A comparison of existing and new methods for the analysis of nonlinear variations in coordinate time series

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Comparison of four methods for the analysis of the GPS position time series:

Existing:

**STD** - Standard least-square fitting of periodic function with annual and semiannual frequencies

**SSA** - Singular Spectrum Analysis

New:

**MOD** - Least-Square fitting of a amplitude and phase modulated annual signal using continuous piecewise-linear amplitude and phase models

**ECS** - Least-Square fitting of equidistant cubic splines
Objectives:

- Separate physically meaningful signals from noise in coordinate time series

- Introduce easy-to-use models for future update of the ITRF to include also nonlinear coordinate variation terms
Coordinate Interpolation method:

SSA
Singular Spectral Analysis
**SSA - Singular Spectrum Analysis**

Time series: \( x_1, x_2, \ldots, x_N \)

Lag–window size: \( M \)

\[
C = \begin{bmatrix}
c_0 & c_1 & c_2 & c_3 & \cdots & c_{M-2} & c_{M-1} \\
c_1 & c_0 & c_1 & c_2 & \cdots & c_{M-3} & c_{M-2} \\
c_2 & c_1 & c_0 & c_1 & \cdots & c_{M-4} & c_{M-3} \\
c_3 & c_2 & c_1 & c_0 & \cdots & c_{M-5} & c_{M-4} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
c_{M-2} & c_{M-3} & c_{M-4} & c_{M-5} & \cdots & c_0 & c_1 \\
c_{M-1} & c_{M-2} & c_{M-3} & c_{M-4} & \cdots & c_1 & c_0
\end{bmatrix}
\]

\[
c_i = \frac{1}{N-i} \sum_{k=1}^{N-i} x_k x_{k+i}, \quad 0 \leq i \leq M - 1
\]

\( a_i \): principal components, \( E \): eigenvectors

\[
ad_i^k = \sum_{j=1}^{M} x_{i+j} E_j^k, \quad 0 \leq i \leq N - M
\]

Reconstruction of coordinates:

\[
x_i^k = \frac{1}{i} \sum_{j=1}^{i} a_{i-j}^k E_j^k, \quad 1 \leq i \leq M - 1
\]

\[
x_i^k = \frac{1}{M} \sum_{j=1}^{M} a_{i-j}^k E_j^k, \quad M \leq i \leq N - M + 1
\]

\[
x_i^k = \frac{1}{N - i + 1} \sum_{j=i-N}^{i-M} a_{i-j}^k E_j^k, \quad N - M + 2 \leq i \leq N
\]
Coordinate Interpolation method:

MOD

Amplitude and phase modulation of annual carrier by piecewise linear functions
MOD - Modeling of coordinate time series by amplitude and phase modulation

\[ x(t) = A(t) \cos \left[ 2\pi \omega_0 t - \varphi(t) \right] \]

\[ \omega_0 = \text{annual carrier frequency} \quad A(t) = \text{amplitude} \quad \varphi(t) = \text{phase} \]

\[ t_k \leq t \leq t_{k-1} : \quad A(t) = A_{k-1} + \frac{A_k - A_{k-1}}{t_k - t_{k-1}} (t - t_{k-1}) \]

\[ \varphi(t) = \varphi_{k-1} + \frac{\varphi_k - \varphi_{k-1}}{t_k - t_{k-1}} (t - t_{k-1}) \]
MOD – Modeling of coordinate time series by amplitude and phase modulation

\( t_k \leq t \leq t_{k-1} : \)

\[
x'(t_i) = x(t_i) + e_i = \left[ A_{k-1} + \frac{A_k - A_{k-1}}{t_k - t_{k-1}} (t_i - t_{k-1}) \right] \cos \left[ 2\pi \omega_i t_i - \phi_{k-1} - \frac{\phi_k - \phi_{k-1}}{t_k - t_{k-1}} (t_i - t_{k-1}) \right] + e_i
\]

\[
y = f(x) + e \quad \text{Linearization} \quad b = y - f(x_0) = \frac{\partial f}{\partial x_0} (x - x_0) + e = A\delta x + e
\]

\( e^T Pe = \min \quad \text{leads to large variation of } A_k \text{ and } \phi_k ! \)

**Instead:** Tikhonov regularization solution: \( e^T Pe + x^T Wx = \min \)

\[
x^T Wx = \rho_A \sum_{k=1}^{n} (A_k - A_{k-1})^2 + \rho_\phi \sum_{k=1}^{n} (\phi_k - \phi_{k-1})^2
\]

\( \rho_A, \rho_\phi \): regularization parameters
Coordinate Interpolation method:

ECS
Equidistant Cubic Splines
Version 1: Spline in terms of second derivatives

\[ x(t) = S_i(t) = g_{i-1} + \left[ \frac{g_i - g_{i-1}}{h} - \frac{2M_{i-1} + M_i}{6} \right] (t - \tau_{i-1}) + \frac{M_{i-1}}{2} (t - \tau_{i-1})^2 + \frac{M_i - M_{i-1}}{6h} (t - \tau_{i-1})^3 \]

Constraints for equality of first derivatives

\[ M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2} (g_{i-1} - 2g_i + g_{i+1}) \]

Constraints for equality of first derivatives:

\[ M_i = \frac{d^2 S_i}{dt^2}(\tau_i) = \frac{d^2 S_{i+1}}{dt^2}(\tau_i) \]
Version 2: Spline in terms of first derivatives

\[
x(t) = S_i(t) = g_{i-1} + N_{i-1}(t - \tau_{i-1}) + \left[3 \frac{g_i - g_{i-1}}{h^2} - \frac{2N_{i-1} + N_i}{h}\right](t - \tau_{i-1})^2 + \left[-2 \frac{g_i - g_{i-1}}{h^3} + \frac{N_{i-1} + N_i}{h^2}\right](t - \tau_{i-1})^3
\]

\[
N_i = \frac{dS_i}{dt}(\tau_i) = \frac{dS_{i+1}}{dt}(\tau_i)
\]

\[
M_i = \frac{d^2S_i}{dt^2}(\tau_i) = \frac{d^2S_{i+1}}{dt^2}(\tau_i)
\]

Constraints for equality of second derivatives

\[
hN_{i-1} + 4hN_i + hN_{i+1} = 3(g_{i+1} - g_{i-1})
\]
ECS - Modeling of coordinate time series by equidistant cubic splines

Version 1: Spline in terms of second derivatives

\[ t_{i-1} \leq t_k \leq t_i : \]

\[ x'(t_k) = x(t_k) + e_k = S_i(t_k) + e_k = g_{i-1} + \left[ \frac{g_i - g_{i-1}}{h} - \frac{2M_{i-1} + M_i}{6h} \right] (t_k - \tau_{i-1}) + \frac{M_{i-1}}{2} (t_k - \tau_{i-1})^2 + \frac{M_i - M_{i-1}}{6h} (t_k - \tau_{i-1})^3 + e_k \]

\[ b = Ax + e \]

\[ x = \begin{bmatrix} g_0 & \cdots & g_n & M_0 & \cdots & M_n \end{bmatrix}^T \]

Constraints:

\[ M_{i-1} + 4M_i + M_{i+1} - \frac{6}{h^2} (g_{i-1} - 2g_i + g_{i+1}) = 0 \]

\[ e^T Pe = \min \quad \text{Subject to} \quad Cx = 0 \]

Least squares with constraints

\[ x(t) \]
**ECS - Modeling of coordinate time series by equidistant cubic splines**

**Version 2: Spline in terms of first derivatives**

\[
x'(t_k) = S_i(t_k) + e_k = g_{i-1} + N_{i-1}(t_k - \tau_{i-1}) + \left[ 3 \frac{g_i - g_{i-1}}{h^2} - \frac{2N_{i-1} + N_i}{h} \right] (t_k - \tau_{i-1})^2 + \left[ -2 \frac{g_i - g_{i-1}}{h^3} + \frac{N_{i-1} + N_i}{h^2} \right] (t_k - \tau_{i-1})^3 + e_k
\]

**Constraints:**

\[
hN_{i-1} + 4hN_i + hN_{i+1} - 3(g_{i+1} - g_{i-1}) = 0
\]

**Least squares with constraints**

\[ e^T Pe = \min \quad \text{Subject to} \quad Cx = 0 \]
\[ \Delta E_i, \Delta N_i = \text{East and North coordinate residuals after linear trend removal} \]

Sample dispersion matrix:
\[ S = \begin{bmatrix} s_E^2 & s_{EN} \\ s_{EN} & s_N^2 \end{bmatrix} = \frac{1}{N} \sum_i \begin{bmatrix} (\Delta E_i - m_E)^2 & (\Delta E_i - m_E)(\Delta N_i - m_N) \\ (\Delta E_i - m_E)(\Delta N_i - m_N) & (\Delta N_i - m_N)^2 \end{bmatrix} \]

\[ m_E = \frac{1}{N} \sum_i \Delta E_i, \quad m_N = \frac{1}{N} \sum_i \Delta N_i \]

Diagonalization:
\[ S = \begin{bmatrix} s_E^2 & s_{EN} \\ s_{EN} & s_N^2 \end{bmatrix} = R(-\theta) \begin{bmatrix} s_{\text{max}}^2 & 0 \\ 0 & s_{\text{min}}^2 \end{bmatrix} R(\theta) \]

Principal horizontal components:
\[ \begin{bmatrix} H_i^{\text{max}} \\ H_i^{\text{min}} \end{bmatrix} = R(\theta) \begin{bmatrix} \Delta E_i \\ \Delta N_i \end{bmatrix} \]

Identification of the horizontal direction \( \theta \) with maximal coordinate variation!
Replacement of East, North coordinates with principal horizontal components
17 GNSS stations

Time span 2007-2014
Definition of the reference system

Alignment to a global network (ITRF, IGS):
- Best for global or large scale tectonic studies
- Coordinate time series reflect station motion at global scale

Regional stacking:
- Best for regional scale tectonic studies
- Removes from coordinate series contributions from
  ▪ motion of the region as whole with respect to global system
  ▪ systematic errors in GNSS data
Time-series of Transformation Parameters
(Stacking procedure on 17 stations)
Comparison of Coordinate Interpolation Methods

GPS station: RLSO
(NW Peloponnese)
Coordinate Time Series
Interpolated vs original nonlinear terms

max horizontal (x 2)

STD

SSA

MOD

ECS

[Images of graphs showing time series data for different categories (STD, SSA, MOD, ECS) with time on the x-axis (2009 to 2014) and coordinate values on the y-axis (mm).]
GPS station: RLSO (Achaia, Northwestern Peloponnesus)

Spectrum Analysis

Covariance Functions
Comparison of Coordinate Interpolation Methods

GPS station: KASI (Corfu, Ionian Sea)
Coordinate Time Series
Interpolated vs original nonlinear terms

max horizontal

-10
-5
0
5
10

[mm]

STD

-10
-5
0
5
10

[mm]

SSA

-10
-5
0
5
10

[mm]

MOD

-10
-5
0
5
10

[mm]

ECS

-10
-5
0
5
10

[mm]

vertical

-10
-5
0
5
10

[mm]

Time

2009 2010 2011 2012 2013 2014
GPS station: KASI (Corfu, Ionian Sea)
RMS of residuals after interpolation for all 17 stations (maximal horizontal component)
RMS of residuals after interpolation for all 17 stations (vertical component)
NYA1 - Ny-Alesund, Norway

QAQ1 - Qaqortoq/Julianehaab, Southern Greenland
Coordinate Time Series
Interpolated vs original nonlinear terms

GPS station: NYA1 (Ny-Alesund, Norway)
Coordinate Time Series
Interpolated vs original nonlinear terms

GPS station: NYA1 (Ny-Alesund, Norway)
Spectral Analysis
Interpolated vs original nonlinear terms

GPS station: NYA1 (Ny-Alesund, Norway)
Covariance functions
Interpolated vs original nonlinear terms

GPS station: NYA1 (Ny-Alesund, Norway)

C(0) = 3.52 mm$^2$

C(0) = 57.25 mm$^2$

max horizontal ORIG
STD
SSA
MOD
ECS

vertical ORIG
STD
SSA
MOD
ECS

[Graph showing covariance functions for max horizontal and vertical over weeks]
GPS station: QAQ1 (Qaqortoq/Julianehaab, Southern Greenland)
GPS station: QAQ1 (Qaqortoq/Julianehaab, Southern Greenland)
GPS station: QAQ1 (Qaqortoq/Julianehaab, Southern Greenland)

Spectral Analysis
Interpolated vs original nonlinear terms

max horizontal:

vertical:

Cycles/Year

Cycles/Year

STD

SSA

MOD

ECS

STD

SSA

MOD

ECS

ORIG

ORIG
GPS station: QAQ1 (Qaqortoq/Julianehaab, Southern Greenland)

- Covariance functions
- Interpolated vs original nonlinear terms

C(0) = 6.19 mm$^2$

- horizontal
- vertical

Max horizontal: ORIG

STD
SSA
MOD
ECS

C(0) = 59.06 mm$^2$

STD
SSA
MOD
ECS

STD
ECS
SSA
MOD
ORIG
The “adaptability” issue for a coordinate interpolation-smoothing method

A good method should be adaptive enough to recover the geophysical signal and at the same time not too adaptive to interpret noise as signal.

Optimal separation is not possible in an undisputable way because the noise characteristics are not fully known. This is mainly due to colored noise (effect of systematic errors) and biases, while only the white and zero mean part of the noise can be easily removed.

Within each method various degrees of adaptability are possible by tuning relevant parameters:

STD: frequencies other than annual and semi-annual
SSA: length of lag window
MOD: length of interval of linear amplitude/phase – regularization parameter values
ECS: length of interval for splines
The suitability of a coordinate interpolation-smoothing method for geophysical interpretation

STD: Standard use of annual, semiannual and other frequencies

Limited interpretation: Amplitude of included frequencies

MOD: Amplitude and phase modulation by piecewise linear functions

Most challenging for geophysical interpretation.

Variation in amplitude values reflects magnitude of meteorological & hydrological effects. Variation in phase values reflects different occurrence times of such effects.

In plane words:

Not all winters/summers are severe to the same degree each year!
Severe phenomena do not occur at the same dates every year!
The suitability of the nodulation method for geophysical interpretation - Examples

GPS station: KASI (Corfu island, Ionian Sea) - interval 3 months

Amplitude

Phase

max horizontal

max horizontal

vertical

vertical

Time

Time
The suitability of the nodulation method for geophysical interpretation - Examples

GPS station: RLSO (Achaia, Northwestern Peloponnesus) - interval 3 months

Amplitude

Phase

max horizontal

max horizontal

vertical

vertical

Time

Time

[mm]

[mm]

[°]

[°]
The suitability of the nodulation method for geophysical interpretation - Examples

GPS station: NYA1 (Ny-Alesund, Norway) - interval 3 months

**Amplitude**

- max horizontal
- vertical

**Phase**

- max horizontal
- vertical
The suitability of the nodulation method for geophysical interpretation - Examples

GPS station: QAQ1 (Qaqortoq/Julianeaab, Southern Greenland) - interval 3 months

**Amplitude**

- **max horizontal**

- **vertical**

**Phase**

- **max horizontal**

- **vertical**
Conclusions:

Standard annual and semiannual frequencies:

ANALYTIC MODEL, INSUFFICIENT ADAPTIVITY
GEOPHYSICAL INTERPRETATION?

Singular Spectrum Analysis:

HIGH ADAPTIVITY, NO ANALYTIC MODEL

Amplitude & phase modulation:

GOOD ADAPTIVITY
ANALYTIC MODEL
GEOPHYSICAL INTERPRETATION!
NEED FOR REGULARIZATION

Equidistant cubic splines:

HIGH ADAPTIVITY
ANALYTIC MODEL
Thanks for your attention!