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THE BRUNS FORMULA IN THREE DIMENSIONS

Abstract

The Bruns formula is generalized to three dimensions with the derivation of equations expressing the height anomaly vector or the geoid undulation vector as a function of the disturbing gravity potential and its spatial derivatives. It is shown that the usual scalar Bruns formula provides not the separation along the normal to the reference ellipsoid but the component of the relevant spatial separation along the local direction of normal gravity. The above results which hold for any type of normal potential are specialized for the usual Somigliana-Pizzetti normal field so that the components of the geoid undulation vector are expressed as functions of the parameters of the reference ellipsoid, the disturbing potential and its spatial derivatives with respect to three types of curvilinear coordinates, ellipsoidal, geodetic and spherical. Finally the components of the geoid undulation vector are related to the deflections of the vertical in a spherical approximation.

1. Introduction

A well known equation of physical geodesy is the famous Bruns formula (Heiskanen and Moritz, 1967, p. 85)

$$\zeta = \frac{T}{\gamma} \tag{1}$$

which relates the geoid undulation ζ to the disturbing potential T. This formula for the determination of the shape of the geoid is essentially the last step in the solution of a non-linear free geodetic boundary value problem, where gravity is given on the geoid, an equipotential surface of unknown shape. This boundary value problem is converted into the familiar Stokes'problem on the known surface of the reference ellipsoid (approximated by a sphere) with the help of a linearization procedure. Once the disturbing potential T is determined from the solution of Stokes'problem the Bruns formula can be used for the determination of the separation between the ellipsoid and the geoid, and the shape of the original boundary surface, the geoid, is finally determined.

The same formula applies also to Molodensky's problem (Heiskanen and Moritz, 1967, p. 293) where ζ represents the height anomaly, i.e., the separation between the earth surface and the telluroid (or equivalently between the quasi geoid and the ellipsoid). In both cases ζ is a scalar quantity which represents a separation along the direction perpendicular to the surface of the reference ellipsoid.

Krarup (1973) has presented a rigorous linearization of the geodetic boundary value problem (Molodensky's problem) where every point P on the earth surface is mapped into a point Q of the telluroid by means of a so called telluroid mapping. The scalar ζ is in this case replaced by a three-dimensional vector $\boldsymbol{\zeta} = \mathbf{x}_P - \mathbf{x}_Q$, the "height anomaly vector", which represents the spatial separation between the points P and Q with position vectors $\mathbf{x}_P$ and $\mathbf{x}_Q$ respectively. Krarup's linearization applies also to Stokes'problem where P is a point on the geoid, Q is its image on the reference ellipsoid and $\boldsymbol{\zeta}$ is the "geoid undulation vector".

Here we shall present formulas expressing the vector $\boldsymbol{\zeta}$ as a function of T and its gradient $\delta\mathbf{g} = grad\,T$, which allow the determination of the shape of the earth surface or of the geoid, once the disturbing potential T has been determined in exterior space from the solution of the linearized Molodensky's problem or of the Stokes' problem, respectively. These formulas are in fact a generalization of the classical scalar Bruns formula in three dimensions.

A conceptually different three-dimensional generalization of Bruns formula has already been proposed by Grafarend (1980). He implicitly views the Bruns formula as a finite approximation to the differential transformation

$$dh = -\frac{1}{g}\,dW \tag{2}$$

between a gravity potential differential dW and an orthometric height differential dh and he introduces similar three-dimensional transformations which he calls Bruns transformations, between the differentials of three gravity related quantities, e.g. Λ, Φ, g (curvilinear coordinates in gravity space) or Λ, Φ, W (natural or Marussi coordinates), and the differentials of three coordinates in geometry space.

Our point of view here is quite different from that of Grafarend (1980, 1983) and our results are different for the most relevant case of the Marussi telluroid mapping. For the gravimetric telluroid mapping however, we obtain essentially similar results, i.e., a "Bruns transformation" between $\boldsymbol{\zeta}$ and the gradient of T.

Here all vectors $\vec{x}$ are represented by the 3×1 column matrices $\mathbf{x}$ of their components with respect to the conventional earth-fixed geocentric reference frame, unless otherwise specified.

2. Derivation of the Three-Dimensional Bruns Formula

According to Krarup (1973), the known gravity vector $\mathbf{g}$ and gravity potential W at any point $\mathbf{x}_P$ on the earth surface, are linearized with the help of normal values γ and U at a point $\mathbf{x}_Q$ on the telluroid (Moritz, 1980, p. 341), so that

$$\Delta\mathbf{g} = \mathbf{g}_P - \boldsymbol{\gamma}_Q = \mathbf{M}\boldsymbol{\zeta} + grad\,T \,, \tag{3}$$

$$\Delta W = W_P - U_Q = \boldsymbol{\gamma}^T\boldsymbol{\zeta} + T \tag{4}$$

where

$$\mathit{grad}\,\mathrm{T} = \left(\frac{\partial \mathrm{T}}{\partial \mathbf{x}}\right)^{\mathrm{T}} \tag{5}$$

$$\mathbf{M} = \frac{\partial \boldsymbol{\gamma}}{\partial \mathbf{x}} \tag{6}$$

$$\boldsymbol{\zeta} = \mathbf{x}_{\mathrm{P}} - \mathbf{x}_{\mathrm{Q}} \,. \tag{7}$$

All quantities in $\mathbf{M}$, $\boldsymbol{\zeta}$, $\boldsymbol{\gamma}$, T and *grad* T are considered evaluated at point Q on the telluroid. The position vector $\mathbf{x}_{\mathrm{Q}}$ of point Q is determined from the observed quantities $\mathbf{g}$, W at the point P with unknown position vector $\mathbf{x}_{\mathrm{P}}$, by means of one of two telluroid mappings, either the gravimetric mapping

$$\mathbf{g}(\mathbf{x}_{\mathrm{P}}) = \boldsymbol{\gamma}(\mathbf{x}_{\mathrm{Q}}) \,, \tag{8}$$

or the Marussi mapping

$$\frac{1}{|\mathbf{g}(\mathbf{x}_{\mathrm{P}})|}\,\mathbf{g}(\mathbf{x}_{\mathrm{P}}) = \frac{1}{|\boldsymbol{\gamma}(\mathbf{x}_{\mathrm{Q}})|}\,\boldsymbol{\gamma}(\mathbf{x}_{\mathrm{Q}}) \tag{9a}$$

$$\mathrm{W}(\mathbf{x}_{\mathrm{P}}) = \mathrm{U}(\mathbf{x}_{\mathrm{Q}}) \,. \tag{9b}$$

Solving equation (3) for $\boldsymbol{\zeta}$ we obtain

$$\boldsymbol{\zeta} = \mathbf{M}^{-1}\,\Delta\mathbf{g} - \mathbf{M}^{-1}\,\mathit{grad}\,\mathrm{T} \tag{10}$$

and replacing in equation (4) the boundary condition

$$\mathrm{T} - \boldsymbol{\gamma}^{\mathrm{T}}\,\mathbf{M}^{-1}\,\mathit{grad}\,\mathrm{T} = \Delta\mathrm{W} - \boldsymbol{\gamma}^{\mathrm{T}}\,\mathbf{M}^{-1}\,\Delta\mathbf{g} \tag{11}$$

on the telluroid is derived.

Equation (10) is not the equation sought since $\boldsymbol{\zeta}$ depends on the gravity anomaly vector $\Delta\mathbf{g}$. This equation, which differs from our concept of Bruns formula as explained in the introduction, is well known from the literature, see, e.g., Grafarend (1978), Sansò (1981), Bode and Grafarend (1981), Knickmeyer (1984, 1986). Spherical approximations based on the central normal field $\mathrm{U} = \mathrm{kM}/\mathrm{r}$ can be found in Grafarend (1983) and Grafarend et al., (1985). Grafarend (1978a) includes a term accounting for parameters of the geodetic datum. Knickmeyer (1984, 1986) gives an improvement of equation (10) which includes the effect of second order terms.

For the elimination of $\Delta\mathbf{g}$ we note that for both the gravimetric and the Marussi telluroid mappings we have

$$\Delta\mathbf{g} = \mathbf{g} - \boldsymbol{\gamma} = \frac{\mathrm{g}}{\gamma}\boldsymbol{\gamma} - \boldsymbol{\gamma} = \frac{\mathrm{g} - \gamma}{\gamma}\,\boldsymbol{\gamma} = \frac{\Delta\mathrm{g}}{\gamma}\,\boldsymbol{\gamma} \tag{12}$$

where $\Delta\mathrm{g}$ is the usual scalar gravity anomaly. Now equations (10) and (11) become

$$\boldsymbol{\zeta} = \frac{\Delta\mathrm{g}}{\gamma}\mathbf{M}^{-1}\,\boldsymbol{\gamma} - \mathbf{M}^{-1}\,\mathit{grad}\,\mathrm{T} \tag{13}$$

$$T - \gamma^T M^{-1} \, grad\, T = \Delta W - \frac{\Delta g}{\gamma} \gamma^T M^{-1} \gamma \, . \tag{14}$$

For the Marussi telluroid mapping $\Delta W = 0$ and solving equation (14) for Δg and substituting in equation (13), the three-dimensional Bruns formula is obtained

$$\zeta = \left(\frac{1}{\gamma^T M^{-1} \gamma} M^{-1} \gamma \gamma^T M^{-1} - M^{-1} \right) \delta g - \frac{T}{\gamma^T M^{-1} \gamma} M^{-1} \gamma \tag{15}$$

where

$$\delta g = grad\, T \tag{16}$$

is the gravity disturbance vector.

In the case of the gravimetric telluroid mapping we have $\Delta \mathbf{g} = \mathbf{0}$ and equation (10) becomes directly the (gravimetric) three-dimensional Bruns formula

$$\zeta_G = - M^{-1} \, \delta g = - \frac{\partial x}{\partial \gamma} \delta g \, . \tag{17}$$

This is also a well known equation from the literature, see, e.g., Grafarend (1978b), Sansò (1980, 1981).

An alternative representation of the three-dimensional Bruns formula (15) can be obtained with respect to the local normal astronomic frame, which is an orthonormal frame with its third axis in the direction of $-\gamma$ (Zenith), its second axis in the plane of $-\gamma$ and the third axis of the geocentric frame (North) and its first axis completing the right-handed triad (East). The matrix ζ^* of the components of the height anomaly vector $\vec{\zeta}$ in the local normal astronomic frame are given by

$$\zeta^* = A \, \zeta \tag{18}$$

where

$$A = \frac{\partial x^*}{\partial x} = R_1 (90° - \varphi) \, R_3 (90° + \lambda) \tag{19}$$

is the orthogonal matrix of rotation from geocentric coordinates $\mathbf{x}$ to local normal astronomic coordinates $\mathbf{x}^*$, λ is the normal longitude and φ the normal latitude. For the so called Marussi matrix $\mathbf{M}$ we have

$$M = \frac{\partial \gamma}{\partial x} = \frac{\partial \gamma}{\partial \gamma^*} \frac{\partial \gamma^*}{\partial x^*} \frac{\partial x^*}{\partial x} = A^T E A \tag{20}$$

where $\mathbf{E}$ is the normal Eötvos matrix of second order derivatives of the normal potential U

$$E = \frac{\partial \gamma^*}{\partial x^*} = \frac{\partial}{\partial x^*} \left(\frac{\partial U}{\partial x^*} \right)^T , \qquad \left(E_{ij} = \frac{\partial^2 U}{\partial x_i^* \, \partial x_j^*} \right) \, . \tag{21}$$

Furthermore

$$\gamma^* = \mathbf{A}\,\gamma = [0\ \ 0\ -\gamma]^T = -\gamma\,\mathbf{i}_3 \tag{22}$$

where $\mathbf{i}_3 = [0\ \ 0\ \ 1]^T$ and

$$\gamma = -\gamma\,\mathbf{A}^T\,\mathbf{i}_3\ . \tag{23}$$

Replacing **M** from equation (20) and γ from (23) into equation (15) and taking (18) into account, the height anomaly vector with respect to the local normal astronomic frame becomes

$$\zeta^* = \left[\frac{1}{\mathbf{i}_3^T\,\mathbf{E}^{-1}\,\mathbf{i}_3}\,\mathbf{E}^{-1}\,\mathbf{i}_3\,\mathbf{i}_3^T - \mathbf{E}^{-1}\right]\delta\mathbf{g}^* + \frac{1}{\mathbf{i}_3^T\,\mathbf{E}^{-1}\,\mathbf{i}_3}\,\frac{T}{\gamma}\,\mathbf{E}^{-1}\,\mathbf{i}_3 \tag{24}$$

where

$$\delta\mathbf{g}^* = \mathbf{A}\ \delta\mathbf{g} = \mathbf{A}\ grad\ T \tag{25}$$

is the gravity disturbance vector with respect to the local normal astronomic frame.

3. The Height Anomaly Vector in Terms of Curvatures and Torsion

The elements of the normal Eötvos matrix are related to differential geometric characteristics of the gravity field according to (Marussi, 1951)

$$\mathbf{E} = -\gamma\begin{bmatrix} k_E & t_E & \kappa_E \\ t_E & k_N & \kappa_N \\ \kappa_E & \kappa_N & \dfrac{\gamma_z}{\gamma} \end{bmatrix}\ . \tag{26}$$

k_E and k_N are the curvatures of the normal sections of the normal equipotential surface in the east and north direction respectively, κ_E and κ_N are the east and north components respectively of the curvature vector of the normal plumb line and γ_z is the derivative of the normal gravity γ in the zenith direction. t_E is the relative torsion in the east direction (Guggenheimer, 1963, p. 207) which is usually called geodesic torsion in the geodetic literature. Usually, (see, e.g., Dermanis, 1984, Dermanis and Livieratos, 1984) a zero subscript or upperscript distinguishes the quantities k_E , k_N , t_E , κ_E , κ_N of the normal gravity field from those of the actual field. Such a distinction is not necessary here since we only deal with the normal potential.

Analytical inversion of the matrix in equation (26) gives

$$\mathbf{E}^{-1} = -\frac{1}{\gamma D}\begin{bmatrix} k_N \frac{\gamma_z}{\gamma} - \kappa_N^2 & \kappa_E \kappa_N - t_E \frac{\gamma_z}{\gamma} & \gamma K H_1 \\ \kappa_E \kappa_N - t_E \frac{\gamma_z}{\gamma} & k_E \frac{\gamma_z}{\gamma} - \kappa_E^2 & \gamma K H_2 \\ \gamma K H_1 & \gamma K H_2 & K \end{bmatrix} \tag{27}$$

where

$$K = k_E \, k_N - t_E^2 \tag{28}$$

is the Gaussian curvature of the normal equipotential surface,

$$H_1 = \frac{t_E \, \kappa_N - k_N \, \kappa_E}{\gamma K} \tag{29}$$

$$H_2 = \frac{t_E \, \kappa_E - k_E \, \kappa_N}{\gamma K} \tag{30}$$

$$D = K\left[\frac{\gamma_z}{\gamma} + \gamma \, (\kappa_E \, H_1 + \kappa_N H_2)\right] . \tag{31}$$

Replacing $\mathbf{E}^{-1}$ from equation (27) into (24) and setting

$$\zeta^* = [\zeta_E \quad \zeta_N \quad \zeta_Z]^T \tag{32}$$

$$\delta \mathbf{g}^* = [\delta g_E \quad \delta g_N \quad \delta g_Z]^T , \tag{33}$$

after some rather tedious algebraic calculations the following east, north and zenith components of the height anomaly vector are derived

$$\zeta_E = \frac{k_N}{\gamma K} \delta g_E - \frac{t_E}{\gamma K} \delta g_N + \frac{t_E \, \kappa_N - k_N \, \kappa_E}{K} \frac{T}{\gamma} \tag{34a}$$

$$\zeta_N = -\frac{t_E}{\gamma K} \delta g_E + \frac{k_E}{\gamma K} \delta g_N + \frac{t_E \, \kappa_E - k_E \, \kappa_N}{K} \frac{T}{\gamma} \tag{34b}$$

$$\zeta_Z = \frac{T}{\gamma} . \tag{34c}$$

From the above equations the following conclusions are immediately drawn :

a) The usual Bruns formula gives not a separation along the normal to the reference ellipsoid, but the projection of the relevent separation in space on the direction of the local normal gravity vector.

b) The height anomaly vector is independent of the zenith component of the gradient of the disturbing potential.

c) The height anomaly vector and the gradient of the disturbing potential are interrelated only through their local normal horizontal components, i.e., their components in the plane perpendicular to the local direction of the normal gravity vector.

It must be emphasized that equations (34a), (34b), (34c) and the above conclusions are completely independent of the type of normal potential used for the linearization.

The coefficients in equations (34a), (34b), (34c) can be computed at any specific point from the known normal potential U. If U is expressed as a function $U(\mathbf{q})$ of geometric curvilinear coordinates $\mathbf{q} = [q_1\ q_2\ q_3]^T$, the elements of the Eötvos matrix can be computed by first performing the differentiations in the following equation

$$\mathbf{E} = \mathbf{A}\ \mathbf{M}\ \mathbf{A}^T = \mathbf{A}\ \frac{\partial}{\partial \mathbf{q}} \left[\left(\frac{\partial \mathbf{q}}{\partial \mathbf{x}} \right)^T \left(\frac{\partial U}{\partial \mathbf{q}} \right)^T \right] \frac{\partial \mathbf{q}}{\partial \mathbf{x}}\ \mathbf{A}^T \ . \tag{35}$$

From the elements of E, the parameters k_E, k_N, t_E, κ_E, κ_N, γ_z/γ appearing in equations (34a), (34b), (34c) are defined according to equation (26). The components δg_E, δg_N, δg_Z of *grad* T can be computed from

$$\delta \mathbf{g}^* = \mathbf{A} \left(\frac{\partial T}{\partial \mathbf{x}} \right)^T = \mathbf{A} \left(\frac{\partial \mathbf{q}}{\partial \mathbf{x}} \right)^T \left(\frac{\partial T}{\partial \mathbf{q}} \right)^T \ . \tag{36}$$

Equations expressing the elements of **E** in terms of first and second order partial derivatives of U with respect to curvilinear coordinates can be found in Dermanis (1984) for ellipsoidal coordinates λ, β, u, in Tscherning (1976a) for spherical coordinates λ, $\bar{\varphi}$, r and in Blaha (1980) for both spherical and the spheroidal coordinates of Hotine (1969). The first and second order derivatives of U with respect to ellipsoidal coordinates can be found in Danas and Dermanis (1983) for the case where U is the familiar Somigliana – Pizzetti normal field. When U is a truncated expansion in spherical harmonics the required derivatives with respect to spherical coordinates can be evaluated with algorithms given by Tscherning (1976a, 1976b).

The coefficients of $\frac{T}{\gamma}$ in equations (34a) and (34b) have the following representation according to Hotine (1969), p. 75 :

$$H_1 = -\frac{1}{\gamma^2 \cos\varphi} \frac{\partial \gamma}{\partial \lambda} = -\frac{\gamma_\lambda}{\gamma^2 \cos\varphi} \tag{37a}$$

$$H_2 = -\frac{1}{\gamma^2} \frac{\partial \gamma}{\partial \varphi} = -\frac{\gamma_\varphi}{\gamma^2} \tag{37b}$$

where γ_λ and γ_φ are the derivatives of γ considered as a function $\gamma(\lambda, \varphi, U)$ of the normal natural coordinates λ, φ, U. Taking equations (29), (30), (37a) and (37b) into account, equations (34a) and (34b) take the alternative form

$$\zeta_E = \frac{k_N}{\gamma K}\,\delta g_E - \frac{t_E}{\gamma K}\,\delta g_N - \frac{\gamma_\lambda}{\gamma \cos\varphi}\,\frac{T}{\gamma} \tag{38a}$$

$$\zeta_N = -\frac{t_E}{\gamma K}\,\delta g_E + \frac{k_E}{\gamma K}\,\delta g_N - \frac{\gamma_\varphi}{\gamma}\,\frac{T}{\gamma}\,. \tag{38b}$$

When the normal potential is independent of longitude λ, e.g., the Somigliana–Pizzetti normal field, both t_E and κ_E vanish (Marussi, 1950, Bocchio, 1977). Setting $t_E = \kappa_E = 0$ and $K = k_N k_E$ in equations (34a), (34b), (34c) we obtain

$$\zeta_E = \frac{1}{\gamma k_E}\,\delta g_E \tag{39a}$$

$$\zeta_N = \frac{1}{\gamma k_N}\,\delta g_N - \frac{\kappa_N}{k_N}\,\frac{T}{\gamma} \tag{39b}$$

$$\zeta_Z = \frac{T}{\gamma} \tag{39c}$$

In this particular case the east and north components of ζ depend only on the east and north components respectively of the gradient of the disturbing potential.

4. The Geoid Undulation Vector

When the linearization of the geodetic boundary value problem is carried out for data **g** , W which are somehow reduced to the geoid, the already derived equations hold for the geoid undulation vector, with all relevant parameters refering to points on the reference ellipsoid. The latter is the image of the geoid under the Marussi telluroid mapping when the Somigliana–Pizzetti normal field is used. On the ellipsoid we have (Dermanis, 1984)

$$k_E = \frac{1}{N} \tag{40}$$

$$k_N = \frac{1}{M} \tag{41}$$

$$\kappa_N = -\frac{\sin 2\beta}{M}\,\rho \tag{42a}$$

with

$$\rho = \frac{e^2 a}{2b} - \frac{N\gamma_e}{a\gamma}\,(f^* + f) \tag{42b}$$

where N and M are the transverse and the meridional radii of curvature of the reference ellipsoid, respectively, a and b are its major and minor semiaxes, e is its first eccentricity, γ_e is the equatorial value of gravity, f is the geometric and f* is the dynamic flattening.

Further evaluation of the components of the geoid undulation vector depends on the particular type of curvilinear coordinates used. We consider three types of coordinates, the ellipsoidal coordinates

$$\mathbf{q} = [\lambda \ \beta \ u]^T \tag{43}$$

the geodetic coordinates

$$\mathbf{y} = [\lambda \ \varphi_g \ h]^T \tag{44}$$

and the spherical coordinates

$$\mathbf{z} = [\lambda \ \overline{\varphi} \ r]^T \tag{45}$$

which are related to the geocentric cartesian coordinates $\mathbf{x} = [x_1 \ x_2 \ x_3]^T$ by means of

$$\mathbf{x} = \begin{bmatrix} v\cos\beta\cos\lambda \\ v\cos\beta\sin\lambda \\ u\cos\beta \end{bmatrix} = \begin{bmatrix} (N+h)\cos\varphi_g \quad \cos\lambda \\ (N+h)\cos\varphi_g \quad \sin\lambda \\ [N(1-e^2)+h]\sin\varphi_g \end{bmatrix} = \begin{bmatrix} r\cos\overline{\varphi}\cos\lambda \\ r\cos\overline{\varphi}\sin\lambda \\ r\sin\overline{\varphi} \end{bmatrix} \tag{46}$$

where $v = \sqrt{u^2 + E^2}$, $E = \sqrt{a^2 - b^2}$ being the linear eccentricity.

Evaluating $(x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$, $x_3/(x_1^2 + x_2^2)^{\frac{1}{2}}$ and $2x_3\,(x_1^2 + x_2^2)^{\frac{1}{2}}$ and after some algebraic manipulations the following relations are found to hold between the different types of coordinates

$$w \equiv \sqrt{1 - e^2 \sin^2\varphi} = \frac{\sqrt{1-e^2}}{\sqrt{1 - e^2\cos^2\beta}} = \frac{\sqrt{1-e^2}\sqrt{1-e^2\cos^2\overline{\varphi}}}{\sqrt{1 + (e^4 - 2e^2)\cos^2\overline{\varphi}}} \tag{47}$$

$$r = a\sqrt{1 - e^2\sin^2\beta} = \frac{a\sqrt{1 + (e^4 - 2e^2)\sin^2\varphi}}{\sqrt{1 - e^2\sin^2\varphi}} = \frac{a\sqrt{1-e^2}}{\sqrt{1 - e^2\cos^2\overline{\varphi}}} \tag{48}$$

$$\sin 2\beta = \frac{\sqrt{1-e^2}}{1 - e^2\sin^2\varphi}\sin 2\varphi = \frac{\sqrt{1-e^2}}{1 - e^2\cos^2\overline{\varphi}}\sin 2\overline{\varphi} \tag{49}$$

where we have set $\varphi_g = \varphi$, since geodetic and normal latitude coincide on the reference ellipsoid. We also have

$$N = \frac{a}{w} \tag{50}$$

$$M = \frac{a(1-e^2)}{w^3} = \frac{b^2}{a w^3}\,. \tag{51}$$

For the evaluation of the gravity disturbance vector $\delta\mathbf{g}$ we have

$$\delta \mathbf{g} = grad\, \mathrm{T} = \left(\frac{\partial \mathrm{T}}{\partial \mathbf{x}}\right)^{\mathrm{T}} = \left(\frac{\partial \mathbf{q}}{\partial \mathbf{x}}\right)^{\mathrm{T}} \left(\frac{\partial \mathrm{T}}{\partial \mathbf{q}}\right)^{\mathrm{T}} = \left(\frac{\partial \mathbf{y}}{\partial \mathbf{x}}\right)^{\mathrm{T}} \left(\frac{\partial \mathrm{T}}{\partial \mathbf{y}}\right)^{\mathrm{T}} = \left(\frac{\partial \mathbf{z}}{\partial \mathbf{x}}\right)^{\mathrm{T}} \left(\frac{\partial \mathrm{T}}{\partial \mathbf{z}}\right)^{\mathrm{T}} . \quad (52)$$

The matrices of partial derivatives are derived directly from the differentiation of equation (46) (see also Dermanis, 1984)

$$\frac{\partial \mathbf{x}}{\partial \mathbf{q}} = \mathbf{R}_3 (-90^\circ - \lambda)\ \mathbf{R}_1 (\bar{\beta} - 90^\circ)\ \mathbf{G}_q^{\frac{1}{2}} \quad (53)$$

$$\frac{\partial \mathbf{x}}{\partial \mathbf{y}} = \mathbf{R}_3 (-90^\circ - \lambda)\ \mathbf{R}_1 (\varphi_g - 90^\circ)\ \mathbf{G}_y^{\frac{1}{2}} \quad (54)$$

$$\frac{\partial \mathbf{x}}{\partial \mathbf{z}} = \mathbf{R}_3 (-90^\circ - \lambda)\ \mathbf{R}_1 (\bar{\varphi} - 90^\circ)\ \mathbf{G}_z^{\frac{1}{2}} \quad (55)$$

where $\bar{\beta}$ is the latitude of the normal to an ellipsoid passing through the point in question and having the same foci as the reference ellipsoid, defined by

$$tan\, \bar{\beta} = \frac{\mathrm{v}}{\mathrm{u}}\, tan\, \beta \quad (56)$$

The matrices $\mathbf{G}_q$, $\mathbf{G}_y$, $\mathbf{G}_z$ are the diagonal metric matrices of the corresponding orthogonal curvilinear coordinates, defined by

$$\mathbf{G}_q^{\frac{1}{2}} = \begin{bmatrix} \mathrm{v}\, cos\, \beta & 0 & 0 \\ 0 & \mathrm{L} & 0 \\ 0 & 0 & \frac{\mathrm{L}}{\mathrm{v}} \end{bmatrix} \quad (57)$$

$$\mathbf{G}_y^{\frac{1}{2}} = \begin{bmatrix} (\mathrm{N}+\mathrm{h})\, cos\, \varphi_g & 0 & 0 \\ 0 & \mathrm{M}+\mathrm{h} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (58)$$

$$\mathbf{G}_z^{\frac{1}{2}} = \begin{bmatrix} \mathrm{r}\, cos\, \bar{\varphi} & 0 & 0 \\ 0 & \mathrm{r} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (59)$$

where $\mathrm{L} = \sqrt{\mathrm{u}^2 + \mathrm{E}^2 sin^2 \beta}$. On the ellipsoid we have $\mathrm{h} = 0$, $\bar{\beta} = \varphi_g = \varphi$, $\mathrm{u} = \mathrm{b}$, $\mathrm{v} = \mathrm{a}$ and taking equation (25) into account we have

$$\delta \mathbf{g}^* = \mathbf{G}_q^{-\frac{1}{2}} \left(\frac{\partial \mathrm{T}}{\partial \mathbf{q}}\right)^{\mathrm{T}} = \mathbf{G}_y^{-\frac{1}{2}} \left(\frac{\partial \mathrm{T}}{\partial \mathbf{y}}\right)^{\mathrm{T}} = \mathbf{R}_1 (\bar{\varphi} - \varphi)\, \mathbf{G}_z^{-\frac{1}{2}} \left(\frac{\partial \mathrm{T}}{\partial \mathbf{z}}\right)^{\mathrm{T}} \quad (60)$$

or explicitly

$$\delta g_E = \frac{1}{a\,cos\beta}\,\frac{\partial T}{\partial \lambda} = \frac{1}{N\,cos\varphi}\,\frac{\partial T}{\partial \lambda} = \frac{1}{r\,cos\bar{\varphi}}\,\frac{\partial T}{\partial \lambda} \tag{61}$$

$$\delta g_N = \frac{a}{b\,N}\,\frac{\partial T}{\partial \beta} = \frac{1}{M}\,\frac{\partial T}{\partial \varphi} = \frac{a^2}{N\,r^2}\,\frac{\partial T}{\partial \bar{\varphi}} - \frac{e^2\,a^2\,r}{2\,N\,b^2}\,sin\,2\bar{\varphi}\,\frac{\partial T}{\partial r} \tag{62}$$

$$\delta g_Z = \frac{a^2}{b\,N}\,\frac{\partial T}{\partial u} = \frac{\partial T}{\partial h} = \frac{e^2\,a^2}{2\,N\,b^2}\,sin\,2\bar{\varphi}\,\frac{\partial T}{\partial \bar{\varphi}} + \frac{a^2}{N\,r}\,\frac{\partial T}{\partial r}\;. \tag{63}$$

Substituting δg_E, δg_N, δg_Z from the above equations and k_E, k_N, κ_N from equations (40), (41), (42) respectively into equations (39a), (39b) and (39c) we obtain the following relations for the components of the geoid undulation vector

$$\zeta_E = \frac{N}{\gamma\,a\,cos\beta}\,\frac{\partial T}{\partial \lambda} = \frac{1}{\gamma\,cos\varphi}\,\frac{\partial T}{\partial \lambda} = \frac{N}{\gamma\,r\,cos\bar{\varphi}}\,\frac{\partial T}{\partial \lambda} \tag{64}$$

$$\begin{aligned} \zeta_N &= \frac{a\,M}{\gamma\,b\,N}\,\frac{\partial T}{\partial \beta} + \rho\,sin\,2\beta\,\frac{T}{\gamma} = \\ &= \frac{1}{\gamma}\,\frac{\partial T}{\partial \varphi} + \rho\,\frac{aM}{bN}\,sin\,2\varphi\,\frac{T}{\gamma} = \\ &= \frac{M\,a^2}{\gamma\,N\,r^2}\,\frac{\partial T}{\partial \bar{\varphi}} - \frac{e^2\,M\,a^2\,r\,sin\,2\bar{\varphi}}{2\,\gamma\,N\,b^2}\,\frac{\partial T}{\partial r} + \rho\,\frac{r^2}{ab}\,sin\,2\bar{\varphi}\,\frac{T}{\gamma} \end{aligned} \tag{65}$$

$$\zeta_Z = \frac{T}{\gamma}\;. \tag{66}$$

Since for points on the reference ellipsoid the normal gravity vector γ is perpendicular to the ellipsoid surface, the classical Bruns formula gives the component of the geoid undulation in the direction normal to the reference ellipsoid which is not the same as the geoid-to-ellipsoid separation in the same direction, due to the presence of the non-zero components ζ_E, ζ_N in the plane tangent to the ellipsoid.

If N, M and r are substituted from equations (47), (48), (50) and (51), the components of the geoid undulation vector, ζ_E along the parallel circle, ζ_N along the meridian and ζ_Z along the normal to the ellipsoid become

$$\begin{aligned} \zeta_E &= \frac{\sqrt{1-e^2\,cos^2\beta}}{\gamma\,cos\beta\sqrt{1-e^2}}\,\frac{\partial T}{\partial \lambda} = \frac{1}{\gamma\,cos\varphi}\,\frac{\partial T}{\partial \lambda} = \\ &= \frac{\sqrt{1+(e^4-2e^2)\,cos^2\bar{\varphi}}}{\gamma\,cos\bar{\varphi}\,(1-e^2)}\,\frac{\partial T}{\partial \lambda} \end{aligned} \tag{67}$$

$$\zeta_N = \frac{1 - e^2 \cos^2\beta}{\gamma\sqrt{1-e^2}} \frac{\partial T}{\partial \beta} + \rho \sin 2\beta \frac{T}{\gamma} =$$

$$= \frac{1}{\gamma}\frac{\partial T}{\partial \varphi} + \rho \frac{\sqrt{1-e^2}}{1 - e^2 \sin^2\varphi} \sin 2\varphi \frac{T}{\gamma} = \tag{68}$$

$$= \frac{1 + (e^4 - 2e^2)\cos^2\bar{\varphi}}{\gamma} \left\{ \frac{1}{1-e^2}\frac{\partial T}{\partial \bar{\varphi}} - \frac{a e^2 \sin 2\bar{\varphi}}{2\sqrt{1-e^2}\,(1 - e^2\cos^2\bar{\varphi})^{3/2}} \frac{\partial T}{\partial r} \right\} +$$

$$+ \rho \frac{\sqrt{1-e^2}}{1 - e^2\cos^2\bar{\varphi}} \sin 2\bar{\varphi} \frac{T}{\gamma}$$

$$\zeta_Z = \frac{T}{\gamma} . \tag{69}$$

The normal gravity on the reference ellipsoid can be obtained using the formula of Somigliana (Heiskanen and Moritz, 1967, p. 70).

A spherical approximation can be obtained by letting $N, M, a, r \to R$, where R is a mean radius of the earth and neglecting terms of the order of e^2 and higher, in which case

$$\zeta_E \approx \frac{1}{\gamma \cos\varphi} \frac{\partial T}{\partial \lambda} = - R\,\eta \tag{70}$$

$$\zeta_N \approx \frac{1}{\gamma} \frac{\partial T}{\partial \varphi} = - R\,\xi \quad . \tag{71}$$

The horizontal components ζ_E, ζ_N of the geoid undulation vector are therefore directly related to the deflections of the vertical ξ and η in a spherical approximation.

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