network-fixed and the other source-fixed, leads to the more familiar concept of Eulerian angles (or other similar parameters) describing the relative orientation of the two frames. We shall refer to such parameters as geometric rotation parameters. There are three for each epoch, i.e., a total of $3i$.

In contrast the instantaneous rotation vector depends not only on the instantaneous values of geometric rotation parameters but also on their kinematic behaviour at a time-neighbourhood of the observation epoch. This fact is well expressed by the Euler kinematic equations, e.g. Magnus (1971, pp. 35–39), expressing rotation vector components in terms of Eulerian angles and time derivatives.

The additional use of source configuration and geometric rotation parameters introduces a total of $3i + 2h - 3$ parameters while the previous treatment required only $2i$ parameters.

A similar analysis as in the previous section will reveal the necessary number of observational epochs for the number of observations to be equal or greater than the number of unknowns. However, such conditions are in this case only necessary and not sufficient. Lack of sufficiency can be easily proved by the following argument. Consider the simplest case where the source configuration is a priori known. At each observation the station network can be rotated about the instantaneous direction of the observed source without any effect on the observations. There exist therefore infinitely many geometric network rotation parameters giving rise to the same observations and consequently such parameters are not estimable.

It therefore becomes obvious that a different pattern of observations is needed for astrometric and geometric rotation parameters to be estimable.

For the determination of source configuration, angles between pairs of source directions must be first determined. This is possible if each of the co-observing stations observes at each epoch not one but two or more sources simultaneously. With a sufficient number of observations both network shape and size and instantaneous source directions with respect
to the network can be estimated. From the known directions the angle between the two sources observed at each epoch can be estimated. This situation is not absolutely impossible. Two nearby antennae simultaneously co-observing two different sources may be locally connected through high precision surveying and after appropriate reductions they may be considered as a single station in the further analysis. However, it is unlikely that such an arrangement will be materialized in practice, and this concept is not further pursued here.

Consider a network of simultaneously co-observing stations, consisting of two subnetworks where each subnetwork co-observes a different source. A sufficient number of observation epochs leads to the estimation of the shape and size of each subnetwork independently plus the instantaneous directions of sources with respect to the subnetwork observing them. If the relative position of the two subnetworks were known, then at each epoch the directions of the two sources observed could be interrelated for the estimation of their angle. To find the relative position of the two subnetworks additional observations are needed with subdivision of the network to new subnetworks.

The total set of observations where stations co-observe according to the same subdivision to subnetworks is said to constitute a phase. Two subnetworks from different phases are said to be directly connected if they have at least three stations in common. Two subnetworks are said to be connected if they belong to a sequence of successively directly connected subnetworks. If all the subnetworks of a network are connected in an observational design, its shape and size is determined when the shape and size of each subnetwork is determined.

A careful choice and arrangement of sources observed may yield a sufficient number of angles among them for the determination of their relative configuration. Then at each epoch the two observed source directions are sufficient for the determination of the orientation of the network with respect to the source configuration. One direction determines the orientation of the network up to a rotation about it and the second removes this degree of freedom. Therefore, in such a design not only parameters for the shape and size of the network (geodesy), but also relative source configuration (astrometry), instantaneous geometric network rotation parameters, and instantaneous rotation vector values for all observational epochs are estimable.

The orientation of the instantaneous rotation vector with respect to the network constitutes instantaneous ‘polar motion’ values. From the estimated instantaneous orientation of the network with respect to the sources, the orientation of the instantaneous rotation vector with respect to the sources can be derived, constituting instantaneous ‘precession—nutation’, values. Thus time-discrete values of polar motion, precession—nutation as well as of rotational velocity of the network (related to the concept of length-of-day) are also estimable.

The above concept of observational design may be better understood by means of an example:

Example (Network of 12 stations co-observing at eight epochs): epochs are divided into two phases of four epochs each. In each phase the network is divided into two subnetworks, stations of each subnetwork co-observing the same source at each epoch. Fig. 1 is a scheme for the subdivision of the network into subnetworks for each of the two phases. The arrangement of the stations is purely schematical and unrelated to their real arrangement on the Earth surface. Stations connected by lines belong to the same subnetwork. Numbers in circles designate the different subnetworks. Shaded parts designate the common parts between subnetworks which maintain direct connections. Table 2 depicts the observational pattern to be followed, i.e. which subnetworks observe which source at each epoch. Five sources are participating in the observations.
Figure 1. Division of a network into subnetworks.

Table 2. Observational pattern – sources observed.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Epochs</td>
<td>$t_1$</td>
<td>1</td>
<td>2</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>$t_2$</td>
<td>–</td>
<td>–</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$t_3$</td>
<td>2</td>
<td>3</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>$t_4$</td>
<td>–</td>
<td>–</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$t_5$</td>
<td>3</td>
<td>4</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>$t_6$</td>
<td>–</td>
<td>–</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$t_7$</td>
<td>4</td>
<td>5</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>$t_8$</td>
<td>–</td>
<td>–</td>
<td>4</td>
</tr>
</tbody>
</table>

Epochs $t_1$, $t_3$, $t_5$, $t_7$ constitute phase 1 and epochs $t_2$, $t_4$, $t_6$, $t_8$ phase 2. Each phase constitutes four epochs which are sufficient for the determination of the shape and size of the subnetworks when both delay and delay derivatives are observed while clock offsets are included as parameters. Reference is made to Table 1.

Subnetwork 3 is directly connected with subnetwork 1 through their three common stations (1, 2, 6). Subnetwork 2 is directly connected with subnetwork 3 through stations (3, 4, 7) and subnetwork 4 with subnetwork 2 through stations (8, 11, 12). These direct connections are sufficient for the establishment of the shape and size of the whole network from the individual shapes and sizes of the subnetworks. There exists an additional superfluous direct connection namely that of subnetworks 4 and 1 through stations (5, 9, 10). Subnetwork 4 could contain three new stations instead of 5, 9 and 10, thus allowing for the determination of shape and size of a network of as much as 15 stations in addition to the determination of other estimable parameters with the same set of observations. The above design was chosen with a view towards minimization of the number of stations rather than the number of observations. From the determined instantaneous source directions with respect to the whole network, one angle between sources is determined at each epoch. From the determined angles between source pairs (1, 2), (2, 3), (1, 3), (2, 4), (3, 4), (3, 5), (4, 5), (4, 1), the source configuration is determined as seen in Fig. 2 where source directions are

Figure 2. Determination of source configuration.
illustrated as points on a unit sphere and angles as great circle arcs. In fact angle (4, 1) is superfluous. The illustrated position of the sources on the unit sphere is purely schematic and unrelated to their true position on the celestial sphere.

**General criteria**

We consider the case of a network with \( g = 2g_s \) stations co-observing separated at each phase into two subnetworks each with \( g_s \) stations. The choice of subnetworks with the same number of stations is not necessary, but it leads to the determination of source angles at each observational epoch. At each phase the minimum number \( i_s \) of observation epochs necessary for the determination of the shape and size of each subnetwork with \( g_s \) stations is given in Table 1. Considering only the case where delays and delay derivatives are observed and clock offset are included as parameters, the relevant inequality is

\[
i_s \geq (4g_s - 7) (2g_s - 7)^{-1}.
\]  

(1.5)

If \( f \) is the number of phases necessary for the connection of the subnetworks to be connected the total minimum number of necessary observation epochs is

\[
i = fi_s.
\]  

(1.6)

The total number of angles between sources determined is also \( i \). Since for the configuration of \( h \) sources \( 2h - 3 \) parameters at least are needed we must have

\[
i \geq 2h - 3, \quad \text{always } h < (i + 3)/2.
\]  

(1.7)

The number \( f \) of necessary phases for subnetworks connection is determined as follows: at the first phase we have subnetworks 1 and 2, at the second phase subnetworks are introduced having three common stations with subnetwork 1 and \( g_s - 3 \) new stations from subnetwork 2. The intermediate subnetwork advances by \( g_s - 3 \) stations at each phase change. For the connection to be obtained the total number of stations advanced \((f - 1)(g_s - 3)\) must be at least three so that a direct connection with three common stations from subnetwork 2 takes place. From \((f - 1)(g_s - 3) \geq 3\) it follows that

\[
f \geq 3/(g_s - 3).
\]  

(1.8)

Table 3 reviews the number of necessary phases according to the number \( g_s \) of subnetwork stations. In contrast, Table 4 summarizes the above results giving the minimum number of observations \( i \) required for the determination of geodetic, astrometric and rotation parameters, as well as the maximum number of sources for which a catalogue may be constructed from the above observations. If superfluous connections are avoided, each new

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**Table 3.** Number of observation phases \( f \) necessary for the connection of networks observing in subnetworks of \( g_s \) stations.

<table>
<thead>
<tr>
<th>( g_s )</th>
<th>( f \geq )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>( \geq 6 )</td>
<td>2</td>
</tr>
</tbody>
</table>
Table 4. Minimum number of observation epochs \( i \) necessary for a network of \( g \) stations co-observing in subnetworks with \( g_s \) stations at \( f \) observational phases of \( i_s \) observation epochs each. \( h \) is the maximal allowed number of sources and \( g_{\text{max}} \) the maximal number of stations that can be determined from the same number of observation epochs \( i \).

<table>
<thead>
<tr>
<th>( g )</th>
<th>( g_s )</th>
<th>( i_s )</th>
<th>( f )</th>
<th>( i ) (min)</th>
<th>( h ) (max)</th>
<th>( g ) (max)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>4</td>
<td>9</td>
<td>4</td>
<td>36</td>
<td>19</td>
<td>11</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>5</td>
<td>3</td>
<td>15</td>
<td>9</td>
<td>14</td>
</tr>
<tr>
<td>12</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td>8</td>
<td>5</td>
<td>15</td>
</tr>
<tr>
<td>14</td>
<td>7</td>
<td>3</td>
<td>2</td>
<td>6</td>
<td>4</td>
<td>18</td>
</tr>
<tr>
<td>( 2g_s )</td>
<td>( g_s &gt; 7 )</td>
<td>3</td>
<td>2</td>
<td>6</td>
<td>4</td>
<td>( 3(g_s - 1) )</td>
</tr>
</tbody>
</table>

phase after the first introduces \( g_s - 3 \) new stations with a total of \((f - 1)(g_s - 3)\). The total number of network stations becomes

\[
g_{\text{max}} = 2g_s + (f + 1)(g_s - 3) = g_s + 3 + f(g_s - 3).
\] (1.9)

The most general case that can be examined is of a network with \( g \) stations subdivided into \( k \) subnetworks with unequal number of stations \( g_1, g_2, \ldots, g_k \) respectively and the subnetwork subdivision may also be different for each phase. No general results can be stated for such a case. For the case of \( k \) subnetworks of \( g_s \) stations each we have \( g = kg_s, i_s \) and \( f \) are determined from \( g_s \) as before, \( i = fi_s \), while at each epoch \( k \) directions are observed allowing for the determination of \( 2k - 3 \) angles. The total number of angles is \( i(2k - 3) \) and from \( i(2k - 3) > 2h - 3 \) we obtain the number of sources allowed being determined by the inequality

\[
h < \lfloor i(2k - 3) + 3 \rfloor / 2.
\] (1.10)

The same type of design above can be repeated with the same network but a new set of sources having only two common sources with the old set. After the relative configuration of the new source set is determined the two sets old and new are interrelated through their two common sources.

2 Estimable quantities for a deformable network of stations

2.1 Geodetic, Astronomic and Rotation Estimables

In the previous section, Table 1, it has been shown that at least three observation epochs are necessary for establishing the geometry of a rigid network. In the case of a deformable network, network geometry is different at every observational epoch. It follows that deformation geometry is not estimable in the case of observations where all stations co-observe one source at each epoch.

From a practical point of view this only means that spectral resolution of deformation cannot be brought to the level of the Nyquist frequency corresponding to the interval \( \Delta t \) of successive observations. Instead, if \( i \) is the minimal necessary number of observation epochs, only frequencies up to the Nyquist frequency corresponding to the time interval \( i \Delta t \) can be resolved.

In order to make the estimation of network geometry possible for each observation epoch we introduce the concept of a network of stations co-observing simultaneously not one but \( h \) radio sources at each epoch. In this case we have for each epoch the following situation of unknowns and observations:

For instantaneous network geometry (shape and size) there are \( m_g = 3g - 6 \) parameters, for instantaneous source directions with respect to the network \( m_g = 2h \) parameters and
$m_c = g - 1$ parameters for clock offsets. When, in addition, delay rates are observed a set of $m_r = 3g - 3$ parameters has to be introduced. The reason is that due to deformation no global vorticity for the whole network exists, but individual ones for station subnets. The characteristic number $m_r$ is derived by the following arguments.

(i) Looking at the observational equations it is obvious that delay rate depends on the average vorticity materialized by three parameters referring to an arbitrarily moving network frame and also on the velocity field of individual stations with respect to chosen frame materialized by $3g$ parameters. Since only velocity differences appear within the observational scalar product, observations are invariant with respect to frame translations and rotations bringing the number of parameters necessary to describe the velocity field to $3g - 6$. Thus the total number of parameters sums up to $(3g - 6) + 3 = 3g - 3$.

(ii) Each individual delay rate observation refers to a single deformable baseline and depends on the rate of change of orientation and length of the baseline. Change of orientation involves two parameters since rotation about the baseline leaves observations invariant. Adding one parameter for the rate of change of baseline length one arrives at three parameters for each baseline and observation, i.e., a total of $3(g - 1)$ parameters since at each epoch there exist $g - 1$ independent delay rate observations.

(iii) Shape and size of a network of $g$ stations are determined by means of $3g - 6$ appropriately chosen side lengths. The rate of change in shape and size is similarly determined by $3g - 6$ rates of change of the same side lengths. Since delay rate observations depend not only on the rate of change of shape and size but also on the orientation of the network-3 parameters — a total of $3g - 6 + 3 = 3g - 3$ new parameters is introduced.

The situation of the observations is as follows: for time delay observations there are $g - 1$ independent observations for each source, e.g., a total of $n_d = h(g - 1)$ observations if $h$ sources are observed. The number of delay rate observations is exactly the same, namely $n_r = h(g - 1)$.

Next we determine the number of sources $h$ necessary for the determination of instantaneous network geometry as a function of the number of stations $g$ in the network for all four special cases considered in Section 1.1. Two side conditions are again imposed by the fact that only the $n_d = h(g - 1)$ delay observations are sensitive to $m_c = g - 1$ parameters, and only the $n_r = h(g - 1)$ delay rate observations are sensitive to additional $m_r = 3g - 3$ parameters.

The first condition is $n_d > m_c$ or $h(g - 1) > g - 1$ which is trivially satisfied by $h > 1$. The second is $n_r > m_r$ or $h(g - 1) > 3g - 3$ leading to the a priori condition $h > 3$!

2(a) Observations of delays only, no clock offsets:

Unknowns: $m = 3g - 6 + 2h$

Observations: $n = h(g - 1)$

$n > m \Rightarrow h(g - 1) > 3g - 6 + 2h \Rightarrow$

$h > (3g - 6)(g - 3)^{-1}$, always $h > 4$. \hfill (2.1)

2(b) Observations of delays only with clock offsets:

Unknowns: $m = 3g - 6 + 2h + g - 1 = 4g - 7 + 2h$

Observations: $n = h(g - 1)$

$n > m \Rightarrow h(g - 1) > 4g - 7 + 2h \Rightarrow$

$h > (4g - 7)(g - 3)^{-1}$, always $h > 5$. \hfill (2.2)
Table 5. Necessary number of sources $h$ necessary to be simultaneously observed in a single epoch from $g$ stations for the determination of estimable parameters.

<table>
<thead>
<tr>
<th></th>
<th>No clock offsets</th>
<th>With clock offsets and clock drifts</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$g$</td>
<td>$h \geq$</td>
</tr>
<tr>
<td>Delay observations</td>
<td></td>
<td></td>
</tr>
<tr>
<td>only</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>$\geq 6$</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>$\geq 8$</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>6</td>
</tr>
<tr>
<td>Delay and delay rate</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>observations</td>
<td>$\geq 4$</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>$\geq 6$</td>
<td>4</td>
</tr>
</tbody>
</table>

2(c) Observations of delays and delay rates, no clock offsets:

Unknwonns: $m = 3g - 6 + 2h + 3g - 3 = 6g - 9 + 2h$

Observations: $n = 2h(g - 1)$

$n \geq m \rightarrow 2h(g - 1) \geq 6g - 9 + 2h \rightarrow$

$h \geq (6g - 9)(2g - 4)^{-1}$, always $h \geq 4$ \hfill (2.3)

2(d) Observations of delays and delay rates with clock offsets:

Unknwonns: $m = 3g - 6 + 2h + 3g - 3 + g - 1 = 7g - 10 + 2h$

Observations: $n = 2h(g - 1)$

$n \geq m \rightarrow 2h(g - 1) \geq 7g - 10 + 2h \rightarrow$

$h \geq (7g - 10)(2g - 4)^{-1}$, always $h \geq 4$ \hfill (2.4)

The results of the above inequalities for the various values of $g$ are summarized in Table 5. In addition to instantaneous network geometry the instantaneous source directions with respect to the network are also estimable from time delay observations. In cases where time delay derivatives are also observed parameters describing the instantaneous rate of network deformation and rotation are estimable.

Since estimated source directions refer to the same epoch the relation configuration of $h$ observed sources is also determined. Since the sources are more than $2 -$ in fact always $h \geq 4$ -- the instantaneous rotation of the network with respect to the specific source frame is also determined. From observations over a sequence of epochs, the same sources or completely independent source group may be observed as two extreme cases.

In the first case the network rotation parameters refer to the same source frame at each epoch and network rotation is determined for all epochs. In the second case rotation parameters refer to independent source frames and they cannot be interrelated for different epochs. Consequently network rotation is not estimable.

In the most general case network stations co-observe at each epoch $i_j$, $j = 1, \ldots, i$, a group $H_j$ consisting of $h_j$ sources. If the configuration of the whole set

$$ \bigcup_{j=1}^{i} H_j $$
Table 6. Maximum number $h_{\text{max}}$ of sources for which a catalogue may be constructed by observations from a deformable network of stations co-observing at $i$ epochs $h$ sources per observation epoch.

<table>
<thead>
<tr>
<th>$h$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>$i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>$2i + 2$</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>8</td>
<td>11</td>
<td>14</td>
<td>17</td>
<td>$3i + 2$</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>10</td>
<td>14</td>
<td>18</td>
<td>22</td>
<td>$4i + 2$</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>12</td>
<td>17</td>
<td>22</td>
<td>27</td>
<td>$5i + 2$</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>14</td>
<td>20</td>
<td>26</td>
<td>32</td>
<td>$6i + 2$</td>
</tr>
<tr>
<td>$h$</td>
<td>$h$</td>
<td>$2h - 2$</td>
<td>$3h - 4$</td>
<td>$4h - 6$</td>
<td>$5h - 8$</td>
<td>$i(h - 2) + 2$</td>
</tr>
</tbody>
</table>

of sources involved, as well as network rotation parameters for each epoch $t_i$, are to be estimable, the source groups (subframes) $H_j$ must be interconnected through common sources.

We say that two subframes $H_j, H_k$ are directly connected if they have at least two sources in common. We say that two subframes are connected if they belong to a sequence of successively directly connected subframes. We say that the frame

$$\bigcup_{j=1}^{i} H_j$$

is connected if any two of its subframes are connected.

To obtain the maximum number of sources $h_{\text{max}}$ such that their relative configuration on the celestial sphere can be determined by a sequence of $i$ observation epochs, we proceed as follows:

$h_1$ sources are introduced at the first epoch. At the second epoch only a maximum of $h_2 - 2$ new sources can be introduced, since two must be common for connection purposes. The same holds for all subsequent epochs. Thus the maximal total number of sources becomes

$$h_{\text{max}} = h_1 + (h_2 - 2) + (h_3 - 2) + \ldots + (h_i - 2) = \sum_{j=1}^{i} h_j - (i - 1)2 = \sum_{j=1}^{i} h_j - 2i + 2.$$  \hspace{1cm} (2.5)

The most simple case appears when the same number of $h$ sources is observed at each epoch. For $h_1 = h_2 = \ldots = h_i = h$ we obtain

$$h_{\text{max}} = i(h - 2) + 2.$$  \hspace{1cm} (2.6)

Table 6 gives $h_{\text{max}}$ as a function of $h$ and $i$.

2.2 Geodetic and Rotation Estimables When Astrometric Parameters Are A Priori Known

On the assumption that radio sources have no proper motions on the celestial sphere it seems more reasonable that a source catalogue constructed on the basis of some initial observations will be available for the analysis of following VLBI observations. In the case of a rigid network the use of such a catalogue makes no real change to the estimability analysis.

In view of the fact that only one source has been considered as being co-observed by network stations it was found that the treatment of its instantaneous direction with respect to the network adds two new unknown parameters per epoch. In order to exploit the
a priori known relative configuration of sources, three parameters describing the instantaneous orientation of the network with respect to the celestial sphere are needed.

It is therefore preferable to ignore the catalogue information and proceed as in the previous Section 1.1. The results of Table 1 remain unchanged. In the case of co-observing subnetworks discussed in Section 1.2, at least two sources must be co-observed, one from each of two subnetworks. The reason is not to obtain the already known angle between the two sources, but to determine the instantaneous orientation of the network with respect to the celestial sphere. Observation to one source only per epoch would not lead to the determination of the network orientation due to the remaining degree of freedom of rotation around the instantaneous direction of source observed.

In the case of a deformable network the use of an a priori constructed source catalogue has a dear effect. In each epoch only three network orientation parameters are needed instead of the $2h$ source parameters previously considered. The results are modified as follows.

2(e) Observations of delays only, no clock offsets:

Unknowns: $m = 3g - 6 + 3 = 3(g - 3)$
Observations: $n = h(g - 1)$

$$n \geq m \rightarrow h(g - 1) \geq 3(g - 3) \rightarrow$$

$$h \geq 3(g - 3)(g - 1)^{-1}, \quad \text{always } h \geq 3 \quad (2.7)$$

2(f) Observations of delays only with clock offsets:

Unknowns: $m = 3g - 6 + 3 + g - 1 = 4(g - 1)$
Observations: $n = h(g - 1)$

$$n \geq m \rightarrow h(g - 1) \geq 4(g - 1) \rightarrow$$

$$h \geq 4(g - 1)(g - 1)^{-1}, \quad \text{always } h \geq 4 \quad (2.8)$$

2(g) Observations of delays and delay rates, no clock offsets:

Unknowns: $m = 3g - 6 + 3 + 3g - 3 = 6(g - 1)$
Observations: $n = 2h(g - 1)$

$$n \geq m \rightarrow 2h(g - 1) \geq 6(g - 1) \rightarrow$$

$$h \geq 6(g - 1)2^{-1}(g - 1)^{-1}, \quad \text{always } h \geq 3 \quad (2.9)$$

2(h) Observations of delays and delay rates with clock offsets:

Unknowns: $m = 3g - 6 + 3 + 3g - 3 + g - 1 = 7(g - 1)$
Observations: $n = 2h(g - 1)$

$$n \geq m \rightarrow 2h(g - 1) \geq 7(g - 1) \rightarrow$$

$$h \geq 7(g - 1)2^{-1}(g - 1)^{-1}, \quad \text{always } h \geq 4 \quad (2.10)$$

These results replace those of Table 5 while the results of Table 6 become irrelevant: instead we introduce Table 7. Following the developments of Section 2 we are now prepared to analyse a deformable network for a single epoch:
### Table 7. Minimum number of observation epochs $i$ necessary for a network of $g$ stations co-observing in subnetworks with $g_s$ stations at $f$ observational phases of $i_s$ observation epochs each. $h$ is the maximal allowed number of sources and $g_{\text{max}}$ the maximal number of stations that can be determined from the same number of observation epochs $i$, case astrometric parameters *a priori* known.

<table>
<thead>
<tr>
<th>$g$</th>
<th>$g_s$</th>
<th>$i_s$</th>
<th>$f$</th>
<th>$i$ (min)</th>
<th>$h$ (max)</th>
<th>$g$ (max)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>4</td>
<td>12</td>
<td>4</td>
<td>48</td>
<td>25</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>3</td>
<td>18</td>
<td>10</td>
<td>10</td>
<td>14</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>2</td>
<td>10</td>
<td>6</td>
<td>6</td>
<td>15</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>2</td>
<td>8</td>
<td>5</td>
<td>5</td>
<td>18</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>2</td>
<td>8</td>
<td>5</td>
<td>5</td>
<td>21</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>2</td>
<td>8</td>
<td>5</td>
<td>5</td>
<td>24</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>2</td>
<td>8</td>
<td>5</td>
<td>5</td>
<td>27</td>
</tr>
<tr>
<td>11</td>
<td>4</td>
<td>2</td>
<td>8</td>
<td>5</td>
<td>5</td>
<td>30</td>
</tr>
<tr>
<td>&gt;13</td>
<td>3</td>
<td>2</td>
<td>6</td>
<td>4</td>
<td>4</td>
<td>36</td>
</tr>
</tbody>
</table>

#### 2(i) Observations of delays only:

Unknowns: $m = 3g - 7 + 2h + g - 1 + g = 5g - 8 + 2h$

Observations: $n = h(g - 1)$

$n > m \rightarrow h(g - 1) > 5g - 8 + 2h \rightarrow$

$h > (5g - 8 + 2h)(g - 1)^{-1}$, always $h \geq 6$ \hspace{1cm} (2.11)

#### 2(j) Observations of delays and delay rates:

Unknowns: $m = 3g - 7 + 2h + 3g - 3 + g - 1 + g = 8g - 11 + 2h$

Observations: $n = 2h(g - 1)$

$n > m \rightarrow 2h(g - 1) > 8g - 11 + 2h \rightarrow$

$h > (8g - 11 + 2h)2^{-1}(g - 1)^{-1}$, always $h \geq 5$ \hspace{1cm} (2.12)

The results have been included in Table 5. Instead for a rigid network observed at $i$ epochs we arrive at:

#### 2(k) Observations of delays only:

Unknowns: $m = 3g - 7 + 2i + g - 1 + g = 5g - 8 + 2i$

Observations: $n = 2i(g - 1)$

$n > m \rightarrow 2i(g - 1) > 5g - 8 + 2i \rightarrow$

$i > (5g - 8 + 2i)2^{-1}(g - 1)^{-1}$, always $i \geq 3$ \hspace{1cm} (2.13)

These results have been already included in Table 1. The above relations between $g$ and $i$ drive relations between $g_s$ and $i_s$ according to the concept of co-observing subnetworks developed in Section 1.2. Table 4 is modified accordingly and the relevant results for the present case are written into Table 7.

The deformability of the Earth makes the approach not directly applicable. The invariance result of Appendix B indicates that it is impossible to discriminate between change of scale of the network and common secular drift in clocks. Observations in an expanding or shrinking network may also be explained in terms of clock deceleration or
acceleration. One might consider the reading of e.g. one clock (master clock) as providing the time unit and interpret any variation as changes of scale or vice versa.

From the point of view of the number of unknowns in the observational equations there are similarly two choices: either consider absolute clock drifts as estimable \(-g\) parameters for a \(g\)-station network \(-\) and scale now \(-\) estimable \(-3g - 7\) parameters for the shape of the network only; or scale estimable \(-3g - 6\) parameters for both shape and size \(-\) and only relative clock drifts estimable \(-g - 1\) parameters, the same number as for clock offsets.

The modern approach, extended perhaps by relativity theory, is to \textit{a priori} fix the value of light velocity to a certain constant close to its experimental value related to the old concept by a standard ‘metre’ and a standard ‘second’. The unit of time is then chosen by relation to a certain highly stable frequency of oscillation, e.g. atomic time. The definition of length unit follows from the relation between chosen unit of time and light velocity value. This is the concept in geodetic work utilizing laser or Doppler observations. If the Earth were rigid, VLBI would offer us the third obvious alternative. Fixing the length between two far-away stations to a certain value close to its experimental value in terms of the ‘old’ metre, combination with the fixed velocity of light will result in the definition of unit of time without need to resort to any standard clock frequency. Clock readings would then be not only synchronized but also related to the above independently established unit of time. Absolute clock drifts with respect to fictitious clock ‘ticking’ according to the unit of time should be experimentally determined from observations.

3 Estimables in case of clocks with relative drift

The case where clocks lack not only synchronization but they also differ in their standard frequencies deserves special attention. In this case the reading of two clocks differ not only by a relative offset, but also have a relative drift with respect to each other.

Except for the number of additional parameters required, a new problem arises from the relevant invariance analysis of VLBI observations in Appendix B. It is proved there that observations are invariant with respect to a change of scale of the network coupled by an equal change in the rate of clocks. In order to understand the situation better, we have to go back to the well-known interrelation between three fundamental physical quantities, namely length, time and velocity of light. The numerical value of light velocity depends on both unit of length and unit of time. The traditional approach was to select arbitrarily both units by means of standards and determine the value of light velocity experimentally.

We leave the discussion here, more details can be taken from Appendix B. For the problem of estimability we cannot do more but pointing on the fundamental unit problem. There is no observational scheme to overcome it!

Acknowledgment

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References


Appendix A: set-up of the observational model for very long baseline interferometry

Based on the model of a deformable Earth we will set up a model for observations of very long baseline interferometry. Emphasis is on a coordinate-free approach.

Let us start with some explicit physical assumptions: the observations are \textit{a priori} reduced for relativistic effects so that an inertial frame and Newtonian mechanics can be approximately used. Reductions for refraction are \textit{a priori} applied so that we may assume wave propagation in vacuum. Wavefronts are plane in the neighbourhood of the Earth, thus omitting the effect of diurnal parallax, while observations are reduced for annual and other parallaxes. This allows us to refer radio source directions to the origin of the chosen quasi-inertial frame. Changes in the direction to radio sources as seen from the origin of the inertial frame are negligible at least for the time span of a set of geodetic observations.

According to Fig. A1 we introduce the placement diagram ('P'-diagram) of two terrestrial points called topocentres as it can be seen from the inertial origin at two different time instants. Let us call \( x_\alpha, x_\beta \) placement vectors of two points \( \alpha, \beta \) at epoch \( t \), \( X_\alpha, X_\beta \) placement vectors at epoch \( T \). We define \textit{transplacement vectors} \( \delta x_\alpha, \delta x_\beta \) by

\[
\delta x_\alpha = X_\alpha - x_\alpha + \delta x_\alpha,
\delta x_\beta = X_\beta - x_\beta + \delta x_\beta
\]

such that

\[
\delta x_\beta = \delta x_\alpha + (\text{grad } \delta x)(x_\beta - x_\alpha) = \delta x_\alpha + F_\alpha(x_\beta - x_\alpha)
\]

is a \textit{coordinate-free transformation close to the identity}. Let us denote the unit observational vector \( e(\gamma) \) directed from the topocentre to the radio source \( \gamma \). Then we can write the coordinate-free observation of time delay \( \tau \) and of time delay rates \( \tau^*, \tau^{**}, \ldots \) of radio source signals which travel from the topocentre \( x_\alpha \) at epoch \( t \) to the topocentre \( X_\beta \) at epoch \( T = t + \tau \) and scaled by the velocity of light \( c \) with respect to the inertial centre

\[
d_{\alpha\beta} = c\tau = |X_\beta - x_\alpha, e(\gamma)|,
\]

\[
d_{\alpha\beta}^* = c\tau^* = |X_\beta - x_\alpha, e(\gamma)| + |X_\beta - x_\alpha, e^*(\gamma)|,
\]

\[
d_{\alpha\beta}^{**} = c\tau^{**} = |X_\beta^* - x_\alpha^*, e(\gamma)| + 2|X_\beta^* - x_\alpha^*, e^*(\gamma)| + |X_\beta - x_\alpha, e^{**}(\gamma)|,
\]

etc., where \( \langle \cdot, \cdot \rangle \) is the Dirac symbol for the inner or scalar product (bra and ket notation).
The placement difference vector $X_\beta - x_\alpha$ and its velocity can be represented by the transplacement vector and its velocity, namely

$$X_\beta - x_\alpha = x_\beta - x_\alpha + \delta x_\beta = x_\beta - x_\alpha + \delta x_\alpha + F_\alpha (x_\beta - x_\alpha),$$  

(A5)

$$X_\beta^* - x_\alpha^* = x_\beta^* - x_\alpha^* + \delta x_\beta^* = x_\beta^* - x_\alpha^* + F_\alpha^*(x_\beta - x_\alpha) + F_\alpha (x_\beta^* - x_\alpha^*),$$  

(A6)

$$X_\beta^{**} - x_\alpha^{**} = x_\beta^{**} - x_\alpha^{**} + \delta x_\beta^{**} = x_\beta^{**} - x_\alpha^{**} + F_\alpha^{**}(x_\beta - x_\alpha) + 2F_\alpha (x_\beta^* - x_\alpha^*) + F_\alpha (x_\beta^{**} - x_\alpha^{**}),$$  

(A7)

etc., if the transplacement vectors transform close to the identity. There are two classical decompositions of the operator $F$. The first one, called the Cartesian decomposition, is based on

$$F = S + A$$  

(A8)

such that $S$ is a symmetric operator, $S = S^T$, $A$ an antisymmetric operator, $A = -A^T$. The second one, called polar decomposition, introduces right and left stretch operators $S_r, S_l$ and the rotation operator $R$ such that

$$F = R S_r = S_l R,$$

(A9)

where $S_r = S_r^T$ and $S_l = S_l^T$. Equations (A5)–(A7) can now be written

$$X_\beta - x_\alpha = \delta x_\alpha + (I + S + A)(x_\beta - x_\alpha),$$  

(A10)

$$X_\beta^* - x_\alpha^* = \delta x_\alpha^* + (I + S + A)(x_\beta^* - x_\alpha^*) + (S^* + A^*)(x_\beta - x_\alpha),$$  

(A11)

$$X_\beta^{**} - x_\alpha^{**} = \delta x_\alpha^{**} + (I + S + A)(x_\beta^{**} - x_\alpha^{**}) + 2(S^* + A^*)(x_\beta^* - x_\alpha^*) + S^{***} + A^{***}$$

$$\times (x_\beta^* - x_\alpha^*),$$  

(A12)

e tc., or

$$X_\beta - x_\alpha = \delta x_\alpha + (I + R S_r)(x_\beta - x_\alpha) = \delta x_\alpha + (I + S_l R)(x_\beta - x_\alpha),$$  

(A13)

$$X_\beta^* - x_\alpha^* = \delta x_\alpha^* + (R^* S_r + R S_l^*)(x_\beta - x_\alpha) + (I + R S_r)(x_\beta^* - x_\alpha^*),$$

$$= \delta x_\alpha^* + (S_l^* R + S_l R^*)(x_\beta - x_\alpha) + (I + S_l R)(x_\beta^* - x_\alpha^*),$$  

(A14)
VLBI observations

\[ X_{\beta}^\prime - x_{\alpha}^\prime = \delta x_{\alpha}^\prime + (R^\prime \cdot S_{\beta} + 2R^* \cdot S_{\beta}^* + RS_{\beta}^*) (x_{\beta} - x_{\alpha}) + (I + RS_{\beta} + R^* S_{\beta} + RS_{\beta}^*) \]
\[ \times (x_{\beta} - x_{\alpha}) + (I + RS_{\beta}) (x_{\beta}^\prime - x_{\alpha}^\prime) = \delta x_{\alpha}^\prime + (S_{\beta}^\prime \cdot R + 2S_{\beta}^* \cdot R^* + S_{\beta} R^*) \]
\[ \times (x_{\beta} - x_{\alpha}) + (I + S_{\beta} R + S_{\beta}^* R + S_{\beta}^* R^*) (x_{\beta}^\prime - x_{\alpha}^\prime) + (I + S_{\beta} R) (x_{\beta}^\prime - x_{\alpha}^\prime), \quad (A15) \]

etc. \( S_{\beta}^* \) and \( S_{\beta}^\prime \) are called stretch rates, \( R^* \) spin (if \( R^* = 0 \) the transformation is called "irrotational", otherwise rotational, \( \sqrt{2|R|} = |\text{rot } x^*| \) is called vorticity) \( \text{tr } S_{\beta}^* = \text{div } x^* \); if \( S_{\beta}^* = 0 \) the transformation is called isonoric. \( C = S_{\beta}^2, B = S_{\beta}^4 \) will be called right and left deformation of Cauchy—Green type.

In addition, let us assume that locally we observe time \( s \) with respect to fictitious Newtonian time \( t \), namely
\[ s_{\alpha}(t) = (1 + \delta a_{\alpha}) t + b_{\alpha}, \quad s_{\beta}(t) = (1 + \delta a_{\beta}) t + b_{\beta}, \quad (A16) \]
where \( b_{\alpha}, b_{\beta} \) are called time offset, \( a_{\alpha} = 1 + \delta a_{\alpha}, a_{\beta} = 1 + \delta a_{\beta} \) frequency drift. Now we are prepared to transplant (A13)–(A15) into the characteristic observation (A2)–(A4) such that
\[ d\gamma = c(s_{\alpha} - s_{\alpha}) = cT + c(b_{\beta} - b_{\alpha}) + (\delta a_{\beta} - \delta a_{\alpha}) + c\delta a_{\beta}T', \quad (A17) \]
\[ d\gamma = c(x_{\beta} - x_{\alpha}, e(\gamma)) (1 + \delta a_{\beta}) + (A_{\alpha} (x_{\beta} - x_{\alpha}), e(\gamma)) (1 + \delta a_{\beta}) \]
\[ + c(x_{\beta} - x_{\alpha}, e(\gamma)) (1 + \delta a_{\beta}) + (S_{\alpha} (x_{\beta} - x_{\alpha}), e(\gamma)) (1 + \delta a_{\beta}) \]
\[ + c(\delta a_{\beta} - \delta a_{\alpha}), \quad (A18) \]
\[ d\gamma = (x_{\beta} - x_{\alpha}, e(\gamma)) (1 + \delta a_{\beta}) + (\text{RS}_{\beta})_{\alpha} (x_{\beta} - x_{\alpha}), e(\gamma)) (1 + \delta a_{\beta}) \]
\[ + d\gamma = (x_{\beta} - x_{\alpha}, e(\gamma)) (1 + \delta a_{\beta}) + (c\delta a_{\beta} - \delta a_{\alpha}), \quad (A19) \]

and
\[ d\gamma = c(s_{\beta} - s_{\beta}) = c(1 + \delta a_{\beta}) T^* + c(\delta a_{\beta} - \delta a_{\alpha}), \quad (A20) \]
\[ d\gamma = d/dt (X_{\beta} (s_{\alpha} + T_s) - x_{\alpha}(s_{\alpha})), e(\gamma)) = (X_{\beta} (t + \tau) - x_{\alpha}(t), e(\gamma)) (1 + \delta a_{\beta}) \]
\[ + (S_{\alpha} (x_{\beta} - x_{\alpha}), e(\gamma)) (1 + \delta a_{\beta}) + (\delta x_{\alpha}, e(\gamma)) (1 + \delta a_{\beta}) \]
\[ + c(\delta a_{\beta} - \delta a_{\alpha})T, \quad (A21) \]
\[ d\gamma = (x_{\beta} - x_{\alpha}, e(\gamma)) (1 + \delta a_{\beta}) + (\text{RS}_{\beta})_{\alpha} (x_{\beta} - x_{\alpha}), e(\gamma)) (1 + \delta a_{\beta}) \]
\[ + d\gamma = (x_{\beta} - x_{\alpha}, e(\gamma)) (1 + \delta a_{\beta}) + (c\delta a_{\beta} - \delta a_{\alpha}), \quad (A22) \]

The observation equations can be interpreted as follows: for instance, in (A19) the first term contains the station-to-station baseline vector, the second one the baseline retardation due to a rotation, the third one baseline deformation, the fourth one baseline aberration due to a
translation, finally the fifth one time offset and the sixth one frequency drift. In contrast, (A24) includes the relative baseline velocity vector within the first term, the stretched and rotated relative baseline velocity vector in the second one, spin and stretch rate in the third one, translational velocity in the fourth one, finally relative drift rate of the clocks in the fifth one and the absolute change of the source direction in the sixth one.

Let us present a numerical analysis of terms within (A19). If we assume a value of velocity of light approximately $3 \times 10^8$ m s$^{-1}$, a baseline length of approximately $6.371 \times 10^6$ m, the approximate radius $r$ of the Earth, we will find a time delay of approximately $0.022$ s = 22 ms. In addition let us introduce a rotational speed of the Earth $\omega = \pi/12 \times 3600$ s$^{-1}$ approximately. Then the maximum baseline retardation at the equator for a baseline projection onto the source direction is about $\omega r \tau = \pi (12 \times 3600)^{-1} \times 0.022 \times 6.371 \times 10^6 \approx 10$ m. Baseline aberration cannot be disregarded with respect to its magnitude. A reader being familiar with essentials of special relativity theory will immediately realize that this effect is the well-known 'ether wind' of pre-relativistic Newtonian mechanics. It shows up in the observational model by considering velocity of light to be constant in an inertial frame. If we use a quasi-Lorentzian frame following the Earth in its almost linear path during the small observational interval, we know from special relativity that velocity of light remains unchanged for such a frame. Therefore common translation with respect to the quasi-Lorentzian frame is zero and so is the baseline aberration effect. For more details we refer, for instance, to Bergmann (1942).

The term-by-term analysis proves that only a simplified model for estimability problems is needed. Let us finally present an observation model for Sections 1, 2 and 3. Besides Appendix B we use

$$d_{\alpha \beta \gamma}^k = \langle x_\beta - x_\alpha, e(\gamma) \rangle + c (b_\beta - b_\alpha) + c (\delta a_\beta - \delta a_\alpha) t, \quad (A25)$$

$$d_{\alpha \beta \gamma}^k = \langle x_\beta^* - x_\alpha^*, e(\gamma) \rangle + \langle A_\alpha^* (x_\beta - x_\alpha), e(\gamma) \rangle + c (\delta a_\beta - \delta a_\alpha), \quad (A26)$$

**Figure A2. (a) Base vector diagram ('E'-diagram).**

- $t$ reduced vector $x_\beta - x_\alpha$, $\omega$ global vorticity or the rotation vector, $\Delta = \omega + \delta \omega$ local vorticity vector, $\psi$ e.g. ecliptic normal vector, $Y_Gr$ gravity vector at Greenwich, norm $t = t / \| t \|$. 
- $e_s = \text{norm} (\omega)$
- $e_s^* = \text{norm} (\psi \times \omega)$
- $e_s = e_s \times e_s$
- $f_s = \text{norm} (\omega)$
- $f_s^* = \text{norm} (Y_Gr \times \omega)$
- $f_s = f_s \times f_s$
- $f_s^* = \text{norm} (\omega \times t)$
- $f_s^* = f_s^* \times f_s^*$. 

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where $A_\alpha$ is taken as uniform rotation of the model Earth, thus neglecting local rotation (vorticity) within the observation interval. All the other terms are transferred to the model of reductions.

Finally let us introduce coordinates that is a system of reference frames according to Fig. A2 within the orthonormal base vector diagram ('E-diagram') following Grafarend

$$
\begin{align*}
\mathbf{e}_1 &= \mathbf{e}^* \\
\mathbf{e}_2 &= \mathbf{e}_* \\
\mathbf{e}_3 &= \mathbf{e}_+ \\
\mathbf{e}_+ &= \mathbf{e}_2 \times \mathbf{e}_3 \\
\mathbf{E}_+ &= \text{norm} (\mathbf{e}_+ ) \\
\mathbf{E}_2 &= \text{norm} (\psi_1 \times \mathbf{A}_1 ) \\
\mathbf{E}_3 &= \text{norm} (\mathbf{A}_1 ) \\
\mathbf{E}_+ &= \text{norm} (\mathbf{e}_+ \times \mathbf{A}_1 ) \\
\mathbf{E}_* &= \text{norm} (\psi_1 \times \mathbf{A}_1 ) \\
\mathbf{E}_* &= \mathbf{E}_+ \times \mathbf{E}_+ \\
R_C(\alpha, \beta, \gamma) &= R_3(\gamma) R_2(\beta) R_1(\alpha).
\end{align*}
$$
\( f' = R_E(\alpha - 90^\circ, \delta)[I + \delta_t R(\delta_t \alpha, \delta_t \delta)]e^*, \)  

where \( \alpha, \delta \) are conventional right ascension, declination at epoch \( t_0 \), \( \alpha + \delta_t \alpha, \delta + \delta_t \delta \) the corresponding angular parameters at epoch \( t \) under time-like variations. In general \( \delta R_E, \delta R_C \) are infinitesimal anti-symmetric rotation matrices for Eulerian, Cardanian angles, respectively. The time-like variations \( \delta_t \alpha, \delta_t \delta \) are caused by precession and nutation. With respect to the local equatorial frame \( E' \) the global equatorial frame \( e^* \) is related by plate rotations or an inhomogeneous vorticity field materialized within \( \delta_3 R_C \), namely

\[
e'^* \rightarrow E^*' = R_C(M^*)e'^* = R_C(M^*)[I + \delta_3 R_C]e^*. \tag{A28}
\]

Space- and Earth-fixed frames are related by

\[
e^* \rightarrow f^* = R_3(\theta_{Gr} - 90^\circ)e^*, \tag{A29}
\]

where \( \theta_{Gr} \) is the conventional Greenwich apparent sidereal time angle. Terrestrial coordinates in the instantaneous Earth-fixed \( f^* \)-frame are conventionally related to an Earth-fixed \( f^* \)-frame at some reference epoch \( t_0 \) according to

\[
f_* \rightarrow f^* = R_C(\eta_*)f_* = (I + \delta_t R_C)f. \tag{A30}
\]

such that \( \delta_t R_C \) is caused by polar motion. While we have expressed the observation direction \( f_3* = e_3* = e(\gamma) \) of (A27) in terms of the 'fixed' frame \( e^* \), we have to apply a similar process for \( f_* \) in order to arrive at

\[
f_* = (I - \delta_t R_C)R_3(\theta_{Gr} - 90^\circ)e^*. \tag{A31}
\]
Now we are prepared to compute as an example
\[
\langle x_\beta - x_\alpha, e(\gamma) \rangle = (x_\beta^t - x_\alpha^t)(f_1, f_3) = (x_\beta - x_\alpha)^T f_1 f_3 p_3,
\]
where \( p_3 = (0, 0, 1) \) such that \( f_1 p_3 = f_3 = e(\gamma) \). \((x_\beta^t - x_\alpha^t)\) are the coordinate differences of the points \( \alpha \) and \( \beta \) in the Earth-fixed reference frame \( f \), at reference epoch \( t_0 \). Introducing (A27) and (A31) the inner product (A32) can be represented by
\[
\langle x_\beta - x_\alpha, e(\gamma) \rangle = (x_\beta - x_\alpha)(I + \delta R)R_3(\theta_G t - 90^\circ)R_3^T(M,)
\times [I - \delta R_E(\delta_4, \delta_5, \delta_6)]R_3^T(\alpha - 90^\circ, \delta) p_3.
\]
In a first order approximation of (A25), this is the observational equation for time delay (up to the velocity of light \( c \)) including the effects of polar motion and precession–nutation and indirectly of plate rotation.

**Appendix B: invariance characteristics of very long baseline observations**

Grafarend & Schaffrin (1974, 1976) have shown the interrelation between estimability of parameters and the groups of coordinate transformations leaving geodetic observations invariant. We shall examine the invariance of VLBI observations under a coordinate transformation in four dimensions or in space-time. We work with the simplified model
\[
T - t = c^{-1} \langle x_\beta - x_\alpha, e(\gamma) \rangle,
\]
where \( t, T \) are the epochs of signal arrival at the two antennae, \( c \) the velocity of the electromagnetic signal, \( e(\gamma) \) the unit vector in source direction and \( x_\beta - x_\alpha \) the station-to-station baseline vector. The two vectors \( e(\gamma) \) and \( x_\beta - x_\alpha \) must refer to the same frame and \( \langle , \rangle \) denotes the standard Euclidean inner product in \( \mathbb{R}^3 \).

Within the framework of Newtonian mechanics only the Galilean group of space-time transformation is allowed, especially
\[
\begin{bmatrix} x' \\ t' \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} + \begin{bmatrix} x_0 \\ t_0 \end{bmatrix},
\]
where \( \{x, t\} \) are the old, \( \{x', t'\} \) the new coordinates, respectively, \( R \) the 3 x 3 orthogonal or proper rotation matrix (det \( R = 1 \)) and \( \{x_0, t_0\} \) are fixed. Within the Galilean group space coordinates transform according to the group of rigid motions \( x \to x' = Rx + x_0 \) and independently of time coordinates which transform only by shifts \( t \to t' = t + t_0 \), i.e. by changes in origin.

We shall investigate invariance of observations under transformations which are actually somewhat richer than those of the Galilean group, without essentially departing from the limitations of Newtonian mechanics. The reasons for such a generalization are as follows: equations (B2) may be interpreted in two ways, first as a transformation of coordinates of one and the same event (place and epoch) due to a change of reference frame, second as a change from one event to another, i.e. a motion in space-time changing both place and epoch but keeping the reference frame fixed. The Galilean group is exactly the group of such motions which are admissible within the framework of Newtonian mechanics. However, the restriction relies on the \textit{a priori} metrization of space-time, i.e. the \textit{a priori} choice of the unit of length and the unit of time. According to this concept the velocity of light is a frame-dependent parameter whose value should be determined by experiment.

The theory of relativity gave the velocity of light the character of an absolute, frame-independent physical constant. Since we rely here on Newtonian mechanics — the concept
of a quasi-Lorentzian frame — two alternative approaches are suggested for metrization of space-time, both relying on the frame-indifference of the velocity of light. The alternative most familiar to the geodesist is the definition of the unit of time. The unit of length is a quantity derived from the fixed velocity of light and the unit of time. This strategy is common practice in geodesy, especially in the field of electromagnetic distance measurements, laser tracking of satellites etc. Obviously the other alternative is the a priori definition of length. The unit of time is derived from the fixed velocity of light and the unit of length.

Both alternatives have their shortcomings. The first relies on the existence of a perfectly stable clock whose recordings will provide the time unit, while the second relies on the existence of a perfectly invariant length within the observing network, i.e., in essence, a rigid network. The result is — as is proven later — that in the absence of both perfect clocks and rigid networks, changes of scale in length cannot be separated from changes in scale of time by means of VLBI observations.

We shall investigate the invariance of VLBI observations under the group of transformations

\[
\begin{bmatrix}
  x' \\
  t'
\end{bmatrix} = \begin{bmatrix} sR & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix}
  x \\
  t
\end{bmatrix} + \begin{bmatrix}
  x_0 \\
  t_0
\end{bmatrix},
\]

where \( s \) is the scale factor for lengths and \( a \) the one for time. The transformations are admissible due to the change of reference frames, not only under space-time motions.

In our model there are other parameters which in addition implicitly depend on coordinates. These are the clock offsets and drifts.

Let the observations \( z \) depend on a set of parameters \( x \), i.e., in functional form \( z = f(x) \). Invariance of the observational functional under the transformation of parameter space, i.e., \( x \rightarrow x' = h(x) \), is postulated by

\[
f(x) = f(x') \iff f(x) = (f \circ h)(x),
\]

or

\[
f = f \circ h.
\]

The invariance postulate is illustrated in Fig. B1 where id denotes 'identity'. In our case observations \( z \) depend on two sets of parameters, namely coordinates of events denoted by \( x \) and coordinate dependent clock parameters \( y \), in terms of functional forms \( z = f(x, y) \),

![Figure B1. Commutative diagram of the invariance postulate of a one-parameter-set observational functional.](image-url)
Figure B2. Commutative diagram of the invariance postulate of a two-parameter-set observational functional.

\[ y = g(x). \] Invariance of the observational functional under the transformation of parameter spaces, i.e. \( x \rightarrow x' = h(x) \), \( y \rightarrow y' = g(x') = g[h(x)] = goh(x) \) is postulated by

\[ f(x, y) = f(x', y') \] (B5, i)

If \( f[x, g(*)] \) denotes the mapping \( x \rightarrow f[x, g(x)] \) we arrive at

\[ f[x, g(*)] = f[x, g(*)]oh \] (B5, ii)

illustrated in Fig. B2.

In order to examine the invariance of the VLBI observational functional under space-time transformations of type

\[ x = sRx' + x_0, \]
\[ t = \alpha t' + t_0, \] (B6)

we consider the clock reading \( s_\alpha \) at the \( \alpha \)th station which is related to fictitious Newtonian time \( t \) by

\[ s_\alpha(t) = a_\alpha t + b_\alpha, \] (B7)

(see equation A16) where \( a_\alpha, b_\beta \) are constants. The actually observed time delay of two stations co-observing will be rigorously approximated (see equation A25)

\[ s_\beta - s_\alpha = t_\beta - t_\alpha + a_\beta(t_\beta - t_\alpha) + (a_\beta - a_\alpha)t_\alpha. \] (B8)

The actual time delay \( t_\beta - t_\alpha = \tau \) is related to the instantaneous geometry of the approximated baseline vector \( x_\beta - x_\alpha \) and the radio source direction \( e(\gamma) \) by

\[ \tau = c^{-1}a_\alpha(x_\beta - x_\alpha, e(\gamma)) + b_\beta - b_\alpha + (a_\beta - a_\alpha)t_\alpha. \] (B9)

Let us denote by \( \tau' \) the same time delay in terms of space-time coordinates \( (x', t') \) related to the space-time coordinates \( (x, t) \) by (B6), that is

\[ \tau' = c^{-1}a'_\alpha(x'_\beta - x'_\alpha, e'(\gamma)) + (b'_\beta - b'_\alpha) + (a'_\beta - a'_\alpha)t'_\alpha. \] (B10)

In order to find the relation between the parameters \( a_\alpha, b_\alpha \) and \( a'_\alpha, b'_\alpha \) we note that clock readings have fixed numerical values which are independent of the time frame used and that
the effect of change of frame must be accommodated by a change of these parameters. (B7) has to be written

\[ s_\alpha = a_\alpha t + b_\alpha = a'_\alpha t' + b'_\alpha, \]  

(B11, i)

\[ a'_\alpha t' + b'_\alpha = (a_\alpha t_0 + b_\alpha) + a_\alpha \alpha t', \]  

(B11, ii)

\[ a'_\alpha = \alpha a_\alpha, b'_\alpha = a_\alpha t_0 + b_\alpha. \]  

(B11, iii)

Since the baseline \( x_\beta - x_\alpha \) transforms according to \( x_\beta - x_\alpha = s R (x'_\beta - x'_\alpha) \) while the direction of the radio source is only affected by \( R \), namely \( e = Re' \), we find for

\[ \langle x_\beta - x_\alpha, e(\gamma) \rangle = \langle s R (x'_\beta - x'_\alpha), Re' \rangle = s \langle R (x'_\beta - x'_\alpha), Re' \rangle = s \langle x'_\beta - x'_\alpha, e' \rangle \]

since the inner product is invariant under orthogonal transformations. Referring to (B11) and the inverse relation \( t'_\alpha = (t_\alpha - t_0)\alpha^{-1} \) we rewrite (B10)

\[ \tau' = c^{-1} a_\beta s^{-1} (x_\beta - x_\alpha, e(\gamma)) + (b_\beta - b_\alpha) + (a_\beta - a_\alpha) t_\alpha. \]  

(B12)

A necessary and sufficient condition for \( \tau = \tau' \) is \( \alpha = s \), i.e. a change of length unit by a scale factor \( s \) must be compensated by a change of time unit by the same scale factor: VLBI time delay observations cannot distinguish between secular changes of network size — expansion or shrinking — and a corresponding common secular drift — acceleration or retardation — of the system of clocks used.

The case of observation of time delay derivatives is similar. Differentiation with respect to time occurs with a third time frame not coinciding with either \( t \) or \( t' \), and is denoted by an upper dot. Considering the inner product in a frame where the source directions are fixed we derive

\[ \tau'^* = c^{-1} a_\beta (x_\beta - x_\alpha)^*, e(\gamma)) + (a_\beta - a_\alpha) t'^*, \]  

(B13)

\[ \tau'^{*'} = c^{-1} a'_\beta (x'_\beta - x'_\alpha)^*, e'(\gamma)) + (a'_\beta - a'_\alpha) t'^{*'}, \]  

(B14)

Differentiation of (B6) yields \( x^* = s R x'^* \) and \( \tau^* = \alpha \tau'^* \). From \( (x_\beta - x_\alpha)^* = s R (x'_\beta - x'_\alpha)^* \) and \( e = Re' \) we derive

\[ \langle (x_\beta - x_\alpha)^*, e(\gamma) \rangle = \langle s R (x'_\beta - x'_\alpha)^*, Re' \rangle = s \langle (x'_\beta - x'_\alpha)^*, e'(\gamma) \rangle. \]  

(B15)

Referring to (B11) and \( t'^* = t'^*/\alpha \) we obtain for (B14)

\[ \tau'^* = c^{-1} a_\beta s^{-1} (x_\beta - x_\alpha)^*, e(\gamma)) + (a_\beta - a_\alpha) t'^*, \]  

(B16)

A necessary and sufficient condition for the invariance of the observational functional, namely \( \tau^* = \tau'^* \) is \( \alpha = s \).

In summarizing we have shown that both time delay and time delay derivative VLBI observations are invariant under space-time coordinate transformations of the form

\[
\begin{bmatrix}
  x \\
  t
\end{bmatrix} =
\begin{bmatrix}
  sR & 0 \\
  0 & s
\end{bmatrix}
\begin{bmatrix}
  x' \\
  t'
\end{bmatrix} +
\begin{bmatrix}
  x_0 \\
  t_0
\end{bmatrix}.
\]  

(B17)