Strain-type representation of the potential anomalies, with an example for the eastern Mediterranean

A. Dermanis, A. Filaretou, E. Livieratos, and I. N. Tziavos

Department of Geodesy and Surveying, University of Thessaloniki, Univ. Box 474, 54006 Thessaloniki, Greece

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Summary. The use of the geoidal representation of the potential anomalies is not only the basis for a variety of subjects for study in geodesy but also a traditional tool of research in geophysics. In this second field of geosciences, the geoid gives also hints for discussion, e.g., on the internal density distribution of the earth, on the convection hypotheses, crust faultings, etc. In this paper we present a different way of representing the potential anomalies instead of their classical geoidal form. Applying differential geometry methods we express the potential anomalies in terms of surface strain dilatation and surface maximum shear. The term strain here refers to the geometry of a non-material surface, i.e., model surface of potential anomalies. Using a specific example concerning the eastern Mediterranean area we demonstrate how the strain representation of the potential anomalies gives some sharper evidences than the geoidal counterpart.

Introduction

It is a common practice in geosciences to use a traditional geodetic feature, the geoidal representation of the potential anomalies, in the discussion of various geodynamical hypotheses and interpretations. For example, the interpretations relating higher geoidal heights to higher material densities in the interior of the earth and lower geoidal heights to lower densities; the convection hypotheses which use geoidal information for investigations on the behaviour of convection currents in the earth's mantle, etc. (see, e.g., Runcorn 1967; Turcotte and Schubert 1982; Gadomska and Teisseyre 1984).

The classical geometric depiction of the geoid is that of an abstract potential surface, \( W = W(\Lambda, \Phi) \), separated from a model potential surface \( U = U(\lambda^*, \varphi^*) \) along normals from points \( (\Lambda, \Phi) \) on the geoid to points \( (\lambda^*, \varphi^*) \) on the model surface. The length of the normals from the geoidal surface to the model surface are the well-known geoidal heights \( \zeta \). The \( \Lambda, \Phi \) are non-orthogonal curvilinear coordinates (astronomic longitude and latitude) parametrizing the geoidal surface; the \( \lambda, \varphi \) are orthogonal curvilinear coordinates (model longitude and latitude) parametrizing the model surface. A standard technique for the computation of \( \zeta \) is based on the spherical harmonics expansion of the anomalous potential \( W - U \), which divided by scalar gravity gives length, i.e., the geoidal heights.

If we like to see a material equivalence of the above geometric description we can realize a model curvilinear membrane with known material properties. This elastic membrane, in its undeformed (unstrained) state, corresponds to the model potential surface, and its strained counterpart, in a somehow deformed state, corresponds to the geoidal surface. A material point on the unstrained elastic membrane is "mapped" on its strained counterpart inducing a displacement in space in a similar way a nor material point \( (\lambda^*, \varphi^*) \), on the model potential surface, is separated in space from the point \( (\Lambda, \Phi) \), on the geoidal surface, in terms of the geoidal height \( \zeta \). If the strain induced, during the deformation process of the elastic membrane, is known then the strain energy dissipated during the process can be computed.

A consequent idea which can be applied in representing anomalous potential surfaces, i.e., the geoid, is the use of strain analysis, which indeed can be used in a variety of geodetic problems (Dermanis and Livieratos 1983a), as well as in cartographic representations (Dermanis and Livieratos 1983b). Looking for an application of the case of material membrane deformation into its conceptually equivalent case of the geoidal surface representation, two main shortcomings have to

Correspondence to: I. N. Tziavos
be faced: first, the difficulty to define a one-to-one correspondence between the unknown points on the geoidal surface and their counterparts on the model potential surface and second, the abstract (non-material) nature of the potential surfaces involved. The first problem is solved by using well-known geodic mappings, i.e., the mapping along the normal to the model surface (Helmert projection) and the isoparametric mapping, widely used in treating intrinsic geodic problems (see, e.g., Marussi 1974; Grafarend 1978; Bocchio 1979; Danas and Dermanis 1983; Dermanis et al. 1983; Bakker 1985; Dermanis and Livieratos 1984; Amalvict and Livieratos 1988). The second problem can be only simulated by assigning, to the model surface, test elastic parameters related, e.g., to ideal Poisson material (Lamé parameters are unity) or to indicative elastic parameters, similar to those of the earth’s crust.

The surface strain-type representation of the potential anomalies is done, using tools from differential geometry and deformation analysis, by comparing metric tensor differences of the geoidal surface and its model (normal) counterpart bringing into a one-to-one correspondence positions on the geoid with those on the reference surface. From the surface metric tensor differences we obtain the strain tensor which is a function of scalar gravity, curvatures and torsion computed at geoidal points. From the strain tensor, invariant quantities are deduced, i.e., areal dilatation and maximum shear, which are more sensitive to the detailed structure of the geoid than the geoid itself, since they are functions of the second derivatives of W. The above procedure has been tested for global scale solutions (Livieratos 1987; 1991), resulting good agreement between the strain features of the geoid and areas of major geodynamic interest.

In this paper, we present a numerical strain elaboration of the potential anomalies in an area of regional scale (eastern Mediterranean). The strain induced by the deformation of the model potential surface into its geoidal counterpart is represented in terms of a dilatation (isotropic) strain map and a maximum shear (anisotropic) strain map. The strain maps derived here show an interesting correspondence between the zero areal dilatation zones of the geoid and the tectonic faults at the area.

Underlying theory and computations

Metric and strain tensors

Let two surface curvilinear coordinate systems

\[
\begin{bmatrix}
  u^1 \\
  u^2
\end{bmatrix}
\]

\[
\begin{bmatrix}
  v^1 \\
  v^2
\end{bmatrix}
\]

assigned to two surfaces with metric tensors \( G_u \) and \( G_v \), respectively associated to the coordinates chosen. Assume a one-to-one point-wise correspondence between the two surfaces

\( \mathbf{v} = \mathbf{v}(\mathbf{u}) \)

which should satisfy the conditions of a homeomorphism. Then, the strain tensor \( \mathbf{S} \) is given by (Dermanis and Livieratos 1983a)

\[
\mathbf{S}' = \frac{1}{2} [G_v^{-1} \left( \frac{\partial \mathbf{u}}{\partial v} \right)^T G_u \frac{\partial \mathbf{u}}{\partial v}] \quad (\text{Eulerian formulation})
\]

The dimensionless version of (3) concerning the strain components (unitless strain components) is given by

\[
\mathbf{S} = K \mathbf{S}' K
\]

where \( K \) is the diagonal matrix with elements corresponding to the curvatures of the coordinate lines. This is important when the coordinates (usual practice in geodesy) are not of the same nature (e.g., angular, linear). Since we deal with two dimensional spaces, namely the geoidal surface and the reference rotational ellipsoidal surface, the strain tensor is represented by a 2x2 symmetric matrix

\[
\mathbf{S} = \begin{bmatrix}
  S_{11} & S_{12} \\
  S_{12} & S_{22}
\end{bmatrix}
\]

from which the principal invariant quantities \( I_1(\mathbf{S}) \) and \( I_2(\mathbf{S}) \) are computed

\[
I_1(\mathbf{S}) = \text{tr}(\mathbf{S})
\]

\[
I_2(\mathbf{S}) = \text{det}(\mathbf{S})
\]

and the required strain parameters, at the vicinity of the points, are obtained as functions of \( I_1(\mathbf{S}) \) and \( I_2(\mathbf{S}) \). The strain parameters used here are the invariants

\[
\Delta = I_1(\mathbf{S}) = \text{tr}(\mathbf{S}) = S_{11} + S_{22}
\]

\[
\gamma^2 = I_1(\mathbf{S})^2 - 4 I_2(\mathbf{S}) = \text{tr}(\mathbf{S})^2 - 4 \text{det}(\mathbf{S}) = (S_{11} - S_{22})^2 + (2 S_{12})^2
\]
in which $\Delta$ represents the isotropic part of deformation, the areal change per unit area (positive for expansion, negative for compression) and $\gamma$ is the anisotropic or directional part of deformation, the shear across the direction of its maximum value (always positive).

The metric tensor of the geoid

Parametrizing the geoidal surface ($W=\text{const.}$) with the curvilinear, non orthogonal coordinates

\[
(9) \quad u = \left[ \begin{array}{c} \Lambda \\ \Phi \end{array} \right]
\]

where $\Lambda$, $\Phi$ the astronomical longitude and latitude, respectively, obtain the metric tensor of the geoid (Derimanis and Livieratos 1983a)

\[
(10) \quad G_u = \frac{1}{k_G^2} \begin{bmatrix} (k_N^2 + t_E^2) \cos^2 \Phi & -(k_E + k_N) t_E \cos \Phi \\ -(k_E + k_N) t_E \cos \Phi & k_E^2 + t_E^2 \end{bmatrix}
\]

where $k_E$, $k_N$ are the curvatures of the geoidal surface, $W=\text{const.}$, to the astronomic east and north directions respectively, $t_E$ is the torsion to the astronomic east direction, $k_G$ is the Gauss curvature (function of the first and second fundamental forms of the geoidal surface)

\[
(11) \quad k_G = k_E k_N - t_E^2
\]

and $W$ is the constant potential of the geoidal surface. It is known from differential geodesy (Marussi 1949) that

\[
(12.1) \quad k_E = \frac{1}{g} \frac{\partial^2 W}{\partial E^2} \\
(12.2) \quad k_N = \frac{1}{g} \frac{\partial^2 W}{\partial N^2} \\
(12.3) \quad t_E = \frac{1}{g} \frac{\partial^2 W}{\partial E \partial N}
\]

where $g$ is the gravity at the geoidal points. The curvatures and torsion of the geoid are sensitive to location of anomalous masses, since they are functions of the horizontal derivatives of the disturbing potential.

The second derivatives of $W$ with respect to the astronomic east and north directions are simply the gravity gradients of the geoidal surface to the astronomic east and north directions. Gauss and mean curvatures of the geoid have been first treated by Burša (1973) analyzing satellite orbit dynamics.

The metric tensor of the reference rotational ellipsoid surface, $U=\text{const.}$, parametrized by the curvilinear orthogonal coordinates

\[
(13) \quad v = \left[ \begin{array}{c} \lambda \\ \varphi \end{array} \right]
\]

where $\lambda$, $\varphi$ are the model (ellipsoidal) longitude and latitude, respectively, and is diagonal having the form

\[
(14) \quad G_v = M^2 N^2 \begin{bmatrix} \frac{\cos^2 \varphi}{M^2} & 0 \\ 0 & \frac{1}{N^2} \end{bmatrix} = \begin{bmatrix} N^2 \cos^2 \varphi & 0 \\ 0 & M^2 \end{bmatrix}
\]

where $M$, $N$ are the radii of curvature of the meridian section and of the prime normal section of the ellipsoid, respectively.

Mapping and strain tensor

For the computation of the strain tensor at points $P$ on the geoid, a mapping (2) should be introduced. As a matter of fact the mapping defines the deformation with respect to two, out of three, dimensions. The critical question here is not the mapping itself but what deformation, related to the mapping, is of main interest. The mappings which are mostly used in geodesy are two: the Helmert projection, along the normal, of geoidal points onto the model surface and the isoparametric mapping $\lambda=\Lambda$, $\varphi=\Phi$, $U=W$. The first, is of pure geometric character; the second, has a direct natural meaning, since it is related to Marussi's natural coordinates ($\Lambda, \Phi, W$). We have always to keep in mind that the geoid is an abstract concept and what we are really looking for, is the deformation of the gravity field. This deformation should be described by using natural coordinates particularly because the already selected coordinate $W$ vs. $U$ is a natural coordinate. Actually, this natural coordinate reduces the three dimensions into two dimensions through the concept of the equipotential surfaces (geoid vs ellipsoid).

The Helmert mapping is indeed very simple but has no physical meaning, with respect to the two coordinates involved since it is defined geometrically with no dependence from the gravity field. Actually, this is the reason why, when using the Helmert mapping, we derive the original geoidal map depicted as dilatation only with negligible shears (Derimanis et al. 1983; Livieratos 1987), by simply multiplying geoidal heights by a factor of $2/R$. 
Computation of intrinsic parameters

Though model coordinates $\lambda^o, \varphi^o$ at $Q^o$, on the model surface, can be postulated, the relevant coordinates on the geoid $\Lambda, \Phi$ at $P$, are unknown (Fig.1). In order to define the points on the geoid, where the intrinsic quantities $\xi_N, \xi_E, \xi_T$, should be computed, as well as the scalar gravity $g$, we have to know the surface point position $\Lambda, \Phi$. Defined the $\Lambda, \Phi$ coordinates, a one-to-one correspondence with points $\lambda, \varphi$ on the model surface can be established.

Various algorithms were developed for the computation of the curvatures and torsion mainly with respect to the geocentric frame. Here, we use a simple algorithm (see Appendix) for the computation of curvatures and torsion of the geoid with respect to the local astronomical frame, which is needed for our surface geoidal strain analysis, utilizing Rapp’s (1981) geopotential model of spherical harmonics expansion up to degree and order 180 and Tscherning’s (1976a, 1976b) recursive algorithm for the computation of the first and second order derivatives of the geopotential (see also Tscherning et al. 1983) in the geocentric frame. Numerical computations (Filaretou 1986) have shown that this algorithm is efficient and fast.

From $\lambda^o, \varphi^o$ we compute the Cartesian geocentric coordinates of the relevant point $P$ on the geoid having computed the geoidal height $\xi$ from the spherical harmonics expansion. Then, obtaining the geocentric polar coordinates of $P$, the first and second partial derivatives of the geopotential $W$ are computed with respect to a quasi-astronomical reference system, using Tscherning’s GPOD subroutine with Clenshaw summations (Tscherning and Poder 1982) and finally the first and the second partial derivatives of $W$ are obtained, with respect to the geocentric reference frame. Then the astronomical longitude, latitude and the scalar gravity are easily derived at $P$. Thus, all the necessary quantities for the computation of the second derivatives of $W$ with respect to the local astronomical frame, at the geoid are available, and the curvatures and torsion in (12.1), (12.2) and (12.3) are obtained.

An example

Focusing the analysis here, in the eastern Mediterranean-Caspian sea area ($20^o < \varphi < 50^o, 10^o < \lambda < 60^o$), with associated geotectonics shown in Fig. 2 (compiled from Mantovani et al. 1987, NASA 1983), we observe...
four main sharp gradient concentrations in the geoidal map (Fig. 3), based on Rapp's expansion up to 180, computed on a 1°×1° grid: a) in the Persian gulf NW-belt, b) in the Caspian-Black sea E-W belt, (c) in the Cretan sea-south Asia Minor area and (d) in the Otranto straits zone which is part of the Calabrian arc. Compared to geotectonics, the Persian gulf geoidal belt and the Caspian-Black sea geoidal belt look embedded within the relevant compressional tectonic trenches of the area and the transcurrent Anatolian fault. The same holds for the Cretan-south Asia Minor geoidal feature which looks to be in relation with the tectonic system of the NW compressional trench (Hellenic arc) and the NE transcurrent fault. On the other hand there is no evident relation between the geoidal pattern with the NW compressional trench and the two NE and NNE transcurrent faults in the Cyprus-Middle East coasts area. The same holds for the relations between the geoidal pattern along the Yugoslavian coasts and the Dinarides compressional trench.

Studying the strain representation of the potential anomalies, computed on a 1°×1° grid, we observe some much clearer features, with respect to the geodynamics of the area. The dilatational isotropic strain distribution, in terms of positive (expansion) and negative (shrinkage) values (Fig. 4), corresponds to areas in compressional and tensional tectonic state. The evident linear fronts of these opposite signed dilatational zones, are in very good agreement with all geotectonic trenches and faults shown in Fig. 2. The linear fronts coincide with the compressional trench of Dinarides-Hellenides, the compressional trench and the transcurrent fault of the Hellenic arc, the trench-fault basin system in the area of Cyprus, the north Anatolian fault and the two major Caspian and Persian trenches.

In Fig. 5 the distribution of the anisotropic strain (maximum shear strain) is shown. Significant concentration of anisotropy is found at the borders of areas of isotropic strain with opposite signs. Maximum values of anisotropy is concentrated in the compressional trench of the Hellenic arc. High anisotropy is also shown in the area of Cyprus, in the east end of the Anatolian transcurrent fault, in the SW part of Caspian sea, in south Balkans and the Apennine massif.

It is also interesting to mention here that according to the available recent vertical crustal movements data from Eastern Europe (IAG-CRCM 1971, GRD-IGC 1979), concerning the area of our study, the bands of maximum vertical movement values +12 to +14 mm/yr and -6 to +8 mm/yr coincide with the borders of large negative (shrinkage) and of large positive (expansion) isotropic strain. These areas are, for the first band, the belt between the Caspian and the Black sea, west of Bakou. For the second band, the area north of the Black sea and the central area of the frontiers between Bulgaria and Greece.

Concluding remarks

From the geoid strain analysis it comes out that using either the geometric type of Helmert mapping or the physical type of isoparametric mapping, the geoidal representation of geopotential anomalies are mainly of dilatational character. Especially for the isoparametric case, Livieratos and Tziavos (1991) have shown that for the area of the application treated here, the correlation between the geoidal representation and its dilatational counterpart results a negative 80%-100% correlation at 0 km-90 km wave length respectively, in SE-NW direction, in which the main faulting systems and trenches are extended as well. As far as the maximum shear component is concerned, which it is not negligible in the isoparametric case, a strong positive correlation (80%-90%) has been found for very long wave lengths (1400 km-1600 km) in the NW direction.

Concentrating the discussion on the dilatational representation at the area of our example, we observe that the Helmert type of dilatation (15), is nothing but the depiction of the geoid itself (Fig. 3) which is not in sharp agreement with the tectonics of the area (Fig. 2). On the contrary, the dilatation due to the isoparametric mapping (Fig. 4) shows areas of expansion (positive dilatation) and areas of shrinkage (negative dilatation) as well as front-lines of zero dilatation which are, generally, in surprisingly good agreement with the faulting systems and the trenches of the area.

From the example treated in this paper it is evident how the alternative strain representations of the geoid are much better correlated to the geotectonic map, even in details, than the geoid itself. The decomposition of potential anomalies into invariant elastic components, i.e., the isotropic extensional and shrinking parts, the anisotropic shear part, etc., could give major possibilities in the interpretation of the geoidal representation of geopotential anomalies with respect to geotectonics and geodynamics in general, since they contain horizontal derivatives of the disturbing geopotential which are sensitive to location of anomalous internal density distribution.
Fig. 2. Major geotectonic features.

Fig. 3. The geoidal representation of the geopotential anomalies, according to 180×180 Rapp's global model. Isarithmics in meters.
Fig. 4. Isotropic strain (areal expansion and shrinkage) representation of the geoid in Fig. 3. Choropleths×10^{-3}.

Fig. 5. Anisotropic strain (maximum shear) representation of the geoid in Fig. 3. Choropleths×10^{-3}.
Appendix:

Computation of the strain tensor elements

**Step 1.** For the given points Q° on the reference GRS80 ellipsoid with known ellipsoidal coordinates λ°, φ° the associated geoidal heights ζ

(A.1) \( \xi = \xi(\lambda^o, \phi^o) \)

are computed using the spherical harmonics expansion of Rapp's 81 model. With the set of geodetic coordinates \( \lambda^o, \phi^o, \xi \), the geocentric Cartesian coordinates \( x \) of a point P on the geoid can be computed from the well known transformation

(A.2) \[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = 
\begin{bmatrix}
(N^o+\xi) \cos\phi^o \cos\lambda^o \\
(N^o+\xi) \cos\phi^o \sin\lambda^o \\
[(1-e^2)N^o+\xi] \sin\phi^o
\end{bmatrix}
\]

where \( e^2 \) the eccentricity of the GRS80 ellipsoid and \( N^o \) the radius of curvature of the ellipsoidal prime normal section to the meridian section at the point \( \lambda^o, \phi^o \) on the ellipsoid.

**Step 2.** Using the \( x \) coordinates of P the geocentric polar coordinates \( \lambda', \phi' \) are easily computed

(A.3) \( \lambda' = \lambda^o \); \( \phi' = \arctan \left( \frac{Z}{Y} \right) \sin \lambda^o \)

as well as the rotation matrix \( R' \)

(A.4) \( R' = R_2(90° - \phi') P R_3(\lambda') \)

where \( P \) is a reflection matrix

(A.5) \[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Rotation (A.5) relates the geocentric reference frame (\( x \)) with the quasi-astronomic reference frame (\( x' \)): \( x' \) to a quasi astronomic east, \( y' \) to the astronomic north and \( z' \) radially upwards

(A.6) \( x' = R'^T x \)

**Step 3.** Running the Tscherne GIIOTDR subroutine the first and the second partial derivatives of the geopotential \( W \) at P are computed with respect to \( x' \), namely

(A.7) \( g' = \frac{\partial W}{\partial x'} \); \( \mathbf{B}' = \frac{\partial^2 W}{\partial x' \partial x'} \)

where \( g' \) is the 3×1 matrix with the gravity vector components with respect to the \( x' \) coordinates (the first derivatives of \( W \)) and \( \mathbf{B}' \) is the 3×3 symmetric matrix with the gravity gradients (the second derivatives of \( W \)) with respect to \( x' \) as well.

**Step 4.** From the transformation (A.7) we obtain

(A.8) \( \mathbf{g} = R'^T \mathbf{g}' = \frac{\partial W}{\partial x} \)

are the components of the gravity vector with respect to the geocentric base. Similarly the transformation of \( \mathbf{B}' \) to the geocentric base is given by

(A.9) \( \mathbf{B} = R'^T \mathbf{B}' R' \)

where \( \mathbf{B} \) is the 3×3 symmetric matrix containing the second derivatives of \( W \) with respect to the geocentric reference frame.

**Step 5.** Astronomic longitude \( \Lambda \), latitude \( \Phi \) and gravity vector magnitude \( g \) at the point P are easily derived from (A.10) by the well known relations (see, e.g., Heiskanen and Moritz 1967)

(A.10) \[
\Lambda = \arctan \left( \frac{\partial W}{\partial x} \right), \quad \Phi = \arctan \left( \frac{\partial W}{\partial z} \right)
\]

\[
\frac{\partial W}{\partial x} = \sqrt{\left( \frac{\partial W}{\partial x} \right)^2 + \left( \frac{\partial W}{\partial y} \right)^2 + \left( \frac{\partial W}{\partial z} \right)^2}
\]

where

(A.11) \( R' = R_1(90° - \Phi) R_3(90° + \Lambda) \)

is computed which relates the geocentric reference frame with the local astronomic frame

**Step 6.** The first partial derivatives of \( W \), at \( P \), are now computed with respect to the astronomic frame, from the obvious relation

(A.12) \( \mathbf{g}^* = R^* R'^T \mathbf{g}' = \left[ \frac{\partial W}{\partial E}, \frac{\partial W}{\partial N}, \frac{\partial W}{\partial Z} \right]^T \)

as well as the Eotvos 3x3 symmetric matrix containing the second derivatives of \( W \) with respect to the astronomic frame
\[ E^* = R^* R'^T B' (R^* R'^T)^T \]

from which the upper left 2×2 submatrix is considered

\[
E^* = \begin{bmatrix}
\frac{\partial^2 W}{\partial E^2} & \frac{\partial^2 W}{\partial E \partial N} & E_{13} \\
\frac{\partial^2 W}{\partial E \partial N} & \frac{\partial^2 W}{\partial N^2} & E_{23} \\
E_{13} & E_{23} & E_{33}
\end{bmatrix}
\]

The curvatures and torsion of the geoidal surface at \( P \) can be now computed from (11), (12), (A.10) and (A.14) and consequently the metric tensor of the geoid (10).

**Step 7.** Due to the chosen isoparametric mapping (17), the relevant radii of curvature \( N \) and \( M \) can be computed for the GRS80 ellipsoid thus the metric tensor of the ellipsoid (14) is easily derived.

**Step 8.** Having now all the necessary quantities required for the computation of the strain tensor (19) the relevant strain parameters from equation (8) are easily obtainable.

**References**


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