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GEODETIC ESTIMABILITY OF CRUSTAL DEFORMATION PARAMETERS

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Η δυνατότητα εκτίμησης των παραμέτρων παραμόρφωσης του γήινου φλοιού με γεωδαιτικές μεθόδους

Summary

The geodetic estimability of crustal deformation parameters as computed by the classical coordinate method, is examined. Estimability conclusions are drawn from the invariance properties deformation parameters, when the two coordinate sets under comparison undergo independent coordinate transformations leaving the relevant geodetic observations invariant.

Περίληψη

Εξετάζεται η δυνατότητα εκτίμησης των παραμέτρων παραμόρφωσης του γήινου φλοιού από τη διαχρονική μεταβολή των συντεταγμένων ενός δικτύου. Η εκτιμησιμότητα ή μη τέτοιων παραμέτρων θεμελιώνεται εξετάζοντας με ποιο τρόπο μεταβάλλονται όταν οι δύο συγκρινόμενες ομάδες συνετατγμένων υπόκεινται σε διαφορετικούς μετασχηματισμούς λόγω αλλαγής του συστήματος αναφοράς, τέτοιους ώστε οι σχετικές γεωδαιτικές παρατηρήσεις να παραμένουν αναλλοίωτες.

1. Introduction

It is well known to geodesists (Grafarend and Schaffrin, 1976) that the estimability of quantities related to the geometry of geodetic networks depends strongly on their invariance under coordinate transformations corresponding to changes in scale, orientation and origin of the utilized reference frames.

A certain quantity can be estimable from an available set of observations only if it remains invariant under any coordinate transformation that leaves the observations unchanged.

When two sets of coordinates of the same network corresponding to two different epochs are compared for the derivation of deformation parameters, the results depend on the reference frames used as both epochs. The reason for this is that station coordinates refer to reference frames, which are externally defined and not inherent in the observations.

In the most general case both coordinates sets refer to different and independently defined reference frames. Sometimes it is possible to relate both coordinate sets to the same but still arbitrary reference frame, e.g. when the positions of some of the stations do not change with time.

In order to study the invariance, and therefore the estimability, of deformation related parameters such as strain, shear, rotation and dilatation, we examine the way in which these parameters are altered by changes of the corresponding reference frames.

2. Derivation of the basic equations

According to the coordinate method (Livieratos, 1980) deformation parameters are computed separately for each triangle of neighbouring stations, $i = 1,2,3$, from their coordinates (x_i , y_i) at some initial epoch and those (x'_i , y'_i) at some later epoch.

Assuming a linear field of displacements (u , v) within the triangle, the following equations

$$\begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} x'_i - x_i \\ y'_i - y_i \end{bmatrix} = \begin{bmatrix} e_{xx} & e_{xy} \\ e_{yx} & e_{yy} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \begin{bmatrix} p \\ q \end{bmatrix} \tag{1}$$

hold for the unknown elements $e_{xx} = \dfrac{\partial u}{\partial x}$, $e_{xy} = \dfrac{\partial u}{\partial y}$,

$e_{yx} = \dfrac{\partial v}{\partial x}$, $e_{yy} = \dfrac{\partial v}{\partial y}$ of the relevant Jacobian matrix.

Setting

$$\mathbf{x} = [x_1 x_2 x_3]^T , \qquad \mathbf{y} = [y_1 y_2 y_3]^T$$

$$\mathbf{x}' = [x'_1 x'_2 x'_3]^T , \qquad \mathbf{y}' = [y'_1 y'_2 y'_3]^T$$

$$\mathbf{u} = [u_1 u_2 u_3]^T = \mathbf{x}' - \mathbf{x} , \qquad \mathbf{v} = [v_1 v_2 v_3]^T = \mathbf{y}' - \mathbf{y}$$

$$\boldsymbol{\delta} = [1\ 1\ 1]^T \tag{2}$$

eq. (1) is written

$$[\mathbf{u} \mid \mathbf{v}] = [\mathbf{x} \mid \mathbf{y} \mid \boldsymbol{\delta}] \begin{bmatrix} e_{xx} & e_{yx} \\ e_{xy} & e_{yy} \\ p & q \end{bmatrix} \tag{3}$$

Eliminating the unknowns p , q with the help of the operator

$$\mathbf{D} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \tag{4}$$

eq. (3) becomes

$$[\mathbf{Du} \mid \mathbf{Dv}] = [\mathbf{Dx} \mid \mathbf{Dy}] \begin{bmatrix} e_{xx} & e_{yx} \\ e_{xy} & e_{yy} \end{bmatrix} \tag{5}$$

Solving eq. (5) we have

$$\begin{bmatrix} e_{xx} & e_{yx} \\ e_{xy} & e_{yy} \end{bmatrix} = [\mathbf{Dx} \mid \mathbf{Dy}]^{-1} [\mathbf{Du} \mid \mathbf{Dv}] =$$

$$= \frac{1}{\kappa} \begin{bmatrix} -\mathbf{y}^T \mathbf{D}^T \mathbf{P} \\ \mathbf{x}^T \mathbf{D}^T \mathbf{P} \end{bmatrix} [\mathbf{Du} \mid \mathbf{Dv}] =$$

$$= \frac{1}{\kappa} \begin{bmatrix} -\mathbf{y}^T \mathbf{Su} & -\mathbf{y}^T \mathbf{Sv} \\ \mathbf{x}^T \mathbf{Su} & \mathbf{x}^T \mathbf{Sv} \end{bmatrix} \tag{6}$$

where:

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \mathbf{S} = \mathbf{D}^T \mathbf{P} \mathbf{D} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \tag{7}$$

and

$$\kappa = \mathbf{x}^T \mathbf{S} \mathbf{y} \tag{8}$$

Explicitly we have

$$\kappa\, e_{xx} = -\mathbf{y}^T \mathbf{Su}\,, \qquad \kappa\, e_{xy} = \mathbf{x}^T \mathbf{Su}\,,$$

$$\kappa\, e_{yx} = -\mathbf{y}^T \mathbf{Sv}\,, \qquad \kappa\, e_{yy} = \mathbf{x}^T \mathbf{Sv}\,. \tag{9}$$

Noting that $\mathbf{S}^T = -\mathbf{S}$ and $\mathbf{a}^T \mathbf{Sa} = 0$ for any $\mathbf{a}$ the above equations are simplified to

$$\kappa\, e_{xx} = -\mathbf{y}^T \mathbf{Sx}' - \kappa\,, \qquad \kappa\, e_{xy} = \mathbf{x}^T \mathbf{Sx}'$$

$$\kappa\, e_{yx} = -\mathbf{y}^T \mathbf{Sy}'\,, \qquad \kappa\, e_{yy} = \mathbf{x}^T \mathbf{Sy}' - \kappa\,. \tag{10}$$

Suppose now that both coordinate sets (x_i, y_i) and (x_i', y_i') undergo independent transformations due to the change of origin, orientation and scale of the relevant reference frames, described by

$$\begin{bmatrix} \tilde{x}_i \\ \tilde{y}_i \end{bmatrix} = \lambda \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix}$$

$$\begin{bmatrix} \tilde{x}_i' \\ \tilde{y}_i' \end{bmatrix} = \lambda' \begin{bmatrix} \cos\theta' & \sin\theta' \\ -\sin\theta' & \cos\theta' \end{bmatrix} \begin{bmatrix} x_i' \\ y_i' \end{bmatrix} + \begin{bmatrix} b_x' \\ b_y' \end{bmatrix} \tag{11}$$

Using the abbreviations $c = \cos\theta$, $s = \sin\theta$, $c' = \cos\theta'$, $s' = \sin\theta'$ the above transformations can also be written in the move convenient form

$$\tilde{\mathbf{x}} = \lambda c \mathbf{x} + \lambda s \mathbf{y} + b_x \boldsymbol{\delta}$$

$$\tilde{\mathbf{y}} = -\lambda s \mathbf{x} + \lambda c \mathbf{y} + b_y \boldsymbol{\delta}$$

$$\tilde{\mathbf{x}}' = \lambda' c' \mathbf{x}' + \lambda' s' \mathbf{y}' + b_x' \boldsymbol{\delta}$$

$$\tilde{\mathbf{y}}' = -\lambda' s' \mathbf{x}' + \lambda' c' \mathbf{y}' + b_y' \boldsymbol{\delta} \tag{12}$$

The parameters e_{xx}, e_{xy}, e_{yx}, e_{yy} are affected by the above transformations. Denoting transformed quantities by tilted overbars and noting that $\mathbf{S}\boldsymbol{\delta} = \mathbf{0}$, $\boldsymbol{\delta}^T \mathbf{S} = \mathbf{0}$ it is easy to show that

$$\tilde{\kappa} = \tilde{\mathbf{x}}^T \mathbf{S} \tilde{\mathbf{y}} = \lambda^2 \kappa \tag{13}$$

and

$$\begin{bmatrix} \tilde{e}_{xx} \\ \tilde{e}_{xy} \\ \tilde{e}_{yx} \\ \tilde{e}_{yy} \end{bmatrix} = \mu \begin{bmatrix} cc' & sc' & cs' & ss' \\ -sc' & cc' & -ss' & cs' \\ -cs' & -ss' & cc' & sc' \\ ss' & -cs' & -sc' & cc' \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{xy} \\ e_{yx} \\ e_{yy} \end{bmatrix} + \begin{bmatrix} \mu(cc'+ss')-1 \\ \mu(cs'-sc') \\ \mu(sc'-cs') \\ \mu(cc'+ss')-1 \end{bmatrix} \tag{14}$$

or simply

$$\tilde{\mathbf{e}} = \mathbf{Qe} + \mathbf{d} \tag{15}$$

where

$$\mu = \frac{\lambda'}{\lambda} \tag{16}$$

The relevant deformation parameters are the dilatation Δ, the rotation ω and the shear strains γ_1 and γ_2. These are computed from the elements of the Jacobian by means of the relations

$$\Delta = e_{xx} + e_{yy}\,, \qquad \gamma_1 = e_{xx} - e_{yy}$$

$$2\omega = e_{xy} - e_{yx}\,, \qquad \gamma_2 = e_{xy} + e_{yx} \tag{17}$$

In matrix form the above relations become

$$\begin{bmatrix} \Delta \\ 2\omega \\ \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{xy} \\ e_{yx} \\ e_{yy} \end{bmatrix} \tag{18}$$

or simply

$$\boldsymbol{\xi} = \mathbf{Ae}\,. \tag{19}$$

The inverse transformation is easily show to be

$$\mathbf{e} = \mathbf{A}^{-1} \boldsymbol{\xi} \quad \text{or} \quad \begin{bmatrix} e_{xx} \\ e_{xy} \\ e_{yx} \\ e_{yy} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \Delta \\ 2\omega \\ \gamma_1 \\ \gamma_2 \end{bmatrix} \tag{20}$$

i.e., $2\mathbf{A}^{-1} = \mathbf{A}^T$. The deformation parameters with respect to the new frames are given by

$$\tilde{\boldsymbol{\xi}} = \mathbf{A} \tilde{\mathbf{e}} \tag{21}$$

Combining eq. (15), (20) and (21) it follows that

$$\widetilde{\xi} = \mathbf{AQA}^{-1} + \mathbf{Ad} \tag{22}$$

Carrying out the necessary matrix multiplications and rearranging terms the following transformation laws are derived for the deformation parameters

$$\begin{bmatrix} \widetilde{\Delta} \\ 2\widetilde{\omega} \end{bmatrix} = \begin{bmatrix} \cos(\theta'-\theta) & \sin(\theta'-\theta) \\ -\sin(\theta'-\theta) & \cos(\theta'-\theta) \end{bmatrix} \begin{bmatrix} \Delta \\ 2\omega \end{bmatrix} + \\ + \begin{bmatrix} \frac{\lambda'}{\lambda}\cos(\theta'-\theta) - 1 \\ -2\frac{\lambda'}{\lambda}\mu\sin(\theta'-\theta) \end{bmatrix} \tag{23}$$

$$\begin{bmatrix} \widetilde{\gamma}_1 \\ \widetilde{\gamma}_2 \end{bmatrix} = \frac{\lambda'}{\lambda} \begin{bmatrix} \cos(\theta'+\theta) & \sin(\theta'+\theta) \\ -\sin(\theta'+\theta) & \cos(\theta'+\theta) \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \tag{24}$$

Also of interest is the maximum shear strain $\gamma = \sqrt{\gamma_1^2 + \gamma_2^2}$, the maximum principal strain $e_{\max} = (\Delta + \gamma)/2$, the minimum principal strain $e_{\min} = (\Delta - \gamma)/2$ and the azimuth of the direction of maximum principal strain $\phi = \frac{1}{2}\arctan(-\gamma_2/\gamma_1)$.

It is easily verified that

$$\widetilde{\gamma} = \frac{\lambda'}{\lambda}\gamma$$

while for the angle ϕ we have

$$\tan 2\widetilde{\phi} = \frac{\tan(\theta+\theta') + \frac{-\gamma_2}{\gamma_1}}{1 - \tan(\theta+\theta')\frac{-\gamma_2}{\gamma_1}} = \frac{\tan(\theta+\theta') + \tan 2\phi}{1 - \tan(\theta+\theta')\tan 2\phi} =$$

$$= \tan(2\phi + \theta + \theta') \tag{25}$$

and consequently

$$\widetilde{\phi} = \phi + \frac{\theta + \theta'}{2} \tag{26}$$

3. Conclusions

The derived equations (23), (24), (25) and (26) describing how crustal deformation parameters are affected by changes in frame definitions, can now be used in order to conclude whether such parameters are estimable from geodetic observations.

Only those parameters, which are not affected by coordinate transformations leaving the available observations invariant, are estimable quantities.

All usual geodetic observations (angles, directions, distances) are invariant under changes of the origin and orientation of the reference frame. Additionally angle or direction observations are also invariant under changes of scale.

It can be seen right away that changes of frame origins do not affect any of the deformation parameters.

With the exception of the azimuth ϕ of maximum principal strain, all other parameters are affected by scale changes. This means that a necessary condition for their estimability is that scale is defined in both epochs under comparison, by including distance observations. The condition is also sufficient for the maximum shear strain γ, since this parameter is affected by scale changes only.

The same is true if distance observations are missing but the two coordinate sets are connected and referred to a common frame by assuming that the positions of some of the network stations are time invariant. In this case the arbitrarily introduced scale at one epoch is passes on to the other so that $\lambda = \lambda'$ and $\frac{\lambda'}{\lambda} = 1$.

If the two coordinate sets lack scale definition by independent distant observations and they are not connected by fixed stations, only the ratios between values of γ at different network triangles are estimable.

Concerning the effect of changes in frame orientation, one must again differentiate between the most general case where the two coordinate sets refer to independently defined frames and the special case where they ere connected through the time invariant positions of some of the stations. Of course such a choice must be justified, e.g. by statistical hypothesis tests (Koch and Fritsch, 1981).

If the two coordinate sets are connected any change of orientation and/or scale in one of them is passed on to the other, so that $\theta' = \theta$ and $\frac{\lambda'}{\lambda} = 1$. With these restrictions the transformation parameters give

$$\widetilde{\Delta} = \Delta$$

$$\widetilde{\omega} = \omega$$

$$\begin{bmatrix} \widetilde{\gamma}_1 \\ \widetilde{\gamma}_2 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

$$\widetilde{\gamma} = \gamma$$

$$\widetilde{\phi} = \phi + \theta\,. \tag{27}$$

It follows that dilatation Δ, rotation ω and maximum shear strain γ are in this case estimable quantities, while γ_1, γ_2 and ϕ are not. Also estimable are the maximum and minimum principal strains $e_{\max}$ and $e_{\min}$.

In geophysically active areas the assumption that some stations remain fixed might not be justified. In this case one might choose from the infinity of possible deformation parameters, those corresponding to an optimal fit of the two independent coordinate sets, in the sense that

$$\sum_i \left[(\widetilde{x}_i' - \widetilde{x}_i)^2 + (\widetilde{y}_i' - \widetilde{y}_i)^2\right] = \min \tag{28}$$

Such a fit can be achieved by an appropriate transformation of the coordinates of one of the two sets. Any subsequent change of orientation and scale of one of the sets must be accompanied by an identical change of the second set in order that the optimal fit is preserved. The situation is the same as when the two coordinate sets are connected by fixed stations. Among the deformation parameters referring to such an optimal fit the dilatation Δ_{opt} , the rotation ω_{opt} and the maximum shear strain γ_{opt} are uniquely defined and therefore estimable. The parameters γ_1 , γ_2 and ϕ still depend on the common frame chosen for both coordinate sets.

It must noted that the estimable parameters Δ_{opt} , ω_{opt} and γ_{opt} have no distinct physical meaning but their uniqueness is a consequence of the way they are defined. They describe what is left in the data after any relative rotation (and scale) has been filtered out. Once such relative rotation (and scale) cannot be determined, it is only reasonable that its influence is removed from the final results.

One way to obtain Δ_{opt} , ω_{opt} and γ_{opt} is by carrying out the adjustment of the geodetic observations at one epoch, using the adjusted station coordinates of the other as approximate values and utilizing the technique of inner adjustment. In this technique the coordinate frame is defined by means of inner constraints or by taking the pseudoinverse of the coefficient matrix of the normal equations (Brunner et al, 1981).

Another point that needs clarification is the meaning of the invariance. Here the numerical invariance of isolated scalar parameters under coordinate transformations is examined. Related but not identical, is the notion of invariance for parameters, which are the components of vectors and tensors. Tensorial invariance refers not to the numerical invariance of each of the tensor components, but to their transformation under frame changes with prescribed transformation laws, such that the physical object itself described by the tensor remains invariant and not its representation in one coordinate system or the other.

For connected coordinate sets ($\theta' = \theta$, $\lambda' = \lambda$) it is easy to show from eq. (14) that the elements of the Jacobian form a tensor transforming according to

$$\begin{bmatrix} \tilde{e}_{xx} & \tilde{e}_{xy} \\ \tilde{e}_{yx} & \tilde{e}_{yy} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} e_{xx} & e_{xy} \\ e_{yx} & e_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T \tag{29}$$

The same transformation law holds for the elements of the strain tensor

$$\begin{bmatrix} e_{xx} & \frac{1}{2}\gamma_2 \\ \frac{1}{2}\gamma_2 & e_{yy} \end{bmatrix}$$

Note that Jacobians do not generally have tensor character. In our case this is a consequence of the adopted model of a linear field of displacements in eq. (1).

The parameters γ_1 and γ_2 are not the components of a vector since they transform according to a rotation by 2θ and not by θ as the transformation law for vectors requires.

The direction of maximum principal strain is also a physical invariant for connected coordinate sets, since according to eq. (27) any change in the orientation of the reference frame θ is properly added to its azimuth ϕ .

In general our results here depend on the classical coordinate method used for deriving deformation parameters. Other techniques will be examined in a separate work.

Since deformation parameters depend on the spatial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ of the displacement components, they cannot be estimated in a strict sense from point-discrete information about displacements. The derived estimability arguments depend here, and in any case, on the physical validity of the model used for interpolating displacements, since the values of the above mentioned derivatives depend on the chosen interpolation scheme. In the classical coordinate method examined here a piecewise linear interpolation model is used the displacement components, so that their spatial derivatives and the derived deformation parameters have different but constant values within each triangle (step functions).

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