Deformation Analysis of Geoid to Ellipsoid Mappings

(A. Dermanis, E. Livieratos and S. Pertsinidou: Ανάλυση των Παραμορφώσεων των Απεικονισεων απο το Γεωεδεις στο Ελλειψοειδες)

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Summary: Utilizing the general theory of deformation analysis in Riemmanian spaces, the two-dimensional strain tensors and relevant invariant scalar strain parameters are given for geoid to ellipsoid mappings. Two typical isoparametric mappings are analyzed. The classical mapping associated with the well-known, from practice, Helmert projection and the Marussi mapping arising in the rigorous linearization of the geodetic boundary value problem. Decomposed first order expressions are also given for numerical realizations.

Περίληψη: Χρησιμοποιώντας την γενική θεωρία της ανάλυσης παραμορφώσεων σε χώρους Ριέμον, δίνονται οι τανοστές παραμόρφωσης στις 2 διαστάσεις καθώς και σχετικές αναλογίες υπάρχουσας παράμετρων παραμορφώσης για τις απεικονίσεις του γεωεδείς στο ελλειψοειδος. Αναλύονται δύο τυπικές ισοπαραμετρικές απεικονίσεις. Η κλασσική απεικόνιση, γνωστή από την πράξη, της προβολής του Ηχερμαντ και η απεικόνιση του Μαρουσί που χρησιμοποιείται στην μαθηματικά αυστηρή γεωμετροποίηση του γεωδαιτικού προβλήματος της οριακής τιμής. Δίνονται επίσης εκφράσεις πρώτης τάξης αναλυμένες σε ένα κανονικό και ένα διατατικό μέρος, για αριθμητικές πρακτικές εφαρμογές.

Department of Geodesy and Surveying, University of Thessaloniki
Τομέας Γεωδαισίας και Τοπογραφίας, ΑΠΘ
1. INTRODUCTION

A standard technique in classical ellipsoidal geodesy is the mapping of geometric observable quantities from the geoid onto the reference ellipsoid. This is usually done through the Helmert projection procedure in order to elaborate easier on the ellipsoid various geodetic problems taking advantage of its simple metrical properties compared to the more complicated metrical characteristics of the geoid. The rigorous linearization of the geodetic boundary value problem by Krarup (1973), gave rise to a different type of mapping, the so called Marussi mapping. Due to these mappings an alteration of the geometric quantities occurs, which is studied by describing the deformation involved in terms of strain tensor expressions.

Similar alterations also occur when various geodetic geometric quantities are transformed from one reference ellipsoid to an other, a typical case in datum comparisons. The use of deformation analysis for the study of the mapping of the geoid onto the reference ellipsoid for comparisons between alternative choices of reference ellipsoids, is not new in geodetic literature, see, e.g., Marussi, 1951a, 1951b, 1957, O'Keefe 1953, Caputo 1959, Zadro and Carminelli 1966, Teunissen 1982, but the possibility of expressing the relevant deformations in terms of strain expressions has not been considered up to now.

In this paper the deformation analysis of the geoid-to-ellipsoid mappings is given by the elements of the strain tensor involved and by invariant strain parameters, such as dilatation and maximum shear strain. Two typical mappings are analyzed namely the Marussi and Helmert mappings and decomposition expressions are given for numerical realizations. The general theory of the problem is given in Dermanis and Livieratos, 1983a. Detailed derivations can be found in Pertsinidou, 1983.

2. GENERAL FORMULATION

When a Riemmanian manifold $M$ with coordinates

\[ Q = [q_1, q_2]^T \]  

(1)
and metric

\[ dS^2 = d\mathbf{Q}^T \mathbf{C} \, d\mathbf{Q} \]  \hspace{1cm} (2)

is mapped pointwise onto a Riemmanian manifold \( M' \) with coordinates

\[ \mathbf{q} = [q_1 \ q_2]^T \]  \hspace{1cm} (3)

and metric

\[ ds^2 = dq^T \mathbf{C} \, dq \]  \hspace{1cm} (4)

the relevant deformation is described by means of the Lagrangian strain tensor (Dermanis and Livieratos, 1983a)

\[ E(\mathbf{q}) = \frac{1}{2} \left[ \mathbf{C} - \left( \frac{\partial \mathbf{q}}{\partial \mathbf{Q}} \right)^T \mathbf{C} \frac{\partial \mathbf{q}}{\partial \mathbf{Q}} \right] . \]  \hspace{1cm} (5)

The matrix \( \frac{\partial \mathbf{q}}{\partial \mathbf{Q}} \) results from the differentiation of the equations

\[ \mathbf{q} = \mathbf{q}(\mathbf{Q}) \]  \hspace{1cm} (6)

which define the mapping in question.

In the particular case of isoparametric mappings

\[ \mathbf{q} = \mathbf{Q} \]  \hspace{1cm} (7)

the strain tensor becomes

\[ E = \frac{1}{2} \left[ \mathbf{C} - \mathbf{c} \right] . \]  \hspace{1cm} (8)

In this case the Lagrangian and Eulerian strain tensors are identical. The computation of strain components reduces to the computation of the metric tensors of the two Riemmanian manifolds.
3. THE MARUSSI MAPPING

According to the well known Marussi telluroid mapping a point \( P \) is mapped into a point \( P' \) defined by means of

\[
\begin{align*}
\Lambda(P) &= \Lambda(P') \\
\phi(P) &= \phi(P') \\
W(P) &= U(P')
\end{align*}
\]  

(9)

\( \Lambda, \phi, W \) being the astronomic longitude, astronomic latitude and gravity potential respectively, and \( \lambda, \varphi, U \) their normal counterparts. When this mapping is applied on points \( P \) of the geoid where \( W = W_0 \), the image points \( P' \) lie on the reference ellipsoid which is an equipotential surface \( (U = W_0) \) of the standard Somigliana-Pizzetti normal gravity field.

On the geoid the quantities \( \Lambda, \phi \) can be used as curvilinear coordinates

\[
Q = [\Lambda \phi]^T
\]  

(10)

while on the ellipsoid the corresponding curvilinear coordinates are

\[
q = [\lambda \varphi]^T
\]  

(11)

The mapping from the geoid to the ellipsoid is an isoparametric one and equation (6) becomes the first two of equations (9)

\[
\begin{align*}
\lambda &= \Lambda. \\
\varphi &= \phi.
\end{align*}
\]  

(12)

In order to compute the metric on the geoid, the metric in three dimensional Euclidean space will be first computed in terms of the Marussi coordinates \( \Lambda, \phi, W \) and the geoid metric will follow by simply setting \( dW = 0 \). Setting

\[
y = [\Lambda \phi \, -W]^T
\]  

(13)
it is well known that (Dermanis and Livieratos, 1983b, eq. (36))

\[ \frac{\partial y}{\partial x} = FR \quad \text{and} \quad \frac{\partial x}{\partial y} = RTF^{-1} \quad (14) \]

where

\[ F = \begin{bmatrix} k_E \sec \Phi & t_N \sec \Phi & k_E \sec \Phi \\ t_N & k_N & k_N \\ 0 & 0 & g \end{bmatrix} \quad (15) \]

and

\[ R = R_1(90^\circ - \Phi) R_3(90^\circ + \Lambda) \quad . \quad (16) \]

\( k_E, k_N \) are the curvatures of the normal sections of the geoid in the east and north directions, \( k_E, k_N \) are the curvature components of the plumb line in the east and north direction, \( t_N \) is the geodetic torsion of the geoid in the north direction.

The Euclidean metric expressed in terms of the \( y \) coordinates becomes

\[ dS_E^2 = dx^T dx = dy^T (\frac{\partial x}{\partial y})^T \frac{\partial x}{\partial y} dy = dy^T G dy \quad (17) \]

where taking (14) into account

\[ G = (F^{-1})^T F^{-1} = (FF^T)^{-1} \quad . \quad (18) \]

Analytical inversion of \( F \) gives

\[ F^{-1} = \begin{bmatrix} k_N \cos \Phi & -t_N & t_N k_N - k_N k_E \\ -k_G & K_G & g K_G \\ -t_N \cos \Phi & k_E & t_N k_E - k_E k_N \\ K_G & K_G & g K_G \\ 0 & 0 & 1/g \end{bmatrix} \quad (19) \]

and the elements of \( G \) follow from application of equation (18). Inserting these elements in equations (17), the Euclidean metric in three
dimensions expressed in terms of the curvilinear coordinates Λ, Φ, W, becomes

\[
\begin{align*}
\text{ds}_E^2 &= \frac{(k_N^2t^2 + t_N^2)}{K_G^2} \, d\Lambda^2 + \frac{t_N^2 + k_N^2}{K_G^2} \, d\Phi^2 + \\
&\quad + \frac{k_N^2(t_Nk_N - k_Nk_E)^2 + (t_Nk_E - k_Ek_N)^2}{g^2 K_G^2} \, dW^2 - 2 \frac{t_N(k_N + k_E)}{K_G^2} \cos \Phi \, d\Lambda \, d\Phi + \\
&\quad - 2 \frac{t_N(k_N + k_E)k_N - (t_N^2 + k_N^2)k_E}{g^2 K_G^2} \, d\Lambda \, dW + \\
&\quad - 2 \frac{t_N(k_N + k_E)k_E - (t_N^2 + k_N^2)k_N}{g^2 K_G^2} \, d\Phi \, dW,
\end{align*}
\]  

(20)

where

\[
K_G = k_Ek_N - t_N^2
\]  

(21)

is the Gaussian curvature. Setting \(dW = 0\) in equation (20), the metric on the geoid is obtained. The corresponding metric matrix \(C\) follows by deleting the third row and third column of matrix \(G\)

\[
C = \begin{bmatrix}
\frac{(k_N^2t^2 + t_N^2)}{K_G^2} & \frac{t_N(k_N + k_E)\cos \Phi}{K_G^2} \\
\frac{t_N(k_N + k_E)\cos \Phi}{K_G^2} & \frac{t_N^2 + k_N^2}{K_G^2}
\end{bmatrix}
\]  

(22)

In the case of the normal gravity field \(k_E, k_N, k_N\) are denoted by \(k_{OE}, k_{ON}\), respectively, while \(t_N\) and \(k_E\) become zero. The metric matrix \(c\) follows easily from \(C\) with the above replacements which lead to

\[
\begin{bmatrix}
\cos^2 \Phi & 0 \\
K_{OE}^2 & 0
\end{bmatrix}
\begin{bmatrix}
\cos^2 \Phi N^2 & 0 \\
0 & M^2
\end{bmatrix}
\]  

(23)
where \( M \) and \( N \) are the radii of curvature of the normal sections of the reference ellipsoid in the meridian and prime vertical direction respectively.

Since in this case of isoparametric mapping \( \lambda = \Lambda, \phi = \Phi \) and the distinction between the Lagrangian and Eulerian approach disappears, the strain matrix becomes according to equation (8)

\[
E = \begin{bmatrix}
\cos^2 \phi \left( \frac{k_N^2 + t_N^2}{K_G^2} \right) & - \frac{\cos \phi \ t_N \ (k_N + k_E)}{2 \ K_G^2} \\
- \frac{\cos \phi \ t_N \ (k_N + k_E)}{2 \ K_G^2} & \frac{1}{2} \left( \frac{t_N^2 + k_E^2}{K_G^2} \right)
\end{bmatrix}
\]  

(24)

The strain matrix \( E \) describes the deformation in the neighbourhood of any point on the geoid when it is mapped into a point on the ellipsoid with coordinates \( \lambda \) and \( \phi \). The use of unitless curvilinear coordinates (\( \lambda \) and \( \phi \) are in radians) has the disadvantage that the elements of \( E \) have the dimension of length square. For this reason and also for representation purposes it is more convenient to use linear coordinates \( s_\lambda \), \( s_\phi \) on the plane tangent to the ellipsoid at the point in question. Since

\[
\begin{bmatrix}
ds_\lambda \\
ds_\phi
\end{bmatrix} = \begin{bmatrix} N \cos \phi & 0 \\
0 & M
\end{bmatrix} \begin{bmatrix} d\lambda \\
d\phi
\end{bmatrix} = B \begin{bmatrix} d\lambda \\
d\phi
\end{bmatrix}
\]  

(25)

application of the tensor transformation rule leads to the unitless strain matrix \( E' \) with respect to the \( s_\lambda, s_\phi \) coordinates

\[
E' = B^{-1} E (B^{-1})^T = \begin{bmatrix}
E_{s_\lambda s_\lambda} & E_{s_\lambda s_\phi} \\
E_{s_\phi s_\lambda} & E_{s_\phi s_\phi}
\end{bmatrix}
\]  

(26)

with elements

\[
E_{s_\lambda s_\lambda} = \frac{1}{2} \left( \frac{k_N^2 + t_N^2}{N^2 K_G^2} - 1 \right)
\]  

(27)
\[ E_{s\lambda} S_{\lambda \phi} = -\frac{t_N (k_E + k_N)}{2N M K_G^2} = E_{s \phi S_{\lambda \phi}} \]  

(28)

\[ E_{s \phi} S_{\lambda \phi} = \frac{1}{2} \left( \frac{\frac{t_N^2 + k_E^2}{M^2 K_G^2}}{1} \right). \]  

(29)

One can also compute dilatation

\[ \Delta = E_{s \lambda} S_{\lambda} + E_{s \phi} S_{\phi} = \frac{1}{2 K_G^2} \left( \frac{k_N^2 + t_N^2}{N^2} + \frac{k_E^2 + t_E^2}{M^2} \right) - 1 \]  

(30)

the shear components

\[ \gamma_1 = E_{s \lambda} S_{\lambda} - E_{s \phi} S_{\phi} = \frac{1}{2 K_G^2} \left( \frac{k_N^2 + t_N^2}{N^2} + \frac{k_E^2 + t_E^2}{M^2} \right) \]  

(31)

\[ \gamma_2 = 2 E_{s \lambda} S_{\phi} = -\frac{t_N (k_E + k_N)}{N M K_G^2} \]  

(32)

the maximum shear strain

\[ \gamma = \left\{ \gamma_1^2 + \gamma_2^2 \right\}^{\frac{1}{2}} = \frac{1}{2 K_G^2} \left[ \frac{(k_N^2 + t_N^2)^2}{N^4} + \frac{(k_E^2 + t_E^2)^2}{M^4} + \frac{2[t_N^2 (k_N + k_E)^2 - K_G^2]}{N^2 M^2} \right]^{\frac{1}{2}} \]  

(33)

and the azimuth of maximum shear strain

\[ \psi = \arctan \left( -\frac{\gamma_2}{\gamma_1} \right) = \arctan \left( \frac{2 N M t_N (k_E + k_N)}{M^2 (k_N^2 + t_N^2) - N^2 (k_E^2 + t_E^2)} \right). \]  

(34)
4. DECOMPOSITION EXPRESSIONS

The computation of the elements of the strain tensor or other strain parameters requires the knowledge of the values of curvatures \( k_N \), \( k_E \) and torsion \( t_N \) at points on the geoid. These values can be computed utilizing the decompositions

\[
\Lambda = \lambda + \varepsilon = \lambda + \frac{\eta}{\cos \phi} \\
\Phi = \phi + \xi \\
W = U + T
\]

at geoid points. In this section \( \lambda \) and \( \phi \) denote normal longitude and latitude at geoid points, in disagreement with other sections where they refer to points on the reference ellipsoid. Thus, \( \varepsilon \) (or \( \eta \)) and \( \xi \) are disturbances of the vertical. Denoting by

\[
y_o = [\Lambda \quad \Phi \quad -U]^T
\]

(36)

the normal counterpart of \( y \), defined in equation (13), an equation analogous to (14) can be written

\[
\frac{\partial y_o}{\partial x} = F_o R_o
\]

(37)

where

\[
R_o = R_1(90^\circ - \phi) R_3(90^\circ + \lambda)
\]

(38)

and

\[
F_o = \begin{bmatrix}
k_{OE} \sec & 0 & 0 \\ 0 & k_{ON} & k_{ON} \\ 0 & 0 & y
\end{bmatrix}
\]

(39)

Differentiation of equations (35) gives
\[ D = \begin{bmatrix} 1 + \frac{\partial \varepsilon}{\partial \lambda} & \frac{\partial \varepsilon}{\partial \varphi} & - \frac{\partial \varepsilon}{\partial U} \\ \frac{\partial \xi}{\partial \lambda} & 1 + \frac{\partial \xi}{\partial \varphi} & - \frac{\partial \xi}{\partial U} \\ - \frac{\partial T}{\partial \lambda} & - \frac{\partial T}{\partial \varphi} & 1 + \frac{\partial T}{\partial U} \end{bmatrix} \]  

(40)

while from equations (14) and (37) follows that

\[ F = \frac{\partial \mathbf{y}}{\partial x} \mathbf{R}^T = \frac{\partial \mathbf{y}}{\partial y_0} \frac{\partial \mathbf{y}_0}{\partial x} \mathbf{R}^T = D \mathbf{F}_0 \mathbf{R}_0 \mathbf{R}^T. \]  

(41)

Replacing from equations (16), (38), (39) and (40) into equation (41) and neglecting second order terms in the small quantities \( k_{ON}, \varepsilon, \xi \) and their derivatives, the following first order approximations of the elements of \( F \) are obtained

\[ F_{11} = \frac{k_E}{\cos \phi} = \frac{k_{OE}}{\cos \phi} \left( 1 + \frac{\partial \varepsilon}{\partial \lambda} \right) \]  

(42)

\[ F_{22} = k_N = k_{ON} \left( 1 + \frac{\partial \xi}{\partial \varphi} \right) \]  

(43)

\[ F_{21} = \tan \phi \varepsilon + k_{ON} \sin \phi \varepsilon \]  

(44)

\[ F_{12} = \frac{\tan \phi \varepsilon + k_{ON} \frac{\partial \varepsilon}{\partial \varphi}}{\cos \phi} \]  

(45)

Since the geoid is closely approximated by the reference ellipsoid, it holds in first order approximation that

\[ k_{OE} = \frac{1}{N} \]  

(46)

\[ k_{ON} = \frac{1}{M} \]  

(47)

where the \( N \) and \( M \) are computed for the latitude \( \varphi \) of the point on the geoid. Finally
\[ k_E = \frac{1 - \tan \phi \xi}{N} \left( 1 + \frac{\Delta \epsilon}{\Delta \lambda} \right) \]  
(48)

\[ k_N = \frac{1}{M} \left( 1 + \frac{\Delta \xi}{\Delta \phi} \right) \]  
(49)

\[ t_N = \frac{1}{N \cos \phi} \frac{\Delta \xi}{\Delta \lambda} + \frac{\sin \phi \epsilon}{M} = \frac{\cos \phi \Delta \epsilon}{M} \frac{\Delta \xi}{\Delta \phi} - \frac{\sin \phi \epsilon}{N} \]  
(50)

With the help of the above relations the elements of the strain tensor become to first order approximation

\[ E_{S_{\lambda \lambda}} = \frac{\Delta \xi}{\Delta \phi} \]  
(51)

\[ E_{S_{\lambda \phi}} = - \left( \sin \phi \epsilon + \frac{1}{\cos \phi} \frac{\Delta \xi}{\Delta \lambda} \right) = \frac{\Delta \epsilon}{\Delta \lambda} - \tan \phi \xi \]  
(52)

\[ E_{S_{\phi \phi}} = \frac{\Delta \epsilon}{\Delta \lambda} - \tan \phi \xi \]  
(52)

and consequently

\[ \Delta = \frac{\Delta \xi}{\Delta \phi} + \frac{\Delta \epsilon}{\Delta \lambda} - \tan \phi \xi \]  
(53)

\[ \gamma_1 = \frac{\Delta \xi}{\Delta \phi} - \frac{\Delta \epsilon}{\Delta \lambda} + \tan \phi \xi \]  
(54)

\[ \gamma_2 = -2 \left( \sin \phi \epsilon + \frac{1}{\cos \phi} \frac{\Delta \xi}{\Delta \lambda} \right) = 2 \left( \frac{\Delta \epsilon}{\Delta \lambda} - \tan \phi \xi \right) \]  
(55)

Note that the decompositions of equations (42) to (45) used in the above analysis are comparable with those in Livieratos, 1976.
5. THE HELMERT MAPPING

A different way of mapping a point of the geoid into a point of the ellipsoid is the well known Helmert's projection along the perpendicular to the reference ellipsoid. For the analytical description of this mapping the geodetic coordinates $\lambda$, $\varphi$, $h$ are used, where $\varphi$ should not be confused with the normal latitude used in the previous sections. These coordinates are related to the vector $x$ of cartesian coordinates through the well known relation

$$
\begin{bmatrix}
(N+h) \cos \varphi \cos \lambda \\
(N+h) \cos \varphi \sin \lambda \\
[N(1-e^2)+h] \sin \varphi
\end{bmatrix}
$$

(56)

where

$$
N = \frac{a}{\sqrt{1-e^2 \sin^2 \varphi}}
$$

(57)

is the radius of curvature of the normal section of the ellipsoid in the prime vertical direction. For points on the ellipsoid $h=0$ holds, while for points on the geoid $h=\zeta$, $\zeta$ being the geoid undulation.

Both points, the original on the geoid and its image on the ellipsoid have the same $\lambda$ and $\varphi$, i.e.,

$$
Q = q = [\lambda \varphi]^T
$$

(58)

and the relevant mapping is trivially isometric when these two coordinates are used as curvilinear coordinates on both Riemmanian manifolds.

The metrics on the geoid and ellipsoid are further needed. For this purpose the Euclidean three-dimensional metric is first expressed in terms of the geodetic coordinates

$$
z = [\lambda \varphi h]^T
$$

(59)

Noting that
\[ \frac{\partial}{\partial \phi} (N \cos \phi) = -M \sin \phi \]  

(60)

and

\[ \frac{\partial}{\partial \phi} [N(1-e^2) \sin \phi] = M \cos \phi \]  

(61)

where

\[ M = a (1-e^2) (1-e^2 \sin^2 \phi)^{-3/2} \]  

(62)

is the curvature of the normal section of the ellipsoid in the meridional direction, it follows that

\[ \frac{\partial x}{\partial z} = R_3(-90^0-\lambda) R_1(\phi-90^0) \hat{G} \]  

(63)

and the metric becomes

\[ ds^2_E = dx^T dx = dz^T \hat{G} dz \]  

(64)

where

\[ \hat{G} = \left( \frac{\partial x}{\partial z} \right)^T \left( \frac{\partial x}{\partial z} \right) = (\hat{G}^z)^2 = \begin{bmatrix} (N+h)^2 \cos^2 \phi & 0 & 0 \\ 0 & (M+h)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]  

(65)

Analytically the Euclidean metric is

\[ ds^2_E = (N+h)^2 \cos^2 \phi \; d\lambda^2 + (M+h)^2 \; d\phi^2 + dh^2 \]  

(66)

In order to obtain the metric \( ds^2 \) on the ellipsoid, \( h = 0 \) and \( dh = 0 \) must be set in equation (66), in which case the relevant metric matrix \( c \) of equation (8) becomes

\[ c = \begin{bmatrix} N^2 \cos^2 \phi & 0 \\ 0 & M^2 \end{bmatrix} \]  

(67)
In order to obtain the metric $dS^2$ on the geoid, it is assumed that the geoid undulation is known as a function

$$
\zeta = \zeta(\lambda, \varphi)
$$

(68)

which is in fact the case when $\zeta$ is computed from gravity anomalies applying Stokes' formula. Differentiation of equation (68) gives

$$
d\zeta = \frac{\partial \zeta}{\partial \lambda} d\lambda + \frac{\partial \zeta}{\partial \varphi} d\varphi = \zeta_\lambda d\lambda + \zeta_\varphi d\varphi
$$

(69)

where the partial derivatives $\zeta_\lambda$, $\zeta_\varphi$ are also known.

Setting $h = \zeta$, $dh = d\zeta$ into equation (66) the metric on the geoid is obtained

$$
dS^2 = \left[ (N+\zeta)^2 \cos^2 \varphi + \zeta_\lambda^2 \right] d\lambda^2 + \left[ (M+\zeta)^2 + \zeta_\varphi^2 \right] d\varphi^2 + 2 \zeta_\lambda \zeta_\varphi d\lambda d\varphi
$$

(70)

and the relevant metric matrix $C$ of equation (8) becomes

$$
C = \begin{bmatrix}
(N+\zeta)^2 \cos^2 \varphi + \zeta_\lambda^2 & \zeta_\lambda & \zeta_\varphi \\
\zeta_\lambda & (M+\zeta)^2 + \zeta_\varphi^2
\end{bmatrix}
$$

(71)

Applying equation (8) the strain matrix in $\lambda, \varphi$ coordinates becomes

$$
E = \frac{1}{2} \begin{bmatrix}
(2N+\zeta) \zeta \cos^2 \varphi + \zeta_\lambda^2 & \zeta_\lambda & \zeta_\varphi \\
\zeta_\lambda & (2M+\zeta) \zeta + \zeta_\varphi^2
\end{bmatrix}
$$

(72)

A strain matrix $E'$ with dimensionless elements is obtained as in the previous sections applying equation (26)

$$
E' = \begin{bmatrix}
E_{\lambda\lambda} & E_{\lambda\varphi} & \zeta_\lambda & \zeta_\varphi \\
E_{\varphi\lambda} & E_{\varphi\varphi} & 2 \frac{N}{M} \cos \varphi & 2 \frac{M}{N} \cos \varphi \\
\zeta_\lambda & \zeta_\varphi & 2 \frac{N}{M} \cos \varphi & 2 \frac{M}{N} \cos \varphi
\end{bmatrix}
$$

(73)

A spherical approximation can be obtained by setting $N = M = R$, where
R is the mean radius of the earth and

\[ \zeta_\varphi = - R \xi \]
\[ \zeta_\lambda = - R \cos \varphi \eta \]  \hfill (74)  \hfill (75)

where \( \xi \) and \( \eta \) are now the deflections of the vertical. The strain matrix \( E' \) becomes

\[
E' = \begin{bmatrix}
\frac{\xi}{R} + \frac{\zeta^2}{2R^2} + \frac{\eta^2}{2} & \frac{\xi \eta}{2} \\
\frac{\xi \eta}{2} & \frac{\zeta}{R} + \frac{\zeta^2}{2R^2} + \frac{\xi^2}{2}
\end{bmatrix}
\]  \hfill (76)

The second order terms \( \zeta^2/R^2, \xi^2, \eta^2, \xi \eta \) can also be deleted so that in first order approximation

\[
E' = \begin{bmatrix}
\frac{\xi}{R} & 0 \\
0 & \frac{\zeta}{R}
\end{bmatrix}
\]  \hfill (77)

As a consequence, in this geoid-to-ellipsoid mapping, maximum shear strain \( \gamma \) is essentially negligible (no shape alteration), while dilatation is given to sufficient accuracy by

\[
\Delta = \frac{2 \xi}{R}
\]  \hfill (78)

This means that any chart of the geoid where \( \zeta \)-isolines are depicted is also a map of dilatation caused by the Helmert projection, if the indications on the isolines are multiplied by a factor of \( 2/R \).

For a geoid undulation of the order of 100 m the corresponding dilatation at the same point is of the order of \( 1.6 \times 10^{-5} \) or 16 ppm.
REFERENCES

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